# CIS002-2 Computational Alegrba and Number Theory

## David Goodwin

david.goodwin@perisic.com



University of
Bedfordshire

10:00, Monday 27<sup>th</sup> February 2012

# Outline

# OUTLINE

# GREATEST COMMON DIVISOR OF TWO POLYNOMIALS

The greatest common divisor of two polynomials $a(x)$, $b(x)$, is a polynomial of highest degree which divides them both.

- This is the largest polynomial dividing both where we measure the "largeness" using the degree.

- The gcd is only determined up to multiplication by a non-zero real number.

- Note that if you calculate the gcd of (say) 2 and 4 as polynomials you get the answer 1, while as integers the result is 2.

The Euclidean algorithm for polynomials says; If $d(x)$ is the gcd of $a(x)$, $b(x)$ there are polynomials $p(x)$, $q(x)$ such that $d = a(x)p(x) + b(x)q(x)$.

University of
Bedfordshire

# GREATEST COMMON DIVISOR OF TWO POLYNOMIALS - ALGORITHM

Find the greatest common divisor of two polynomials $f, g \in \mathbb{R}[x]$. This can be solved by the Euclidean Algorithm:

1. Let $F = f$ and $G = g$.

# GREATEST COMMON DIVISOR OF TWO POLYNOMIALS - ALGORITHM

Find the greatest common divisor of two polynomials $f, g \in \mathbb{R}[x]$. This can be solved by the Euclidean Algorithm:

1. Let $F = f$ and $G = g$.

2. Use the division algorithm to find polynomials $q, r$ with degree $r < G$, such that $F = qG + r$.

# GREATEST COMMON DIVISOR OF TWO POLYNOMIALS - ALGORITHM

Find the greatest common divisor of two polynomials $f, g \in \mathbb{R}[x]$. This can be solved by the Euclidean Algorithm:

1. Let $F = f$ and $G = g$.

2. Use the division algorithm to find polynomials $q, r$ with degree $r < G$, such that $F = qG + r$.

3. If $r = 0$ then we stop: $G = g$ is the greatest common divisor.

# GREATEST COMMON DIVISOR OF TWO POLYNOMIALS - ALGORITHM

Find the greatest common divisor of two polynomials $f, g \in \mathbb{R}[x]$. This can be solved by the Euclidean Algorithm:

1. Let $F = f$ and $G = g$.

2. Use the division algorithm to find polynomials $q, r$ with degree $r < G$, such that $F = qG + r$.

3. If $r = 0$ then we stop: $G = g$ is the greatest common divisor.

4. If $r \neq 0$ then replace $F$ by $G$ and $G$ by $r$ and go to step 2.

University of
Bedfordshire

# EXAMPLE

Find the greatest common divisor, $d$, of the following pairs of polynomials, $f$ and $g$:

$$f(x) = x^3 + x^2 - 3x - 3$$
$$g(x) = x^2 + 3x + 2$$

$$x^3 + x^2 - 3x - 3 = (x^2 + 3x + 2) \cdot (x - 2) + (x + 1)$$
$$x^2 + 3x + 2 = (x + 1) \cdot (x + 2) + 0$$

The last nonzero remainder is a greatest common divisor, which is $(x + 1)$. Expicit division is shown below:

$$
\begin{array}{r}
x - 2 \\
\hline
x^2 + 3x + 2 \overline{)\, x^3 + x^2 - 3x - 3} \\
- x^3 - 3x^2 - 2x \\
\hline
- 2x^2 - 5x - 3 \\
2x^2 + 6x + 4 \\
\hline
x + 1
\end{array}
$$

## EXERCISES

Find the greatest common divisor, $d$, of the following pairs of polynomials, $f$ and $g$:

❶
$$f(x) = -3x - 2x^2$$
$$g(x) = 3x - 2x^2 + 2x^3$$

❷
$$f(x) = -1 + 2x + 3x^2$$
$$g(x) = 1 + 3x + 2x^2$$

❸
$$f(x) = -1 + x^2$$
$$g(x) = 2 + 3x + x^2$$

❹
$$f(x) = 3 - 3x^2$$
$$g(x) = -2 + 2x + 2x^2 - 2x^3$$

❺
$$f(x) = 2x + 2x^2$$
$$g(x) = -2x - 3x - x^3$$

## Exercises - Solutions

Find the greatest common divisor, $d$, of the following pairs of polynomials, $f$ and $g$:

1. $2x^3 - 2x^2 + 3x = \left(-2x^2 - 3x\right) \cdot \left(-x + \frac{5}{2}\right) + \frac{21}{2}x$
   $-2x^2 - 3x = \frac{21}{2}x \cdot \left(-\frac{4}{21}x - \frac{2}{7}\right) + 0$
   last nonzero remainder gives gcd $\frac{21}{2}x$

2. $2x^2 + 3x + 1 = \left(3x^2 + 2x - 1\right) \cdot \frac{2}{3} + \left(\frac{5}{3}x + \frac{5}{3}\right)$
   $3x^2 + 2x - 1 = \left(\frac{5}{3}x + \frac{5}{3}\right) \cdot \left(\frac{9}{5}x - \frac{3}{5}\right) + 0$
   last nonzero remainder gives gcd $\frac{5}{3}(x + 1)$

3. $x^2 + 3x + 2 = \left(x^2 - 1\right) \cdot 1 + \left(3x + 3\right)$
   $x^2 - 1 = \left(3x + 3\right) \cdot \left(\frac{1}{3}x - \frac{1}{3}\right) + 0$
   last nonzero remainder gives gcd $3(x + 1)$

4. $-2x^3 + 2x^2 + 2x - 2 = \left(-3x^2 + 3\right) \cdot \left(\frac{2}{3}x - \frac{2}{3}\right) + 0$
   last nonzero remainder gives gcd $3(1 - x^2)$

5. $-x^3 - 5x = \left(2x^2 + 2x\right) \cdot \left(-\frac{1}{2}x + \frac{1}{2}\right) + -6x$
   $2x^2 + 2x = -6x \cdot \left(-\frac{1}{3}x - \frac{1}{3}\right) + 0$
   last nonzero remainder gives gcd $-6x$

## Exercises - Solutions

Explicit division of the previous exersises are shown below:

❶
$$
\begin{array}{r}
-x + \frac{5}{2} \\
-2x^2 - 3x{\overline{)}\, 2x^3 - 2x^2 + 3x} \\
\underline{-2x^3 - 3x^2} \\
-5x^2 + 3x \\
\underline{5x^2 + \frac{15}{2}x} \\
\frac{21}{2}x
\end{array}
$$

❷
$$
\begin{array}{r}
\frac{2}{3} \\
3x^2 + 2x - 1{\overline{)}\, 2x^2 + 3x + 1} \\
\underline{-2x^2 - \frac{4}{3}x + \frac{2}{3}} \\
\frac{5}{3}x + \frac{5}{3}
\end{array}
$$

❸
$$
\begin{array}{r}
1 \\
x^2 - 1{\overline{)}\, x^2 + 3x + 2} \\
\underline{-x^2 + 1} \\
3x + 3
\end{array}
$$

❹
$$
\begin{array}{r}
\frac{2}{3}x - \frac{2}{3} \\
-3x^2 + 3{\overline{)}\, -2x^3 + 2x^2 + 2x - 2} \\
\underline{2x^3 - 2x} \\
2x^2 - 2 \\
\underline{-2x^2 + 2} \\
0
\end{array}
$$

❺
$$
\begin{array}{r}
-\frac{1}{2}x + \frac{1}{2} \\
2x^2 + 2x{\overline{)}\, -x^3 - 5x} \\
\underline{x^3 + x^2} \\
x^2 - 5x \\
\underline{-x^2 - x} \\
-6x
\end{array}
$$

# Further Exercise

Find $gcd(2x^5 + 7x^4 - 2x^3 - 9x^2 + 6x - 12, x^4 + 2x^3 - 5x^2 - 2x + 4)$

# FURTHER EXERCISE - SOLUTION

Find gcd$(2x^5 + 7x^4 - 2x^3 - 9x^2 + 6x - 12, x^4 + 2x^3 - 5x^2 - 2x + 4)$

$$2x^5 + 7x^4 - 2x^3 - 9x^2 + 6x - 12 = \left(x^4 + 2x^3 - 5x^2 - 2x + 4\right) \cdot (2x + 3) + \left(2x^3 + 10x^2 + 4x - 24\right)$$

$$x^4 + 2x^3 - 5x^2 - 2x + 4 = \left(2x^3 + 10x^2 + 4x - 24\right) \cdot \left(\tfrac{1}{2}x - \tfrac{3}{2}\right) + \left(8x^2 + 16x - 32\right)$$

$$2x^3 + 10x^2 + 4x - 24 = \left(8x^2 + 16x - 32\right) \cdot \left(\tfrac{1}{4}x + \tfrac{3}{4}\right) + 0$$

Where we have the explicit division algorithms:

$$
\begin{array}{r}
\phantom{x^4 + 2x^3 - 5x^2 - 2x + 4)}\quad 2x \;+ 3 \\
\hline
x^4 + 2x^3 - 5x^2 - 2x + 4)\quad 2x^5 + 7x^4 - 2x^3 - 9x^2 + 6x - 12 \\
- 2x^5 - 4x^4 + 10x^3 + 4x^2 - 8x \\
\hline
3x^4 + 8x^3 - 5x^2 - 2x - 12 \\
- 3x^4 - 6x^3 + 15x^2 + 6x - 12 \\
\hline
2x^3 + 10x^2 + 4x - 24
\end{array}
$$

$$
\begin{array}{r}
\tfrac{1}{2}x - \tfrac{3}{2} \\
\hline
2x^3 + 10x^2 + 4x - 24)\quad x^4 + 2x^3 - 5x^2 - 2x + 4 \\
- x^4 - 5x^3 - 2x^2 + 12x \\
\hline
- 3x^3 - 7x^2 + 10x + 4 \\
3x^3 + 15x^2 + 6x - 36 \\
\hline
8x^2 + 16x - 32
\end{array}
$$

$$
\begin{array}{r}
\tfrac{1}{4}x + \tfrac{3}{4} \\
\hline
8x^2 + 16x - 32)\quad 2x^3 + 10x^2 + 4x - 24 \\
- 2x^3 - 4x^2 + 8x \\
\hline
6x^2 + 12x - 24 \\
- 6x^2 - 12x + 24 \\
\hline
0
\end{array}
$$

The last non-zero remainder gives gcd $8x^2 + 16x - 32$.

# OUTLINE

- Bisection is the division of a given curve, figure, or interval into two equal parts.

- A simple bisection procedure for iteratively converging on a solution which is known to lie inside some interval $[a, b]$.

- Evaluating the function at the midpoint of the original interval $x = (a + b)/2$.

- Test to see in which of the subintervals $[a, (a + b)/2]$ or $[(a + b)/2, b]$ the solution lies.

- The procedure is then repeated with the new interval as often as needed to locate the solution to the desired accuracy.

- Let $a_n$ and $b_n$ be the endpoints at the $n^{\text{th}}$ iteration (with $a_1 = a$ and $b_1 = b$) and let $r_n$ be the $n^{\text{th}}$ approximate solution.

- Then the number of iterations required to obtain an error smaller than $\epsilon$ is found by noting that

$$b_n - a_n = \frac{(b-a)}{(2^{(n-1)})}$$

- and that $r_n$ is defined by

$$r_n = \frac{1}{2}(a_n + b_n)$$

- In order for the error to be smaller than $\epsilon$,

$$|r_n - r| \le \frac{1}{2}(b_n - a_n) = 2^{(-n)}(b-a) < \epsilon$$

- Taking the natural logarithm of both sides then gives

$$-n \ln 2 < \ln \epsilon - \ln(b-a)$$

- so

$$n > \frac{(\ln(b-a) - \ln \epsilon)}{(\ln 2)}$$

University of
Bedfordshire

# Matlab File Exchange

File Exchange is a place in the MATLAB Central community where users can find and share content related to MATLAB development. Find example code samples, tutorials or videos to help you with your development tasks. You can:

- Find files related to your programming interest
- Comment on and rate files
- Tag files
- Build your reputation by contributing your own submissions

http://www.mathworks.co.uk/matlabcentral/fileexchange/

## TASK

1. Navigate within a browser to the mathworks file exchange.

2. Find a function within the repository that claims to solve an equation using the Bisection Method

3. Download and test this function, to see if it gives some correct answers.

4. Read the code of the function and try to understand the main points of the code.

5. Use this code to find solution to polynomial equations of increasing degree i.e. your coursework code on Eigenvalues.

University of Bedfordshire