Lecture 6

Stress–energy tensor

Objectives:

• To introduce the stress–energy tensor
• Conservation laws in relativity

Reading: Schutz chapter 4; Hobson, chapter 8; Rindler, chapter 7.

6.1 Number–flux vector

Consider a cloud of particles ("dust") at rest in frame $S_0$, the “instantaneous rest frame” or IRF with number density $n_0$.

Lorentz contraction means that a cube $dx_0$, $dy_0$, $dz_0$ in $S_0$ transforms to $dx = dx_0/\gamma$, $dy = dy_0$, $dz = dz_0$ in a frame $S$ in which the particles move, while particle numbers are conserved, so in $S$ the particle density $n$ is given by

$$n = \gamma n_0.$$ 

$n$ is not a scalar or a four-vector and so cannot be part of form-invariant relations. Consider instead

$$\vec{N} = n_0 \vec{U}.$$ 

This is a four-vector because

• The four velocity $\vec{U} = \gamma(c, v)$ is a four-vector
• $n_0$ is a scalar (defined in the IRF so all observers agree on it).
The time component \( N^0 = \gamma n_0 c = nc \) gives the number density. The spatial components \( N^i = \gamma n_0 v^i = nv^i, \ i = 1, 2, 3 \) are the fluxes (particles/unit area/unit time) across surfaces of constant \( x, y \) and \( z \).

Even \( N^0 \) is a “flux across a surface”, a surface of constant time:

\[ \text{Sketch this:} \]

![Figure: World lines of dust particles travelling at speed \( v \) in the \( x \)-direction crossing surfaces of constant \( t \) (\( A-B \)) and constant \( x \) (\( B-C \)).]

Worldlines crossing \( CB \) represent the flux across constant \( x \), \( N^1 = nv \)

Same worldlines crossing \( AB \) represent flux across constant \( t \). Scaling by ratio of sides of triangle we get a flux:

\[
\frac{N^1_{CB}}{AB} = N^1 \frac{\Delta (ct)}{\Delta x} = N^1 \frac{c}{v} = N^0,
\]

so \( N^0 \) is the particle flux across a surface of constant time.

### 6.2 Conservation of particle numbers

Consider the scalar \( \nabla (\vec{N}) \) (one-form \( \nabla \) acting on \( \vec{N} \)). Written out in full:

\[
\nabla (\vec{N}) = \frac{\partial N^0}{\partial x^0},
\]

\[
\quad = \frac{\partial N^0}{\partial x^0} + \frac{\partial N^1}{\partial x^1} + \frac{\partial N^2}{\partial x^2} + \frac{\partial N^3}{\partial x^3},
\]

\[
\quad = \frac{\partial nc}{\partial ct} + \frac{\partial n v_x}{\partial x} + \frac{\partial n v_y}{\partial y} + \frac{\partial n v_z}{\partial z}.
\]
This can be written as
\[ \frac{\partial n}{\partial t} + \nabla \cdot (nv) . \]

Compare with the continuity equation of fluid mechanics:
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 , \]

based on (Newtonian) conservation of mass.  \( \implies \) if particles are conserved:
\[ \frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0 . \]

Thus conservation of particle numbers can be expressed as:
\[ \hat{\nabla} (\vec{N}) = \frac{\partial N^\alpha}{\partial x^\alpha} = \partial_{\alpha} N^\alpha = N^\alpha_{\alpha} = 0 , \] (6.1)

introducing the short-hand \( \partial_{\alpha} = \partial/\partial x^{\alpha} \), and the even shorter-hand comma notation for derivatives.

## 6.3 Stress–energy tensor

If the mass density in the IRF is \( \rho_0 \), then due to Lorentz contraction and relativistic mass increase, in any other frame it becomes:
\[ \rho = \gamma^2 \rho_0 , \]

Now consider
\[ T^{\alpha\beta} = \rho_0 U^\alpha U^\beta , \]

then since \( U^0 = \gamma c \),
\[ T^{00} = \gamma^2 \rho_0 c^2 = \rho c^2 . \]

From \( E = mc^2 \), \( T^{00} \) must therefore be the energy density.

\( T \) is a tensor because

- The four velocity \( \vec{U} \) is a four-vector
- \( \rho_0 \) is a scalar (defined in the IRF)

\( T \) is called the stress–energy tensor.
6.3.1 Physical meaning

$T^{\alpha \beta}$ is the flux of the $\alpha$-th component of four-momentum across a surface of constant $x^\beta$, so:

- $T^{00} = \text{flux of 0-th component of four-momentum (energy) across the time surface (cf } N^0) = \text{energy density}$
- $T^{0i} = T^{i0} = \text{energy flux across surface of constant } x^i \text{ (heat conduction in IRF)}$
- $T^{ij} = \text{flux of } i\text{-momentum across } j\text{ surface = “stress”}$.

6.4 Perfect fluids

Definition: a perfect fluid has (i) no heat conduction and (ii) no viscosity.

In the IRF (i) implies $T^{0i} = T^{i0} = 0$, while (ii) implies $T^{ij} = 0$ if $i \neq j$.

For $T^{ij}$ to be diagonal for any orientation of axes $\implies T^{ij} = p_0 \delta^{ij}$ where $p_0$ is the pressure in the IRF. Therefore in the IRF:

$$T^{\alpha \beta} = \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix}.$$

But this can be written:

$$T^{\alpha \beta} = \left( \rho_0 + \frac{p_0}{c^2} \right) U^\alpha U^{\beta} - p_0 \eta^{\alpha \beta},$$

and since all terms are tensors, this is true in any frame remembering that $\rho_0$ and $p_0$ are defined in the IRF.

Just as conservation of particles implies $N^{\alpha, \alpha} = 0$, so energy–momentum conservation gives

$$T^{\alpha \beta}_{, \beta} = \frac{\partial T^{\alpha \beta}}{\partial x^\beta} = 0.$$

This equation plays a key role in GR where the stress–energy tensor replaces the simple density, $\rho$, of Newtonian gravity.