Scaling in Opinion Dynamics

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ODE model

- Opinions in $x_i \in [-1,1]$ for $i = 1, \dots, N$.
- A function $\phi : \mathbb{R} \to [0, 1]$ determines how individuals weight each others' opinions.
- Opinions evolve according to:

$$\frac{dx_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} \phi\big(|x_j - x_i|\big)(x_j - x_i)$$

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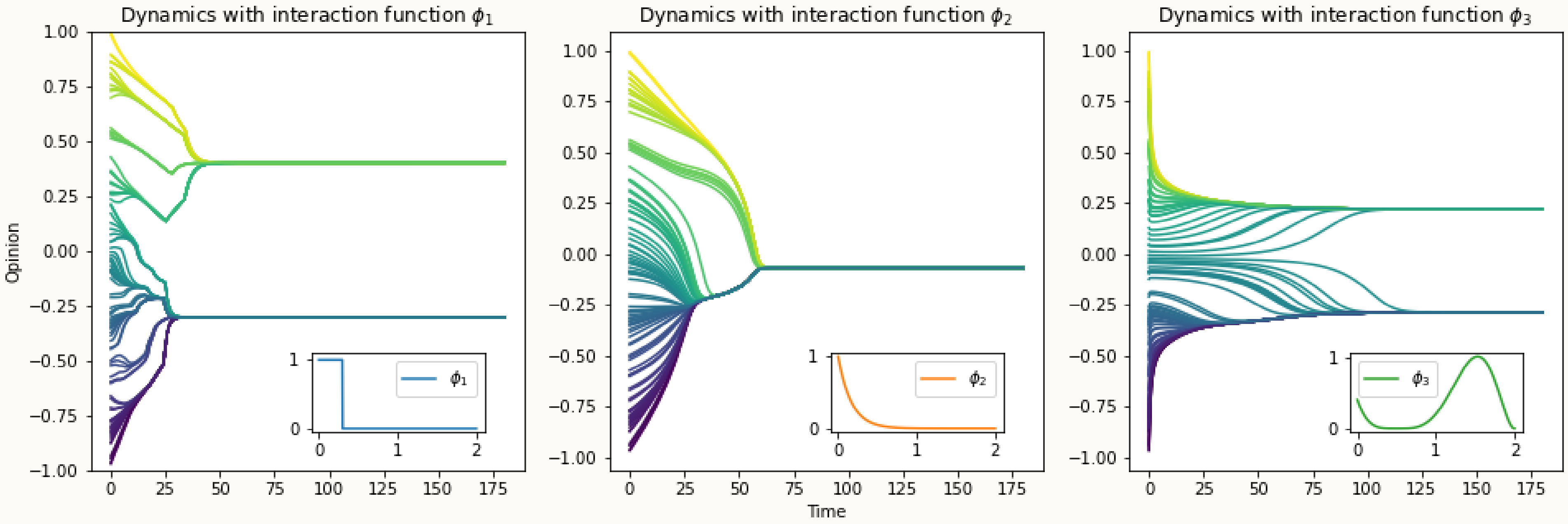


Unfortunately… reality

1. Social interactions are **discrete events**, meaning
2. We cannot interact with **everyone simultaneously**, so
3. The **order** of interactions makes a difference.
4. Also, interactions occur **randomly**.

Unfortunately… reality

1. Social interactions are discrete events, meaning
   - ☒ Social interactions happen **continuously** and
2. We cannot interact with everyone simultaneously, so
   - ☒ Everyone interacts **simultaneously**, so
3. The order of interactions makes a difference.
   - ☒ There is **no sequence** of interactions, and so **no order**.
4. Also, interactions occur randomly.
   - ☒ Interactions are also **deterministic**.

Agent-based model

1. Choose two individuals i and j uniformly at random with replacement.
2. They interact with probability $p_{ij}(x)$, with individual i updating their opinion according to:

$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(x_j(t) - x_i(t)\right) & \text{with probability } p_{ij}(x) \\ x_i(t) & \text{with probability } 1 - p_{ij}(x)\,. \end{cases}$$

3. Repeating until time T is reached.

Closer to reality

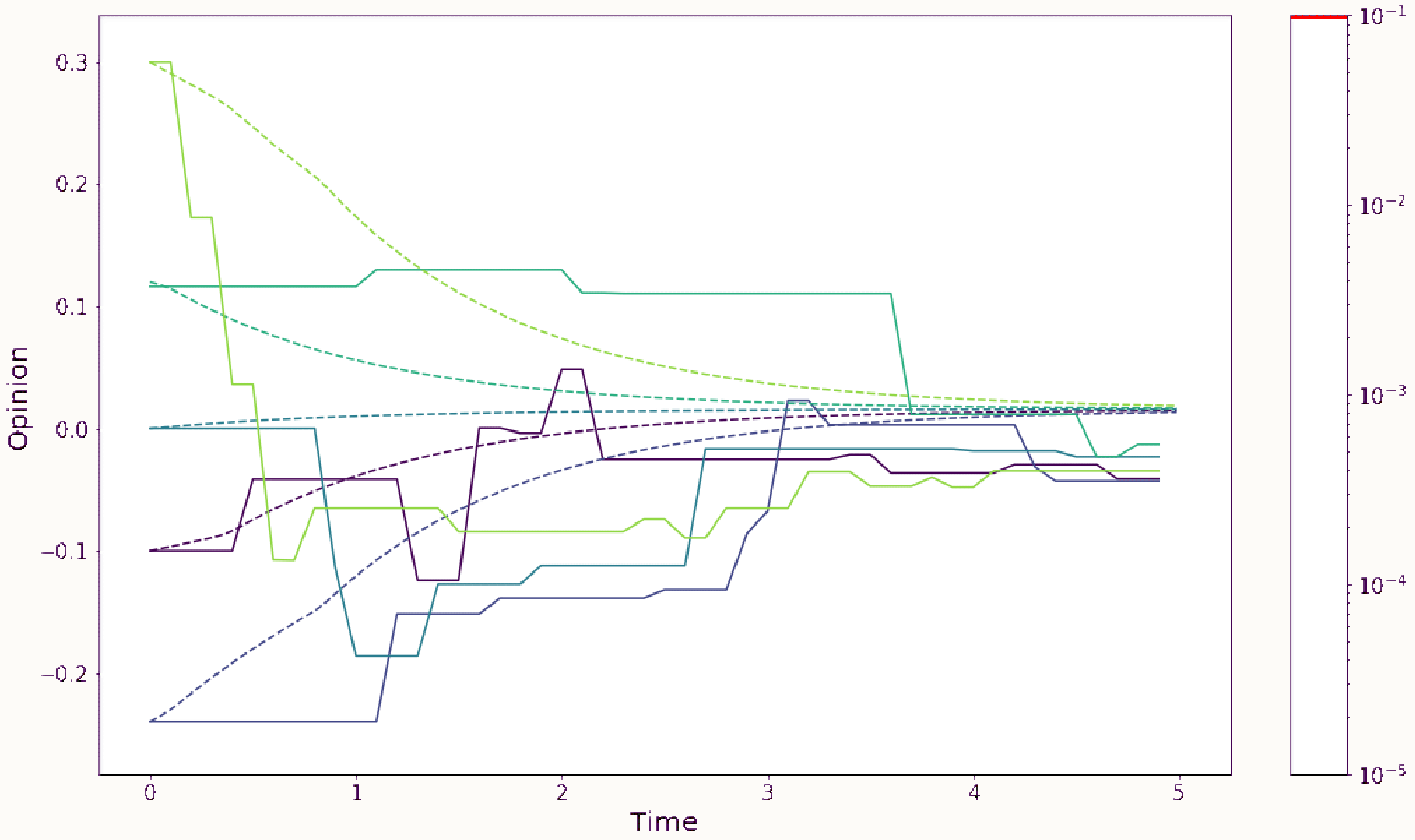
1. Social interactions are discrete events, meaning
   - ☑ Social interactions are **discrete events**.
2. We cannot interact with everyone simultaneously, so
   - ☑ Individuals interact **pairwise**, not globally.
3. The order of interactions makes a difference.
   - ☑ Interactions happen **in an order**.
4. Also, interactions occur randomly.
   - ☑ Interactions occur **randomly**.

Link to the ODE model?

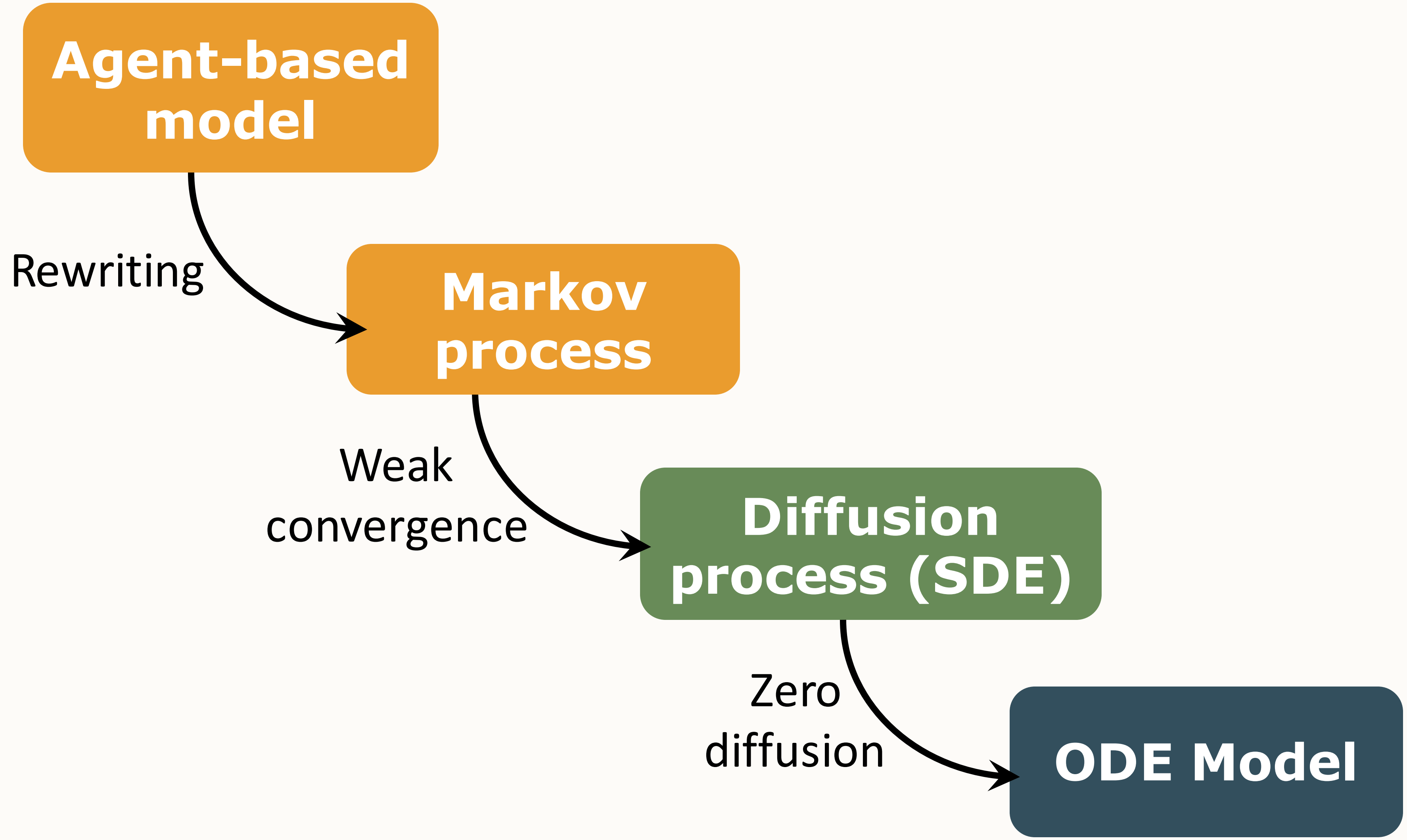
Claim:

1. The ODE model can be obtained by rescaling the timestep h and update distance $\mu^h = Nh$ of the ABM.
2. This means the ODE model approximates **small but frequent pairwise interactions,** so is a realistic model if these assumptions hold.

Link to the ODE model?



Establishing the link



Markov process

Consider a discrete time Markov process with state space $\mathbb{R}^N$, time step $h > 0$ and transition function:

$$\Pi^h(x,y) = \begin{cases} \frac{1}{N^2} p_{ij}(x) & \text{if } y = x + e_i \mu^h(x_j - x_i) \text{ for some } i \neq j, \\ \frac{1}{N} + \frac{1}{N^2} \sum_{i \neq j} \big(1 - p_{ij}(x)\big) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Markov process

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SDE system

$$dX_i = b_i(X)\, dt + \sqrt{a_{ii}(X)}\, dW_i$$

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Approximating generator

SDE system

$$dX_i = b_i(X)\,dt + \sqrt{a_{ii}(X)}\,dW_i$$

Approximating generator

We calculate the mean and variance of increments to **approximate the drift and diffusion terms** of an SDE.

$$b_i^h(x) = \frac{1}{h}\int_{\mathbb{R}^N} (y_i - x_i)\, \Pi^h(x, dy)$$

$$a_{ij}^h(x) = \frac{1}{h}\int_{\mathbb{R}^N} (y_i - x_i)(y_j - x_j)\, \Pi^h(x, dy)$$

A slight detour into generators...

$$
\begin{aligned}
(\mathcal{L}f)(x) &= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}\big[f\big(X(h)\big) - f(x)\,\big|\, X_0 = x\big] \\
&\approx \lim_{h\downarrow 0} \left(f'(x)\,\frac{1}{h}\mathbb{E}\big[X(h) - x \,\big|\, X_0 = x\big] + \frac{f''(x)}{2}\,\frac{1}{h}\mathbb{E}\big[\big(X(h) - x\big)^2 \,\big|\, X_0 = x\big] \right) \\
&\approx \lim_{h\downarrow 0} \left(f'(x)\,\frac{1}{h}\int_{\mathbb{R}} (y - x)\,\Pi^h(x, dy) + \frac{f''(x)}{2}\,\frac{1}{h}\int_{\mathbb{R}} (y - x)^2\,\Pi^h(x, dy) \right) \\
&\approx \lim_{h\downarrow 0} \Bigg(\underbrace{f'(x)\,b^h(x) + \frac{f''(x)}{2}\,a^h(x)}_{dX = b^h(X)\,dt + \sqrt{a^h(X)}\,dW} \Bigg)
\end{aligned}
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\end{aligned}
$$

And we're back

We calculate the mean and variance of increments to **approximate the drift and diffusion terms** of an SDE.

$$b_i^h(x) = \frac{1}{h} \int_{\mathbb{R}^N} (y_i - x_i) \, \Pi^h(x, dy)$$

$$a_{ij}^h(x) = \frac{1}{h} \int_{\mathbb{R}^N} (y_i - x_i)(y_j - x_j) \, \Pi^h(x, dy)$$

Convergence to SDE

By setting $\mu^h = Nh$ we obtain:

$$b_i^h(x) = \frac{1}{N}\sum_{j=1}^{N} p_{ij}(x)\left(x_j - x_i\right),$$

$$a_{ii}^h(x) = h\left(\sum_{j=1}^{N} p_{ij}(x)\left(x_j - x_i\right)^2\right).$$

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Using some results from Durrett's Stochastic Calculus, we have that **as $h \to 0$ the ABM converges** in probability to the solution of

$$\frac{dx_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} p_{ij}(x)\left(x_j - x_i\right)$$

Is this the ODE model?

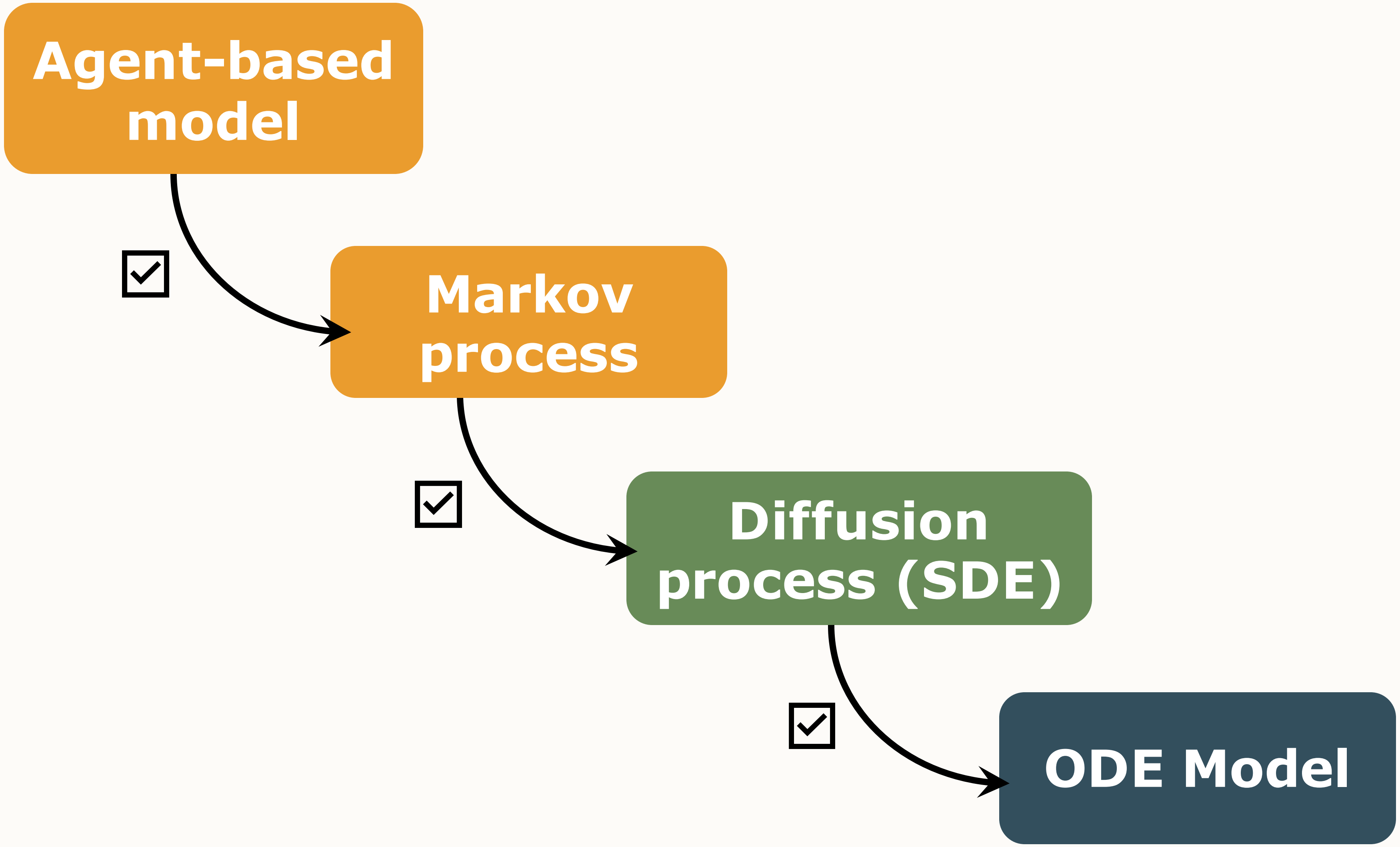
$$\frac{dx_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} p_{ij}(x)\,(x_j - x_i)$$

$$p_{ij}(x) = \phi(|x_j - x_i|)$$

The interaction function is precisely the **probability** of an interaction occurring.

$$\frac{dx_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} \phi\big(|x_j - x_i|\big)(x_j - x_i)$$

Link established!

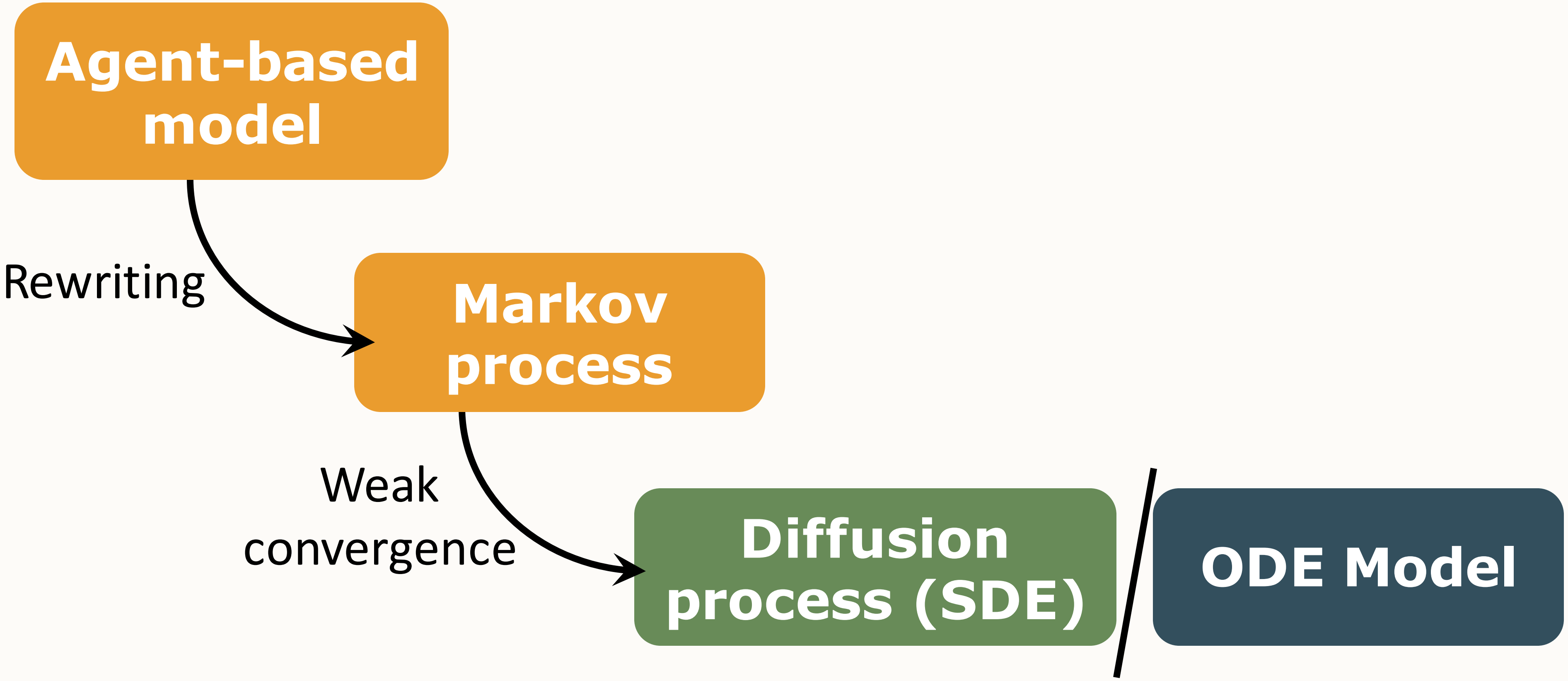


What does this mean?

Result:

1. The ODE model can be obtained by rescaling the timestep h and update distance $\mu^h = Nh$ of the ABM.
2. This means the ODE model approximates **small but frequent pairwise interactions,** so is a realistic model if these assumptions hold.

Following the link



External noise

Change the update rule by adding a new noise term ξ^h with mean zero and variance that decreases with h.

$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(x_j(t) - x_i(t)\right) + \xi^h & \text{with probability } p_{ij}(x) \\ x_i(t) + \xi^h & \text{with probability } 1 - p_{ij}(x)\,. \end{cases}$$

This noise translates directly in the limiting model

$$dX_i = \underbrace{\frac{1}{N}\sum_{j=1}^{N} p_{ij}(X)\,(X_j - X_i)\,\mathrm{d}t}_{\text{Same drift as ODE model}} + \underbrace{\sqrt{\frac{1}{N}\lim_{h\to 0}\left(\frac{\mathbb{E}\left[(\xi^h)^2\right]}{h}\right)}\,\mathrm{d}\beta_i}_{\text{New diffusion term}}$$

Same drift as ODE model

New diffusion term

Noisy update distance

Replace the fixed update distance with a random variable

$$x_i(t+h) = \begin{cases} x_i(t) + \nu^h \left(x_j(t) - x_i(t)\right) & \text{with probability } p_{ij}(x) \\ x_i(t) & \text{with probability } 1 - p_{ij}(x)\,. \end{cases}$$

Again, this noise translates into the limiting model

$$dX_i = \frac{1}{N}\sum_{j=1}^{N} p_{ij}(X)\,(X_j - X_i)\,\mathrm{d}t + \sqrt{\frac{m_2(\nu)}{N^2}\left(\sum_{j=1}^{N} p_{ij}(X)\,(X_j - X_i)^2\right)}\,\mathrm{d}\beta_i$$

Ambiguity noise

Assume that the interaction probability is given by

$$p_{ij}(x) = \phi(|x_j - x_i|)$$

and updates are given by

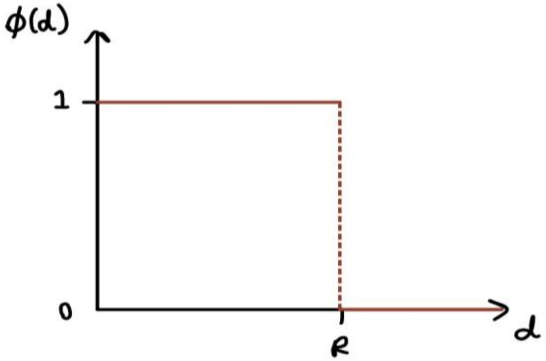
$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(\omega_j(t) - x_i(t)\right) & \text{with probability } \phi(|\omega_j - x_i|) \\ x_i(t) & \text{with probability } 1 - \phi(|\omega_j - x_i|)\,. \end{cases}$$

with $\omega_j = x_j + \eta^h$.

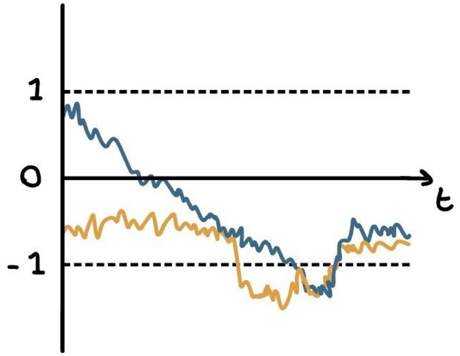
Then we obtain the same ODE as before!

$$\frac{dx_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} p_{ij}(x)\,(x_j - x_i)$$

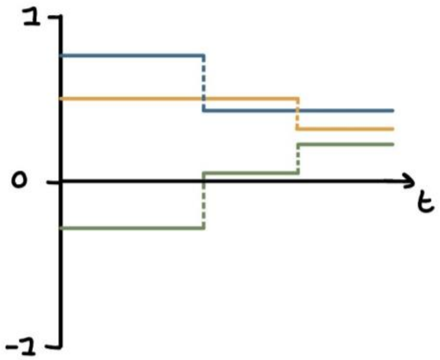
Limitations



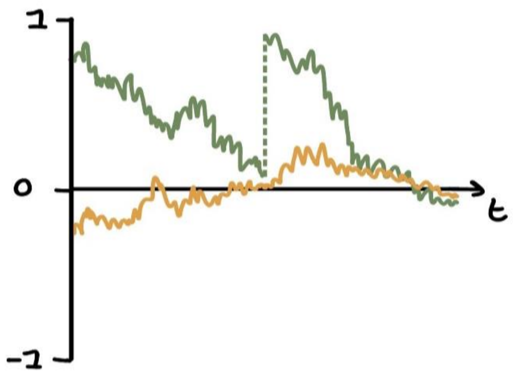
Discontinuous ϕ



Boundary conditions



Invalid assumptions



Additional features in ABM

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