Article

3D Reconstruction of Coronal Loops by the Principal Component Analysis

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Abstract: Knowing the three dimensional structure of plasma filaments in the uppermost part of the solar atmosphere, known as coronal loops, and especially their length, is an important parameter in the wave-based diagnostics of this part of the Sun. The combination of observations of the Sun from different points of observations in space, thanks to the most recent missions including SDO and STEREO, allows us to infer information on the geometrical shape of coronal loops in the 3D space. Here, we propose a new method to reconstruct the loop shape starting from stereoscopically determined 3D points, which sample the loop length, by the Principal Component Analysis. This method is shown to retrieve in an easy way the main parameters that define loop, e.g. the minor and major axes, the loop plane, the azimuthal and inclination angles, for the special case of a coplanar loop.

Keywords: Solar corona; Coronal loop; 3D reconstruction

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1. Introduction

Dynamics of the uppermost part of the atmosphere of the Sun, known as the solar corona, is one of the key elements of the magnetohydrodynamic (MHD) interaction of the Sun with the Earth and other planets, and is important for basic plasma physics. Understanding how the solar corona is structured in the three dimensional (3D) space is a challenging and no trivial task. Excluding solar eclipses, which
allow us to see the corona in the visible light for only a few minutes, routine observations from telescopes or coronagraphs aboard spacecraft (such as SoHO, TRACE, STEREO, Hinode, and more recently SDO) at extreme ultra-violet (EUV) and X-ray bandpasses, are the main source of information about the complex dynamics of the corona. In these bands, the corona is usually an optically thin medium: an emitted photon can go through it without experiencing any noticeable absorption or scattering. This seriously affects the identifications or tracking of important features (e.g. coronal loops and plumes) in imaging data obtained from a single point of observation. A crucial parameter in this sense is the column depth [1;2], that gives a measure of how much the intensity of a given structure is integrated along the line-of-sight (LOS).

The launch of the Solar TErrestrial RElations Observatory (STEREO) in 2006 has opened the possibility to make stereoscopy, thanks to the twin spacecraft observing the Sun at about 1 AU but with two different LOS [3]. The STEREO mission’s aim is to understand the 3D structure of coronal mass ejections (CMEs) and their impact on the solar system and Earth environment [see 4], but it is suitable also for 3D reconstruction of the geometry of loops in coronal active regions (ARs) [for a review of the topic, see 5]. In particular, observations show that coronal loops are subject to transverse (kink) oscillations [e.g. 6–11]. These oscillations are a good tool for the diagnostics of the magnetic field inside the loop that depends on the period of the kink oscillations, its length, the plasma density [12], and of the loop’s sub-resolution structuring [13]. Therefore, a better estimate of the loop length, could easily improve the inference of the field. The value of the field is one of the decisive parameters of AR for answering the enigmatic questions of solar and stellar physics such as the mechanisms for coronal heating and flaring energy releases, coronal mass ejections and the solar and stellar wind acceleration.

A forward-modelling 3D loop reconstruction method has been developed by Verwichte et al. [14,15] in the context of studying kink oscillations of coronal loops and reducing the error in measuring loop lengths. The method works as follows. In one view, the plane-of-sky loop coordinates are traced. A third coordinate that denotes the loop observed depth is added by assuming that the loop is planar (or non-planar, following a simple geometrical model). The problem is thus reduced to finding the value of one (or a few) parameters such as the inclination angle of the loop plane with respect to the normal to the solar surface. For a range of inclination angles, the loop is forward-modelled to a new view and the angle with the best fit is determined visually. In Verwichte et al. [14,15] a second viewpoint was absent. In the first study, a second viewpoint was created from data of several days later, using the change of the LOS by the solar rotation. In many respect this method is then akin to the dynamic stereoscopy method by Aschwanden et al. [16]. In the second study, the inclination angle is found by minimising the variation in curvature. Later, this method has been applied to loops seen simultaneously with SDO and STEREO [10;17–19]. This method has the advantage of not requiring accurate tracing of the loop from both views and does not impose a specific shape of the loop (e.g. semi-circular) except for planarity. Because of the difference in the spatial resolution between SDO/AIA and STEREO/EUVI, tracing a loop in the AIA view and forward-modelling to the EUVI view gives the best results.

In a strict sense, 3D reconstruction is performed by observations of a solar feature from at least two observers placed at different positions: a technique that is called “stereoscopy”, and allows for the determination of the 3D coordinates in the space of an observed point [for an explanation on the princi-
The SolarSoftWare (SSW) package (http://www.lmsal.com/solarsoft) provides useful tools for making 3D reconstruction, e.g. the procedures scc_measure.pro or sunloop.pro. The former includes a widget application (see Fig. 1 for a typical graphical output) that enables the user to select with the cursor a solar feature of interest appearing in both perspectives from one of the STEREO spacecraft (but it is also possible to combine SDO and one of STEREO spacecraft). By selecting a point in one image, the program displays a line in the other image representing the LOS from the first image (or the epipolar line). The user then selects the point along this line with the same feature. The 3D coordinates are then calculated as (Earth-based) Stonyhurst heliographic longitude and latitude, along with the radial distance in solar radii. The coordinates can be easily converted in the Heliocentric Earth Equatorial (HEEQ) coordinates for a Cartesian representation of the data [for a complete review of the coordinate systems, please refer to 22]. As we have seen, fitting a loop with a 1D curvilinear feature (circular or elliptical) depends on 6 parameters, namely: the heliographic longitude and latitude of the midpoint of the loop baseline on the solar surface, the baseline length between the two footpoints, the height of the loop, the inclination and azimuth angle [see Sec. 3.4.4 of 23]. A representation of the loop and its main parameters are shown in Fig. 2. Recently, Gary et al. [24]

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1 A description is given at the following link, as well: http://www.lmsal.com/~aschwand/software/loopgeo/loopgeo_tutorial1.html
2 The procedures can be downloaded with SSW or viewed at http://hesperia.gsfc.nasa.gov/ssw/stereo/secchi/idl/display/scc_measure.pro and http://hesperia.gsfc.nasa.gov/ssw/stereo/secchi/idl/display/sunloop/sunloop.pro

Figure 1. Graphical output of the procedure scc_measure: the windows give the field of view from STEREO-A (right) and STEREO-B (left) spacecraft for a coronal loop on 27th June, 2007, studied by Aschwanden [5]; Verwichte et al. [14].
Figure 2. Schematic representation of a coronal loop relative to the solar surface, with the baseline connecting the footpoints, the inclination angle relative to the normal surface, and the azimuth angle relative to the East-west direction. ** Courtesy of Markus Aschwanden.**

proposed a fitting approach of loops by cubic Bézier curves, in order to distinguish between competing models for the magnetic field topology.

In this paper, we propose a new method allowing to get the 3D shape of a loop by stereoscopic measurements, using the principal component (PC) analysis. It reduces the number of parameters, dependencies and correlations between variables, by calculating a new basis of vectors, which defines the reference system relative to the loop. In the next section, we give an overview of PC analysis in the frame of 3D loop reconstruction, in Section 4 we present some examples, discussing analysis and comparing results with previous methods, and Conclusions are in the last section.

2. Three-dimensional reconstruction by Principal Component Analysis

Given a set of variables, which can be interrelated, PC analysis allows to transform this set into a new one of “uncorrelated” or independent variables, known as principal components, but preserving the variation of the data distribution. The Principal Component Analysis, also know as Minimum Variance Analysis, has been applied in several research fields, e.g. in the determination of the magnetic field geometry by in-situ multi-spacecraft measurements [25]. Algebraically, PCs are particular linear combinations of the set of original variables, and geometrically, this is equivalent to the rotation of the original coordinate system into a new one, where the new axes represent the maximum variability of the data set. Avoiding a rigorous mathematical treatment, that can be found in textbooks, as Jolliffe [26]; Johnson and Wichern [27], we give here a specific step-by-step description of how to fit coronal loops by PC analysis.
Given a set of $N$ 3D data points $x_i = (x_i, y_i, z_i)$ (we use for simplicity the HEEQ system), which sample
the loop shape and is represented by an $N \times 3$ matrix $X$, we can build the sample covariance matrix $S$, defined as

$$X^T X = S = \begin{pmatrix}
\sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\
\sigma_{xy}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\
\sigma_{xz}^2 & \sigma_{yz}^2 & \sigma_{zz}^2
\end{pmatrix}, \tag{1}
$$

where $\sigma_{ij}^2 = \sum_{i=1}^{N}(q_i - \bar{q}_i)(q'_i - \bar{q}_i)$, with $q = (x, y, z)$ and $\bar{q}$ the mean, identified as the centre $C = (\bar{x}, \bar{y}, \bar{z})$ of the loop. Now, we would like to put in a new reference frame, whose axes maximise the variances and minimise the covariances of the points, and have means equal to zero. This is equivalent to find the eigenvalues and eigenvectors that diagonalise the matrix $S$ in a matrix $\Lambda$. Let us obtain the eigenvalues and sort them in the ascending order, as

$$\lambda_n \leq \lambda_a \leq \lambda_b. \tag{2}
$$

The corresponding eigenvectors will be $e_n, e_a, e_b$, whose components are relative to the original frame. Thus, the matrix

$$E = \begin{pmatrix}
e_{n,x} & e_{n,y} & e_{n,z} \\
e_{a,x} & e_{a,y} & e_{a,z} \\
e_{b,x} & e_{b,y} & e_{b,z}
\end{pmatrix} \tag{3}
$$
defines the basis change $X = EX_i + C$, and diagonalises $S$ as $\Lambda = E^T SE$ (since $E$ is the basis change matrix, $E^T = E^{-1}$); $x_i$, with $i = 1, ..., N$, are the coordinates of the data points in the new reference frame (they are also called “scores” in PC analysis); the eigenvector $e_n$ locates the normal to the loop plane, $e_a$ is directed like the minor axis of the ellipse, and $e_b$ like the major one. The axes values are related to the variance as:

$$a = \sqrt{\frac{2\lambda_n}{N}} \quad b = \sqrt{\frac{2\lambda_b}{N}} \tag{4}
$$

Indeed, since we have defined the covariance matrix $S$ without dividing its elements by $N$, this should be done for the eigenvalues (the eigenvectors remain the same despite of normalisation of $S$). A further justification for the presence of the factor $\sqrt{2}$ is given in Appendix A.1. The “fitted” loop can be traced starting from the parametric equations of an ellipse:

$$x'_e = a \cos(t) \quad y'_e = b \sin(t) \tag{5}
$$

with $t = [0, 2\pi]$, then, transformed in HEEQ coordinates $x_e, y_e, z_e$ by the basis change with the matrix $E$.

Those points whose squared distance satisfies the condition $r^2 = x'_e^2 + y'_e^2 + z'_e^2 \geq 1.0R_\odot$, defines the loop curve.

We can look at how much of the total variance of the data points is counted by each single eigenvalue. Let $\Lambda$ be the total variance defined as $\Lambda = \lambda_n + \lambda_a + \lambda_b$. Then, the ratio $\hat{\Lambda}_i = \lambda_i/\Lambda$ (with $i = n, a, b$), gives an indication of how much of the “energy” is carried out by $\lambda_i$. According to Table 1, most of the variance is stored in along the $e_{a,b}$ directions, suggesting that the points are mostly distributed in a 2D plane. This is equivalent to say that we have reduced the dimensionality of the data set, by projecting it into a subspace, that is the 2D plane of the loop.

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3For statistical reasons, when $N$ is small, it can be replaced by $N - 1$. 
We can obtain the inclination and azimuth angles from the orientation of the loop plane in space. The inclination angle is defined as the angle between the normal to the solar surface and the loop plane. By considering the vector $e_n$, normal to the loop plane, we can find the inclination angle as

$$\theta = \frac{\pi}{2} - \cos^{-1}(e_r \cdot e_n)$$  \hspace{1cm} (6)

where $e_r$ is the normal, directed in the radial directions. The azimuthal angle is defined as the angle formed by the intersection of the loop baseline with the East-West solar direction, represented by the longitudinal vector $e_\phi$. So the azimuthal angle can be found as:

$$\alpha = \frac{\pi}{2} - \cos^{-1}(e_\phi \cdot e_n).$$  \hspace{1cm} (7)

The errors of the fitting can be estimated from the “scores” and their distance $d$ from the fitted curve: it is possible to give an estimate of the standard deviation of the $a$ and $b$ direction from the sum of the squared distance (see A.2, for more details). Another possible way is to perform PC analysis for $M$ samples of the 3D points, or construct synthetic samples by varying randomly the 3D original measurements within their errors [priv. comm. Antonio Vecchio]: the best values of the radii, inclinations, and azimuth angles, as well as their standard deviations, will be inferred from their distributions.

3. Analysis and discussion

In this section, we demonstrate some applications of PC analysis to specific coronal loops, with some comparisons with previous analysis.

3.1. The event on 27th June, 2007

The coronal loop in Fig. 3 appeared on the East solar limb, and was subject to kink oscillations, triggered by a flare. An analysis of the loop geometry is given in Verwichte et al. [14], using a forward modelling technique based upon the assumption of the circular shape, and in Aschwanden [5] with a fitting approach of the 3D stereoscopic points. The loop centre is not clearly identified, because of its proximity to the solar limb. The knowledge of the possible footpoints in Verwichte et al. [14], was inferred by looking at the same active region three days later, when it passed the East solar limb. Here, we deduce it directly from the stereoscopic measurements by calculating the centre of “mass” of the distribution of points, and projecting the highest point (that can be regarded as the “apex” of the loop) to the solar surface, crossing the centre. Following this idea, we extrapolate a heliographic longitude of about $\phi_C = -87.76$ deg, and latitude $\theta_C = -11.41$ deg for the candidate loop centre. The results of the PC analysis are given in the first column of Table 1. We got an inclination of $\theta = -2.85$ deg and an azimuth angle $\alpha = -37.21$ deg. The relative errors for the minor and major radii are $\sigma_a/a = 5.6\%$, and $\sigma_b/b = 6.1\%$, respectively. The loop length is about 325 Mm, and it is close to the value estimated by Aschwanden [5]; Verwichte et al. [14]. Moreover, it exhibits a strong eccentricity ($\sim 0.9$), indicating to the elliptical geometry of coronal loops rather than circular.

We overplotted the data points and the reconstructed loop over the AR seen three days later, on 30th June, 2007. The loop seems to agree with the underlying topology of the AR, especially for the position of the footpoint labelled by green in Fig. 4.
**Figure 3.** A coronal loop on the East solar limb during a flare event on 27th June, 2007. The event was studied by Aschwanden [5]; Verwichte et al. [14]. The top figures show FOVs from STEREO-A (right) and STEREO-B (left). The yellow symbols are the 3D data points measured with the routine scc_measure. The light-blue line is the fitted curved loop; at the base the centre is in blue, and the footpoints inferred from the reconstruction are in red and green colours. The middle plots show the reconstructed loop against the yellow data points, for the different orientations of the HEEQ coordinates system. The coloured vectors represent the new set of three axes: in red and blue the minor and major axis, in green the normal to the loop plane. Indeed, the bottom figures show the loop in this new reference frame for different orientations. Similar description is applied for the Figures of the other analysed events.
Figure 4. Image from STEREO-B at 171 Å of the active region which was the site of the analysed coronal loop, on 30th June 2007. The 3D points and the “fitted” loop are overplotted in order to visually prove the consistency of our results. Note that there is a good agreement with the position of the top-right footpoint with the open loops of the active regions. The red meridian marks the position of the solar limb on 27th June, 2007.
3.2. The event on 21st January, 2013

Here we show a loop appearing in the quiet region on 21st January, 2013, close to the Western solar limb. The loop is well seen with SDO/AIA at 171, 193 Å, and visible in STEREO/EUVI-A as well, but with a spatial resolution 2.5 times lower. We were able to take 3D points with scc_measure: we worked on filtered images (we subtracted the frames with those 5 min previous) in order to highlight the loop shape (Fig 5-top). Since the footpoints are well identified, we determined the centre as the midpoint between footpoints. We took 10 independent measurements, and considered their mean as the best value for the centre. The outcomes of PC analysis are listed in Table 1.

3.3. The event on 22nd January, 2013

Here, we present 3D reconstruction of coronal loops belonging to the active region AR NOAA 11654, studied by Anfinogentov et al. [28]. The active region exhibits a large number of coronal loops. From SDO/AIA, the active region is partly off-limb, and loop footpoints are not visible. The same active region is well seen from STEREO-A: the footpoints are easily identified, and the loops seem to span for a wide angle across to the surface normal. So, intuitively, one could assume that the only parameter changing is the inclination angle $\theta$, while the other properties, like the loop lengths and the azimuth angles $\alpha$ should remain almost constant. We use running difference images, in order to identify the loop in both FOVs when we note a simultaneous intensity variation. Fig. 6,7,8 show the reconstruction for some of these loops, which were located with good certainty. The parameters are listed in Table 1: for all three loops note consistency in the parameters, like the minor and major radii, the eccentricity, the azimuth angle, and the loop length. Also, one can see the discrepancy in the estimation of the inclination, suggests that the analysed loops become gradually more perpendicular to the solar surface. The loop in Fig. 9 is different, showing different geometrical values, and a stronger inclination.
Figure 5. The coronal loop event observed with SDO/AIA (left) and STEREO-A (right) in a quiet region on 21st Jan, 2013 in the South hemisphere.
Figure 6. Analysis of the loop (a) on the 22nd January, 2013.
Figure 7. Analysis of the loop (b) on the 22nd January, 2013.
Figure 8. Analysis of the loop (c) on the 22nd January, 2013.
Figure 9. Analysis of the loop (d) on the 22nd January, 2013.
Table 1. The table shows the main quantities used and inferred from PC analysis: the first rows are related to the inputs and show the number of 3D points \( N \) and the heliographic coordinates of the supposed loop centre, then we tabulated the total variance \( \Lambda \) and the percentage relative to each eigenvalue \( \lambda \); the last rows show the estimates of each analysed loop: the minor (\( a \)) and major radii (\( b \)) with their corresponding standard deviations, the standard deviation \( \sigma_n \) of the points from the loop plane, the inclination and azimuth angles, the eccentricity, and finally the loop length.

<table>
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<tr>
<th>Event (Figure)</th>
<th>2007-06-27</th>
<th>2013-01-21</th>
<th>2013-01-22</th>
<th>2013-01-22</th>
<th>2013-01-22</th>
<th>2013-01-22</th>
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</thead>
<tbody>
<tr>
<td>( N )</td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>22</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>( \phi_C ) [deg]</td>
<td>-87.76</td>
<td>69.23</td>
<td>100.35</td>
<td>100.67</td>
<td>100.85</td>
<td>94.11</td>
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<tr>
<td>( \theta_C ) [deg]</td>
<td>-11.41</td>
<td>25.88</td>
<td>9.25</td>
<td>9.43</td>
<td>8.84</td>
<td>6.74</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>0.32</td>
<td>0.21</td>
<td>1.17</td>
<td>0.91</td>
<td>0.60</td>
<td>0.17</td>
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<tr>
<td>( \hat{\lambda}_n ) [%]</td>
<td>0.24</td>
<td>0.80</td>
<td>0.04</td>
<td>0.05</td>
<td>0.09</td>
<td>0.50</td>
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<tr>
<td>( \hat{\lambda}_a ) [%]</td>
<td>11.17</td>
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<td>26.03</td>
<td>20.86</td>
<td>19.29</td>
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<tr>
<td>( \hat{\lambda}_b ) [%]</td>
<td>88.59</td>
<td>61.16</td>
<td>73.93</td>
<td>79.09</td>
<td>80.61</td>
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<td>( \sigma_a ) [R(_\odot)]</td>
<td>±0.004</td>
<td>±0.008</td>
<td>±0.006</td>
<td>±0.006</td>
<td>±0.010</td>
<td>±0.003</td>
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<td>( \sigma_b ) [R(_\odot)]</td>
<td>0.202</td>
<td>0.136</td>
<td>0.244</td>
<td>0.262</td>
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<td>0.130</td>
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<tr>
<td>( \sigma_n ) [R(_\odot)]</td>
<td>±0.014</td>
<td>±0.004</td>
<td>±0.011</td>
<td>±0.010</td>
<td>±0.019</td>
<td>±0.007</td>
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<tr>
<td>( \sigma_a ) [R(_\odot)]</td>
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<td>±0.011</td>
<td>±0.004</td>
<td>±0.004</td>
<td>±0.006</td>
<td>±0.007</td>
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<tr>
<td>( \theta ) [deg]</td>
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<td>0.02</td>
<td>25.06</td>
<td>20.66</td>
<td>16.71</td>
<td>-41.49</td>
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<tr>
<td>( \alpha ) [deg]</td>
<td>37.21</td>
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<td>eccentricity</td>
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<td>0.80</td>
<td>0.86</td>
<td>0.87</td>
<td>0.95</td>
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<td>( L ) [Mm]</td>
<td>325.22</td>
<td>276.62</td>
<td>448.30</td>
<td>457.96</td>
<td>468.77</td>
<td>203.64</td>
</tr>
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</table>
4. Conclusions

In this paper, we propose a new method for the determination of the 3D shape of coronal loops from stereoscopic measurements. The necessity to understand the loop shape is important in order to better estimate for example the loop length, which is a crucial parameter in inferring the magnetic field by coronal seismology (or generally to compare the geometry of loops, which are assumed to trace magnetic field lines in the low-\(\beta\) coronal plasma, with potential field models of the coronal magnetic field [24]). Also, the loop shape determines the variation of the column depth along the loop in different phases of kink oscillations, and hence the distribution and dynamics of the LOS-integrated intensity of the loop. The previous approach to 3D reconstruction is based on fitting stereoscopic 3D tie-points, taken along the loops. A good fitting requires a large number of points, but taking them could be a laborious task because of high observational and instrumental noise and the effect of the LOS integration. Usually, by procedures e.g. scc\_measure the number of tie-points selected is of about 10-20. The method used in Aschwanden et al. [21] and his subsequent papers, is to interpolate the 3D points with a spline, in order to retrieve the loop shape. On the other hand, this operation could add further degrees of freedom (e.g., non-planarity), and the corresponding reconstructed loop could deviate from a regular shape like a circle or ellipse (e.g., see Fig. 9 of Aschwanden et al. [21], and Fig. 7 of Aschwanden [5]), also because of the natural spread of the measurements around the true value. Moreover, fitting the 3D points with a 1D curvilinear feature, requires the determination of the 6 free parameters: the inclination and azimuth angles, the length of the baseline, the loop centre position and the loop height. Conversely, all this information is implicitly stored in the distribution of a sample of 3D measurements, and it can be retrieved easily by the technique of Principal Components analysis. Determination of these parameters can be made by a simple step by step process. The technique we developed can be summarised in the followings steps:

- measurement of a sample of 3D tie-points which outline the loop shape from two different LOS;
- calculation of the sample covariance matrix \(S\) of the 3D points;
- determination of the 3 pairs of eigenvalue/eigenvectors \(\lambda_i/e_i\), which identify the new reference frame relative to the loop, sorted in ascending order;
- the smallest eigenvalues and the corresponding eigenvector is associated with the normal to the loop plane;
- the largest eigenvalue/eigenvector locates the major axis, and the remaining the minor axis according to (4);
- determination of the inclination and azimuth angles according to (6) and (7);
- loop tracing with the parametric equation (5) and transformation in the original coordinates system HEEQ.

The determination of the set of three vectors \(e_n, e_a, e_b\), which constitutes the new basis for the reference frame relative to the loop, depends uniquely on the choice of the loop centre (or also called midpoint),
which plays the role of the decisive parameter of the problem. Moreover, according to the example
of the loop on 27th June, 2007 (Fig. 3), we have shown that the information of the midpoint can be
deduced or guessed directly from the 3D measurements, overcoming the issue of the lack of information
of the loop baseline, when it is too close to the solar limb. The advantage of this method is that it
works for a reasonably small number of data points (10-20), without requiring any intermediate step,
like interpolation. The only assumption made in our analysis is the planarity of the loop, but possible
speculations on it can be made by studying the distribution of the data points around the loop plane,
and searching for some possible patterns like arcs or s-shapes, which could outline respectively the
fundamental and second harmonic in kink oscillations. On the other hand, the relative deviation of the
measurements from the best-fitted loop plane is less the 1%, probably too small in order to appreciate
any non-planarity. Moreover, a further step forward for reasonable modeling of the magnetic field would
be to fit the loop shape by a dipole or stretched dipole line, according as suggested in Hu et al. [29].
These issues, as well as refinements of the technique by PC analysis, could be object of a future work.

A. Appendix

A.1. Coronal loops as a bivariate normal distribution

The distribution of 3D points that identifies coronal loops, can be regarded as a case of multivariate
normal distribution [see Chapter 4 and 8 of 27]. The univariate normal distribution, with the mean \( \mu \) and
variance \( \sigma^2 \), has the probability density function

\[
f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),
\]

(8)

where the term

\[
\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 = \frac{1}{2} (x - \mu)(\sigma)^{-2}(x - \mu)
\]

measures the squared distance from \( x \) to \( \mu \) in the standard deviation units. For \( N \) observations of a vector
\( x \), this can be represented by:

\[
\frac{1}{2} (x - \mu)^\top \Sigma^{-1} (x - \mu)
\]

(10)

with \( \mu \) being the expected value and \( \Sigma \) being the variance-covariance matrix of the \( x \) observations (the
apex indicates the complex conjugate of \( x - \mu \), similarly in the next equations as well). The above
quantity for a multivariate distributions represents the contour of constant density probability at specific
values \( k^2 \), or the surface of an ellipsoid centred in \( \mu \). In our case, given the \( x_i \) observations, the centre \( C \),
and the sample covariance matrix \( S \), we have

\[
\frac{1}{2} (x - C)^\top S^{-1} (x - C) = k^2 = \frac{1}{2} \sum_{i} x_i^\top A^{-1} x_i,
\]

(11)

because of the principal components reduction. So, if we expand the right side of (11), we get the
equation of a 3D ellipsoid:

\[
\frac{x_{1a}^2}{2\lambda_a^2} + \frac{x_{1b}^2}{2\lambda_b^2} + \frac{x_{1n}^2}{2\lambda_n^2} = k^2.
\]

(12)
Since the eigenvalues $\lambda_n$ do not significantly contribute to the total variance, they can be eliminated, and we get the equation of an ellipse in the canonical form, with $k = 1$, $a = \sqrt{2}\lambda_a$, $b = \sqrt{2}\lambda_b$,

$$\frac{x_{la}^2}{a^2} + \frac{x_{lb}^2}{b^2} = 1.$$  \hfill (13)

A.2. Error estimations

The errors in the fitting can be estimated from the “scores” or the distance from the fitted curve. Let $x_l = (x_{la}, x_{lb}, x_{ln})$ be the data point in the loop reference frame, and $x_e = (x_{ea}, x_{eb}, x_{en})$ a point of the fitted curve (that should be determined). The squared distance $d^2$ will be

$$d^2 = d_a^2 + d_b^2 + d_n^2 = (x_{la} - x_{ea})^2 + (x_{lb} - x_{eb})^2 + (x_{ln} - x_{en})^2.$$  \hfill (14)

Let us normalise the distance along the axes to the radii $a$ and $b$,

$$d_a^2 = a(\hat{x}_{la} - \hat{x}_{ea})^2 \quad d_b^2 = b(\hat{x}_{lb} - \hat{x}_{eb})^2.$$  \hfill (15)

Now we have reduced the loop to a circumference of radius 1. The coordinates $\hat{x}_{ea}$ and $\hat{x}_{eb}$ will be respectively equal to the cosine and sine of the angle spotted by the point $(\hat{x}_{la}, \hat{x}_{lb})$, so that, we rewrite as

$$\hat{x}_{ea} = \cos \gamma = \frac{\hat{x}_{la}}{\sqrt{\hat{x}_{la}^2 + \hat{x}_{lb}^2}}$$  \hfill (16)

$$\hat{x}_{eb} = \sin \gamma = \frac{\hat{x}_{lb}}{\sqrt{\hat{x}_{la}^2 + \hat{x}_{lb}^2}},$$  \hfill (17)

and, hence:

$$d_a^2 = a\hat{x}_{la}(1 - \frac{1}{\sqrt{\hat{x}_{la}^2 + \hat{x}_{lb}^2}})^2 \quad d_b^2 = b\hat{x}_{lb}(1 - \frac{1}{\sqrt{\hat{x}_{la}^2 + \hat{x}_{lb}^2}})^2.$$  \hfill (18)

The standard deviation along $a$, $b$, and $n$ can be estimated as

$$\sigma_j = \frac{\sum_i d_{ij}^2}{N - 1} \quad \text{with } j = (a, b, n).$$  \hfill (19)

A.3. Error estimate on the loop length

The loop length can be estimated numerically given the radii $a$ and $b$, and $N_p$ grid points along the loop, as:

$$L = \sum_{i=2}^{N_p} l_i = \sum_{i=2}^{N_p} \sqrt{(x_{ea}(i) - x_{ea}(i - 1))^2 + (x_{eb}(i) - x_{eb}(i - 1))^2}.$$  \hfill (20)
In the polar representation we have

\[ L = \sum_{i=2}^{N_p} \sqrt{a^2(\cos \gamma_i - \cos \gamma_{i-1})^2 + b^2(\sin \gamma_i - \sin \gamma_{i-1})^2} \leq \]

\[ \sum_{i=2}^{N_p} \left( \sqrt{a^2(\cos \gamma_i - \cos \gamma_{i-1})^2} + \sqrt{b^2(\sin \gamma_i - \sin \gamma_{i-1})^2} \right) \]

\[ = a \sum_{i=2}^{N_p} (\cos \gamma_i - \cos \gamma_{i-1}) + b \sum_{i=2}^{N_p} (\sin \gamma_i - \sin \gamma_{i-1}). \]

(21)

(22)

(23)

The uncertainty of the length can be expressed as:

\[ \frac{\Delta L}{L} \approx d \ln L \leq d \ln \left( a \sum_{i=2}^{N_p} (\cos \gamma_i - \cos \gamma_{i-1}) \right) + d \ln \left( b \sum_{i=2}^{N_p} (\sin \gamma_i - \sin \gamma_{i-1}) \right) \]

\[ \approx d \ln a + d \ln b \approx \frac{\Delta a}{a} + \frac{\Delta b}{b}. \]

(24)

For very good fittings, we obtained the uncertainties \( \sigma_a/a \) and \( \sigma_b/b \) of the order of 3–5%, so that the loop length error should be less than 8%.

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Conflict of Interest

The authors declare no conflict of interest.

References


Note that in general the inequality \( \ln(x + y) \leq \ln(x) + \ln(y) \) is true when \( x \geq y/(y - 1) \).


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