Weakly nonlinear ion-sound waves in magnetised plasmas with electron temperature gradients

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Abstract

Low frequency electrostatic waves are studied in magnetised plasmas for the case where the electron temperature varies with position in a direction perpendicular to the magnetic field. We analyse guided waves with characteristic frequencies below the ion cyclotron and ion plasma frequency for the case where the ion cyclotron frequency is below the ion plasma frequency. A particular feature of low frequency electrostatic waves under these conditions is the existence of trapped waveguide modes when the frequency is below the ion cyclotron frequency, while the modes are radiative for higher frequencies. These conditions allow the formation of a new type of electrostatic shocks. The results are illustrated by results from a $2\frac{1}{2}$-D Particle-In-Cell (PIC) code.

Introduction

The existence of a waveguide mode for electrostatic ion acoustic waves is demonstrated in magnetised plasmas, for conditions where the electron temperature is striated along the magnetic field lines. The waveguide mode is confined to frequencies below the ion cyclotron frequency. For a frequency band between the ion cyclotron and the ion plasma frequency we have a radiative mode, escaping from the temperature striation. Based on simple analytical arguments, the formation and propagation of an electrostatic shock is subsequently demonstrated by Particle-In-Cell (PIC) simulations for these conditions. In the classical Burgers' model, the shock represents a balance between viscous dissipation and nonlinear wave steepening [9], and the formation can be understood as a balance between the energy input by an external source (e.g. a piston moving with velocity U) and viscous dissipation, where the kinematic viscosity coefficient is ν. It can furthermore be shown that the shock thickness varies as $\sim \nu/U$.

In principle, Burgers' equation can apply for any continuous viscous fluid media, also plasmas. In the electron temperature striated plasma, the dissipation mechanism is replaced by radiation losses of the harmonics generated by the wave steepening. Experiments preformed in the strongly magnetised plasma of the Risø Q-machine [1] demonstrated that for moderate electron to ion temperature ratios, T_e/T_i, the strong ion Landau damping prohibited the formation of shocks. For large temperature ratios, the ion Landau damping is reduced, and there is a possibility for forming steady state nonlinear shock-like forms, propagating at a constant speed [4]. In this experiment the energy drain was nonlinear particle reflection by the electrostatic field.

We present a novel mechanism of energy dissipation, namely selective radiation of short wavelength ion sound waves. We also demonstrate that electrostatic shocks can form as a balance between these losses and the standard nonlinear wave-steepening as described by the nonlinear term in the "simple wave" equation $\partial u/\partial t + u\partial u/\partial z = 0$ [2, 9]. Studies in two spatial dimensions are sufficient for illustrating the basic ideas, and the analysis of the present paper is restricted to 2D.

Model Equations

Magnetised plasmas are considered here for conditions where the electron temperature T_e varies in the direction perpendicular to an externally imposed homogeneous magnetic field [6, 3]. Such conditions seem to occur often in nature for plasmas out of equilibrium [7]. Here we have the ion cyclotron frequency smaller than the ion plasma frequency, i.e. $\Omega_{ci} < \Omega_{pi}$. The relevant frequencies are assumed to be so low that the electron component can be taken to be in local Boltzmann equilibrium at all times. We assume quasi-neutrality, $n_e \approx n_i$. For a linearised fluid model of the present problem we readily derive a basic equation in the form

$$\frac{\partial^4}{\partial t^4}\psi - \frac{T_e(x)}{M}\frac{\partial^2}{\partial t^2}\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2}\right)\psi + \Omega_{ci}^2\frac{\partial^2}{\partial t^2}\psi - \Omega_{ci}^2\frac{T_e(x)}{M}\frac{\partial^2}{\partial z^2}\psi = 0 \qquad (1)$$

where ψ is the electrostatic potential, related to the relative density perturbations as $e\psi/T_e = \eta \equiv \delta n/n_0$. Since we are here only interested in cases where $T_e/T_i \gg 1$, we set T_i=0 in Eq. (1). In the case T_e=constant, a linear dispersion relation is readily obtained from Eq. (1) by Fourier transform with respect to t and x, z. This dispersion relation contains two branches, one for $\omega < \Omega_{ci}$ and one for $\Omega_{ci} < \omega < \Omega_{pi}$, the latter containing also the ion cyclotron waves. The wave properties may be summarised by the angle between the group velocity and the wave-vector. For very low frequencies, $\omega \ll \Omega_{ci}$, these two vectors are almost perpendicular, while they are close to parallel when $\omega \gg \Omega_{ci}$. In the limit $\boldsymbol{k}_\perp \to 0$, the dispersion relation reduces to $(\omega^2 - \Omega_{ci}^2)(\omega^2 - k^2C_s^2) = 0$ containing ion sound waves and the electrostatic ion cyclotron resonance.

If we now let T_e=$T_e(x)$, with z being along $\boldsymbol{B}$ and x be in the transverse direction, we can still Fourier transform with respect to time and the z-variation. We denote the Fourier transformed quantities by $\widehat\psi$. Normalising frequencies and x-positions so that $\Omega \equiv \omega/\Omega_{ci}$ and $\xi \equiv x\Omega_{ci}/C_s$, respectively, we obtain

$$\frac{d^2}{d\xi^2}\widehat\psi = (\Omega^2 - 1)\left(\frac{1}{\gamma^2} - \frac{T_0}{T_e(\xi)}\right)\widehat\psi \qquad (2)$$

where $\gamma^2 \equiv (\omega/k_z)^2(M/T_0)$ is a normalised propagation speed, and T_0 is a reference temperature. Eq. (2) has the form of an eigenvalue equation (with $1/\gamma^2$ being the eigenvalue). For $T_e(x)$ being piecewise constant we can find exact solutions for Eq. (2). We present some numerical solutions in Fig. 1 for Gaussian variation of $T_e(x)$. We find that in the low frequency limit, $\omega < \Omega_{ci}$, the waves are confined to the electron temperature striation, corresponding to a discrete set of eigenvalues, $\widehat\psi_m$. For such Gaussian variations of $T_e(x)$ the mode number m corresponds to the number of zero-crossings of $\widehat\psi_m(x)$. From $\widehat\psi_m$ we can obtain the corresponding eigenmodes for the $\boldsymbol{B}$-parallel velocity $u_\parallel$. The value of γ depends relatively weakly on Ω. In Fig. 1 we note that the eigenfunctions change only little in spite of the large change in Ω, and the corresponding change in the eigenvalues is also small. Only for Ω close to unity do we see significant variations in $\widehat\psi_0(x)$ and γ_0. For shallow temperature variations and narrow temperature ducts, we have only the lowest order mode $\psi_0(x)$. For γ smaller than $T_e(x)/T_0$ minimum value there is a continuum of eigenvalues.

For $\omega > \Omega_{ci}$ the r.h.s. of Eq. (2) changes sign, and the nature of the eigenmodes changes as well, to become free modes as seen in the left panel of Fig. 1. For uniform T_e, the free modes degenerate to two obliquely propagating plane waves.

We consider now the low frequency limit of the branch of dispersion relation with $\omega < \Omega_{ci}$. For m>1, the waveguide modes can decay for one m-value to modes with other m-values. The m=0 mode has no decay to other forward propagating modes, and will be the one considered here. For this highest phase-velocity mode, with eigenmode $\widehat\psi_0(x)$, we expect wave-steepening to be the dominant nonlinearity. The nonlinear terms couple the various modes to give products of x-modes. These can be expanded as, for instance, $\widehat\psi_i(x)\widehat\psi_j(x) = \sum_q \zeta_{qij}\widehat\psi_q(x)$, where we assume that the set $\widehat\psi_m$ is complete and orthonormal [5]. For the lowest order waveguide mode we have, in particular, $\zeta_{j00} = \int_{-\infty}^{\infty}\widehat\psi_0^2(x)\widehat\psi_j(x)dx$. For a large class of relevant electron temperature profiles we can ignore all higher order modes, and retain only $\widehat\psi_0$, with $\zeta_0 \equiv \int_{-\infty}^{\infty}\widehat\psi_0^3(x)dx$. For the Gaussian and similar $T_e(x)$ studied here, we will have $\zeta_0 > \zeta_j$ for all $j \geq 1$, since $\widehat\psi_0$ is the only eigenfunction that is positive everywhere. To lowest order in the low frequency limit we have the relation $e\widehat\psi/T_e = \widehat\eta \approx \widehat u_\parallel/C_s$ between fluctuations in relative density and the $\boldsymbol{B}$-parallel velocity. Our arguments concerning mode structure apply to velocity variations as well.

Considering the limit of time scales much larger than the ion cyclotron period, we find the result

$$\frac{\partial u_\parallel}{\partial t} + (\zeta_0 u_\parallel \pm \zeta_T C_{s0})\frac{\partial u_\parallel}{\partial z} = 0\,, \qquad (3)$$

with $u_\parallel = u_\parallel(z,t)$, where ζ_T originates from an expansion of $T_e(x)$ in terms of $\widehat\psi_m(x)$, and small polarisation drifts $\perp$ $\boldsymbol{B}$ are ignored. The constant reference sound speed is C_{s0}. We anticipate that ζ_T is not much different from ζ_0, since the form of $\widehat\psi_0(x)$ is close to $T_e(x) - T_e(\infty)$. The solutions of Eq. (3) have the well known steepening of the initial condition. The characteristic time for wave breaking is approximately $\mathcal{L}/\max\{u_\parallel(t=0)\}$, where $\mathcal{L}$ is the characteristic scale length of the initial perturbation and $\max\{u_\parallel(t=0)\}$ is the maximum value of the initial velocity perturbation. The model equation Eq. (3) assumes $\widehat\psi_0(x)$ and also γ_0 being used when $\Omega > 0$. This approximation is acceptable for at least $0 \leq \Omega < 0.75$, as seen from Fig. 1.

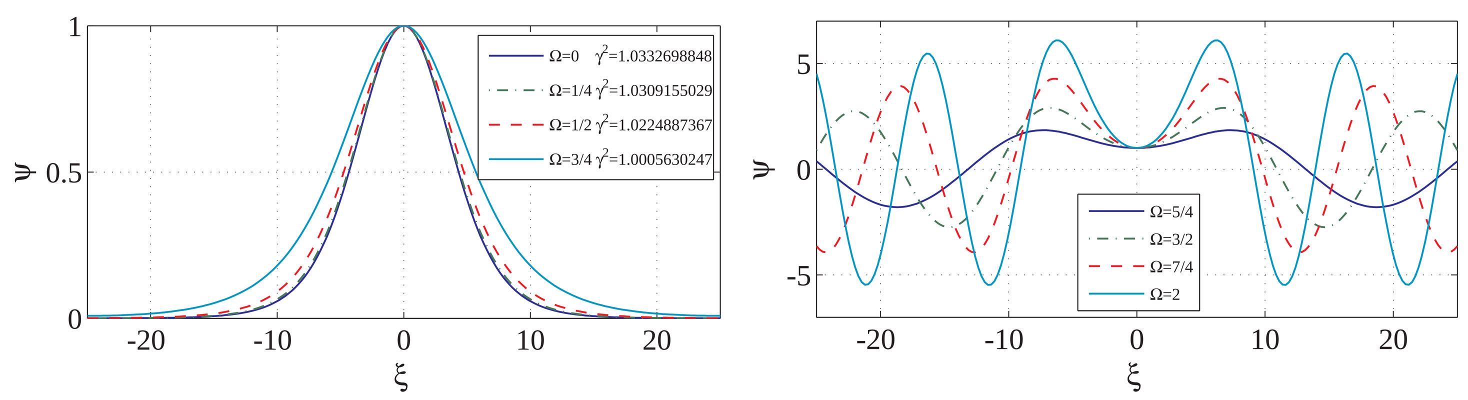

FIG. 1: Numerical solutions of Eq. (2), with $T_e(\xi)/T_0 = 1 - \frac{1}{2}\mathcal{D} + \mathcal{D}\exp(-\xi^2/W^2)$, and W=5, $\mathcal{D}$=1/4 for waveguide modes (left panel) and free modes (right panel) with $\gamma^2 = 1$.

If we initialise the system with characteristic wavelengths corresponding to frequencies $\omega \ll \Omega_{ci}$, i.e. $\mathcal{L} \gg C_s/\Omega_{ci}$, the short time evolution will be governed by Eq. (3), and we will have shorter and shorter scales developing as for the usual breaking of waves. This process is arrested when the characteristic length scales become of the order of the effective ion Larmor radius C_s/Ω_{ci}, when the modes become radiating, and are no longer confined to the waveguide. We propose a phenomenological expression for the process, written in Fourier space in a frame moving with C_{s0} as

$$\frac{\partial u_\parallel(k_\parallel)}{\partial t} + i\frac{\zeta_0 k_\parallel}{2}u_\parallel(k_\parallel) \otimes u_\parallel(k_\parallel) = -u_\parallel(k_\parallel)\frac{C_{s0}}{\mathcal{D}}\mathcal{H}\left(|k_\parallel| - \frac{\Omega_{ci}}{C_{s0}}\right)\,, \qquad (4)$$

where $\otimes$ denotes convolution and $\mathcal{H}$ is Heaviside's step function. The time scale $C_{s0}/\mathcal{D}$ characterises the time it takes for the energy to radiate out of the waveguide with diameter $\mathcal{D}$. Physically we argue that, within the present model, the waveform will steepen uninhibited until the shock width becomes of the order of $2C_s/\Omega_{ci}$. At this width the harmonic frequencies will exceed Ω_{ci} to become radiative and their energy will be lost.

Simulations

The nonlinear propagation of low frequency waves in the striated plasma is studied using a $2\frac{1}{2}$-D PIC code [3]. Our PIC simulator assumes explicitly the electrons to be locally Boltzmann distributed and the resulting nonlinear Poisson equation is solved by iteration.

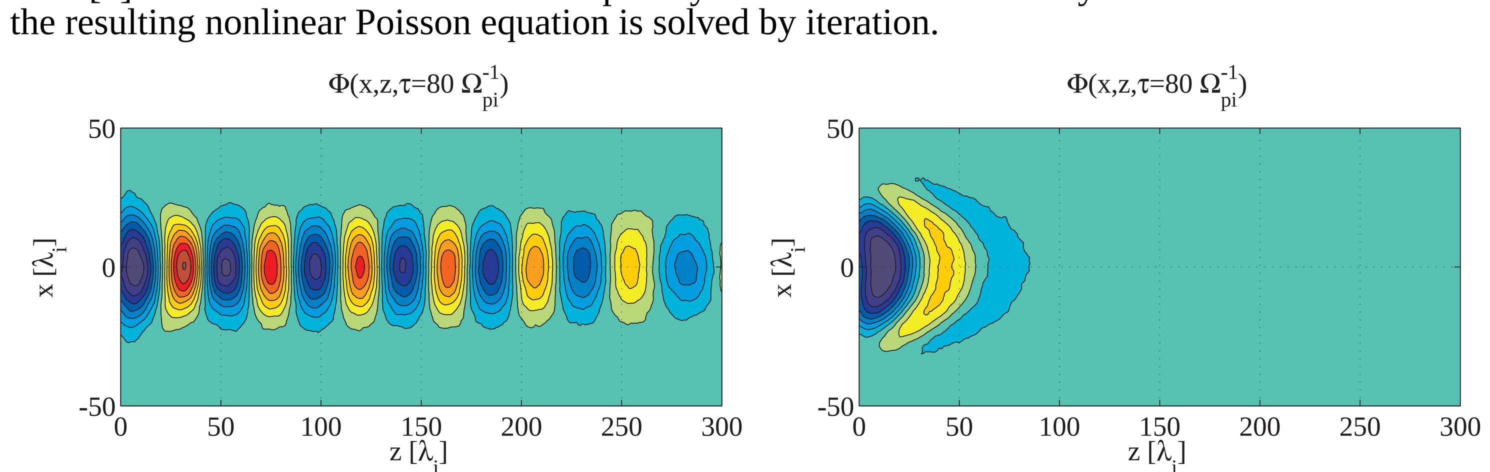

FIG. 2: Electrostatic potential from a PIC simulation The electron temperature enhancement is localised as a Gaussian in the x-direction with $T_{e,\max}/T_i = 50$. $\omega = \Omega_{pi}\pi/5$ in both cases while Ω_{ci}/Ω_{pi}=1 (left panel) and Ω_{ci}/Ω_{pi}=0.05 (right panel).

We use a Gaussian variation for $T_e(x)$ so that $T_e(\pm\infty)/T_i$=1, while $T_e(0)/T_i$>1, where T_i=constant. The width of the electron striation is here $\mathcal{D} = 40\lambda_{Di}$. Results for the waveguide and the free modes are shown in Fig. 2. The properties are clearly different and consistent with the interpretation given before. The waveguide mode remains inside the electron striation, while the high frequency free modes disperse, consistent also with laboratory experimental results [6]. The observed damping is not due to dissipation, but caused by wave energy dispersing in space. In order to emphasise the physical effects we discuss here, we consider only high temperature ratios, $T_e/T_i \geq 25$, in order to reduce the effects of linear as well as nonlinear Landau damping. We note that such high temperature ratios can actually be obtained in discharge plasmas under laboratory conditions [8]. For nonlinear waves described by a Korteweg-de Vries (KdV) equation, a shock formation is followed by Airy-type ripples, originating from the dispersion term in the KdV equation. These ripples are absent in our results shown in Fig. 3. Likewise, at the high temperature ratio used here, we find no ions being reflected by the shock. A backward propagating rarefaction wave is thus of no concern here.

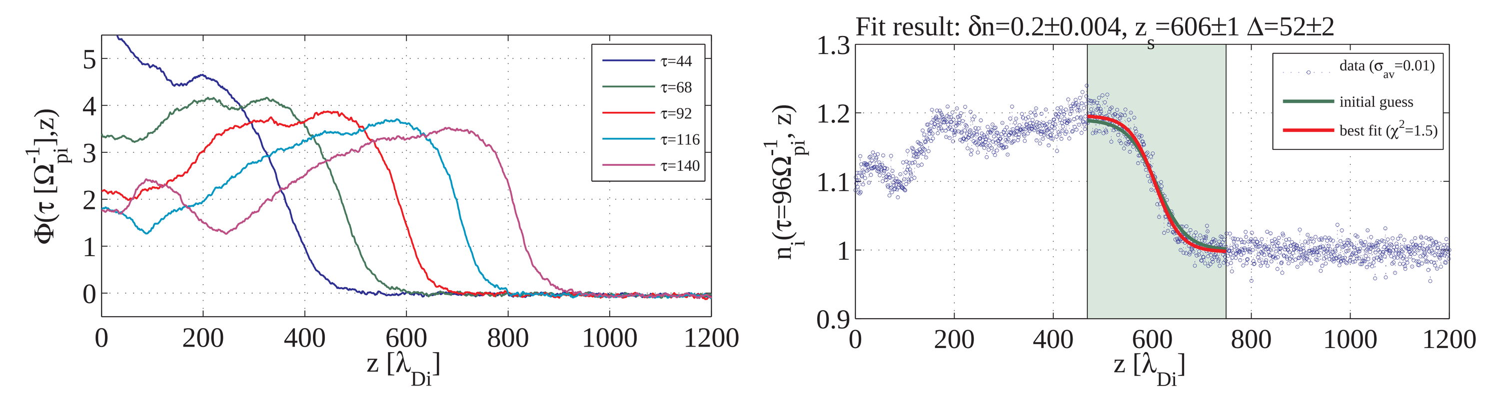

FIG. 3: Numerical simulation showing potential ϕ and density n_i during the formation and propagation of a shock under the conditions $T_e/T_i = 25$, $\Omega_{ci}/\Omega_{pi} = 1/2$ and $\delta n_i/n_0 \approx 0.21$. δn_i is the actual detected density perturbation for fully formed shock.

Fig. 3 illustrates the formation and propagation of a shock under the conditions previously mentioned. The initial condition has an error function type spatial density variation, where ions are continuously injected at the boundary at $y = 0$ to maintain the energy input. We have $T_e/T_i = 25$, $\Omega_{ci}/\Omega_{pi} = 1/2$ and $\delta n_i/n_0 \approx 0.21$, where δn_i refers to the actual detected density perturbation at the time where the shock is fully formed, and not to the initial imposed perturbation. The standard deviation of the parameters is estimated from fitting the shock profile with a nonlinear Levenberg-Marquardt method.

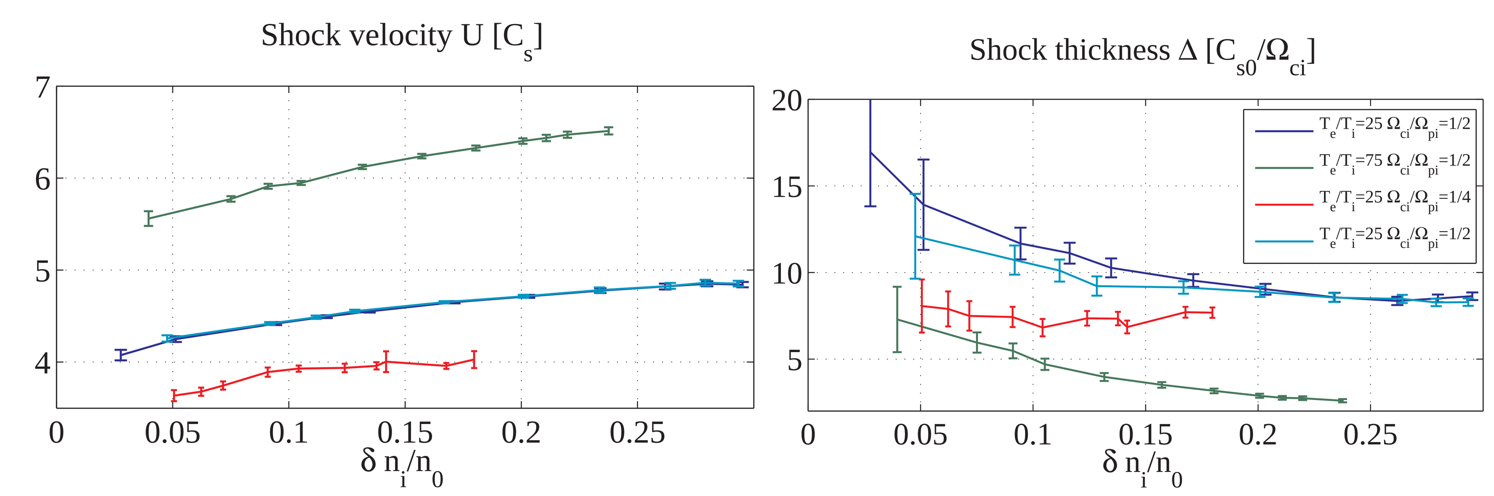

FIG. 4: Shock velocity U (left panel) and shock thickness Δ (right panel), shown for different combinations of the parameters T_e/T_i, Ω_{ci}/Ω_{pi} and $\delta n_i/n_0$.

Fig. 4 presents the shock thickness Δ and velocity U for various combinations of T_e/T_i and Ω_{ci}/Ω_{pi} as a function of $\delta n_i/n_0$. For small or moderate values of $\delta n_i/n_0$ we find a nearly linear relationship with Δ and U. For very large $\delta n_i/n_0$ we find signs of a saturation. We do not expect this limit to be well-covered by any theoretical model.

Discussion

The present results depend critically on the assumption that $\Omega_{ci} < \Omega_{pi}$. We have studied the other limit, considering a case with $\Omega_{ci} = 2\Omega_{pi}$. In this limit all modes are confined to the striation, and relaxing the assumption of quasi-neutrality the wave steepening is inhibited by ion sound wave dispersion as described by a KdV equation [9]. For this case we observed the development of Airy-type dispersive ripples which spread out with time, and in this sense no steady state shock was formed. More generally we can formulate a modified KdV-equation using Eq. (4) as basis. This equation will contain a non-local term, which accounts for the radiation losses.

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