Lecture 10

Parallel transport

Objectives:

- Parallel transport
- Geodesics
- Equations of motion

Reading: Schutz 6; Hobson 3; Rindler 10.

In this lecture we are finally going to see how the metric determines the motion of particles. First we discuss the concept of “parallel transport”.

10.1 Parallel transport

In SR, the equation for force-free motion of a particle is

\[ \ddot{A} = \frac{d\vec{U}}{d\tau} = 0, \]

i.e a straight line through spacetime as well as 3D space with the vector \( \vec{U} \) remaining constant along the line parameterised by \( \tau \).

This is extended to the curved spacetime of GR by the notion of parallel “transport” in which a vector is moved along a curve staying parallel to itself and of constant magnitude.
Figure: Parallel transport of a vector from A to B, keeping it parallel to itself and of constant length at all points.

Consider the change of a vector $\vec{V}$ along a line parameterised by $\lambda$

$$\frac{d\vec{V}}{d\lambda} = \frac{dV^\alpha}{d\lambda} \vec{e}_\alpha + V^\alpha \frac{d\vec{e}_\alpha}{d\lambda}. $$

We can write

$$\frac{d\vec{e}_\alpha}{d\lambda} = \frac{\partial \vec{e}_\alpha}{\partial x^\beta} \frac{dx^\beta}{d\lambda}. $$

Using this and the definition of the connection

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} \vec{e}_\gamma, $$

gives

$$\frac{d\vec{V}}{d\lambda} = \frac{dV^\alpha}{d\lambda} \vec{e}_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta} \frac{dx^\beta}{d\lambda} \vec{e}_\gamma. $$

Swapping dummy indices $\alpha$ and $\gamma$ in the second term finally leads to

$$\frac{d\vec{V}}{d\lambda} = \left( \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} \frac{dx^\beta}{d\lambda} V^\gamma \right) \vec{e}_\alpha. $$

This is a vector with components

$$\frac{DV^\alpha}{D\lambda} = \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} \frac{dx^\beta}{d\lambda} V^\gamma, $$

and is known variously as the “intrinsic”, “absolute” or “total” derivative. One also sometimes sees the vector written as

$$\frac{d\vec{V}}{d\lambda} = \nabla_\theta \vec{V}, $$

where $U^\alpha = dx^\alpha/d\lambda$ is the “tangent vector” pointing along the line ($= \text{four-velocity if } \lambda = \tau$).
The components are very similar to the covariant derivative

\[ V^\alpha;_\beta = V^\alpha,\beta + \Gamma^\alpha\gamma^\beta V^\gamma. \]

In fact if we write

\[ \frac{dV^\alpha}{d\lambda} = \frac{\partial V^\alpha}{\partial x^\beta} \frac{dx^\beta}{d\lambda} = \frac{\partial V^\alpha}{\partial x^\beta} U^\beta, \]

(a cheat: \( V^\alpha \) might only be defined on the line) then we can write

\[ \frac{DV^\alpha}{D\lambda} = V^\alpha;_\beta U^\beta. \]

Parallel transport: if a vector \( \vec{V} \) is “parallel transported” along a line then

\[ \nabla_0 \vec{V} = \frac{d\vec{V}}{d\lambda} = 0, \]

or in component form:

\[ \frac{DV^\alpha}{D\lambda} = \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha\gamma^\beta \frac{dx^\beta}{d\lambda} V^\gamma = 0. \]

\[ \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha\gamma^\beta \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0, \]

More compactly

\[ \ddot{x}^\alpha + \Gamma^\alpha\gamma^\beta \dot{x}^\beta \dot{x}^\gamma = 0, \]

using the “dot” notation for derivatives wrt \( \lambda \).

- These are force-free equations of motion
- Extends SR \( \ddot{A} = d\ddot{U}/d\tau = 0 \) to GR.
• In GR, gravity is not a force but a distortion of spacetime

• Metric $g_{\alpha\beta} \rightarrow \Gamma^\gamma_{\alpha\beta} \rightarrow$ particle motion.

• Straight lines are often called geodesics. “Great circles” are geodesics on spheres.

10.2.1 Affine parameters

We could have defined “straight” by $\nabla \bar{U} = k\bar{U}$, i.e. the tangent vector changes by a vector parallel to itself. However in such cases one can always transform to a new parameter, say $\mu = \mu(\lambda)$, such that $\nabla \bar{U}' = 0$, where $\bar{U}'$ is the new tangent vector. $\mu$ is then called an affine parameter. Proper time $\tau$ is affine for massive particles.

I will always assume affine parameters.

10.3 Example: motion under a central force

Consider motion under Newtonian gravity

$$\frac{d\bar{V}}{dt} = -\frac{GM}{r^2} \hat{r}.$$ 

In general coordinates the left-hand side is

$$\frac{dV^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} V^\beta V^\gamma.$$ 

In polar coordinates $\bar{V} = (\dot{r}, \dot{\theta}).$

From last time $\Gamma^r_{\theta\theta} = -r, \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$ with all others $= 0$. Therefore:

$$\frac{dV^r}{dt} + \Gamma^r_{\theta\theta} V^\theta V^\theta = -\frac{GM}{r^2},$$

and

$$\frac{dV^\theta}{dt} + \Gamma^\theta_{r\theta} V^r V^\theta + \Gamma^\theta_{\theta r} V^\theta V^r = 0.$$ 

These give

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2},$$

and

$$\ddot{\theta} + \frac{2}{r} r\dot{\theta} = 0.$$ 

The second can be integrated to give the well known conservation of angular momentum $r^2 \dot{\theta} = h.$
These two equations are the equations of planetary motion which lead to ellipses and Kepler’s laws. The point here is how the connection allows one to cope with familiar equations in awkward coordinates. In much of physics such coordinates can be avoided, but not in GR where there is no sidestepping the connection. Note here how the centrifugal term, $r\dot{\theta}^2$, appears via the connection.