Introduction to Linear Algebra

CIS002-2 Computational Alegrba and Number Theory

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Outline

1. What is Linear Algebra?
2. LU decomposition
3. Worked example
4. Class exercise
5. Singular Value Decomposition

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What is Linear algebra?

Linear algebra is the study of linear sets of equations and their transformation properties. Linear algebra allows the analysis of rotations in space, least squares fitting, solution of coupled differential equations, determination of a circle passing through three given points, as well as many other problems in mathematics, physics, and engineering.

The matrix and determinant are extremely useful tools of linear algebra. One central problem of linear algebra is the solution of the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

for $\mathbf{x}$. While this can, in theory, be solved using a matrix inverse

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

other techniques such as Gaussian elimination are numerically more robust. In addition to being used to describe the study of linear sets of equations, the term "linear algebra" is also used to describe a particular type of algebra. In particular, a linear algebra $\mathbf{L}$ over a field $\mathbf{F}$ has the structure of a ring with all the usual axioms for an inner addition and an inner multiplication together with distributive laws, therefore giving it more structure than a ring.

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Triangular Matrices

The determinant of a triangular matrix, upper or lower, is simply the product of it's diagonal elements.

$$\begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = \begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix} = abc$$

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LU decomposition

LU decomposition is the procedure for decomposing an $N \times N$ matrix **A** into a product of a lower triangular matrix **L** and an upper triangular matrix **U**,

$$\mathbf{LU} = \mathbf{A}$$

Written explicitly for a 3×3 matrix, the decomposition is

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can find, explicitly, the form of the new matrix, **LU**,

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+l_{22}u_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This gives three types of equations

$$\begin{array}{llll} i<j & l_{i1}u_{1j}+l_{i2}u_{2j}+\cdots+l_{ii}u_{ij} & = & a_{ij} \\ i=j & l_{i1}u_{1j}+l_{i2}u_{2j}+\cdots+l_{ii}u_{jj} & = & a_{ij} \\ i>j & l_{i1}u_{1j}+l_{i2}u_{2j}+\cdots+l_{ij}u_{jj} & = & a_{ij} \end{array}$$

This gives N^2 equations for N^2+N unknowns (the decomposition is not unique)

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LU decomposition of a matrix is frequently used as part of a Gaussian elimination process for solving a matrix equation.
A matrix that has undergone Gaussian elimination is said to be in **echelon form**.
For example, consider the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 7 \end{bmatrix}$$

In augmented form, this becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 4 & 8 \\ 9 & 3 & 4 & 7 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Subtracting 9 times the first row from the third row gives

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 4 & 3 & 4 & | & 8 \\ 0 & -6 & -5 & | & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Subtracting 4 times the first row from the second row gives

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & -1 & 0 & | & -4 \\ 0 & -6 & -5 & | & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Finally, adding -6 times the second row to the third row gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & -5 & 4 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Restoring the transformed matrix equation gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix}$$

which can be solved immediately to give $x_3 = -4/5$, back-substituting to obtain $x_2 = 4$ (which actually follows trivially in this example), and then again back-substituting to find $x_1 = -1/5$.

However, we could use our Gaussian elimination to become the start of an **LU** decomposition:

$$\mathbf{A} = \mathbf{LU}$$

using the matrices from the Gaussian elimination we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

From here, we can directly multiply the upper and lower matrices to find the symbolic elements of the lower triangular matrix.

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+l_{22}u_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix}$$

substituting the values of the elements of the upper triangular matrix we find

$$\begin{bmatrix} l_{11} & l_{11} & l_{11} \\ l_{21} & l_{21}-l_{22} & l_{21} \\ l_{31} & l_{31}-l_{32} & l_{31}-5l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix}$$

Which gives $l_{11}=1,\ l_{21}=4,\ l_{31}=9,\ l_{22}=-3+4=1,$
$l_{32}=-3+9=6,\ l_{33}=\frac{-4+9}{5}=1$

Reconstructing our matrices we find

$$\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & 4\\ 9 & 3 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 9 & 6 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 0 & -5\end{bmatrix}$$

We can check this matrix multiplication:

$$\begin{aligned}
&\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 9 & 6 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 0 & -5\end{bmatrix}\\
&= \begin{bmatrix}(1\cdot 1+0\cdot 0+0\cdot 0) & (1\cdot 1+0\cdot -1+0\cdot 0) & (1\cdot 1+0\cdot 0+0\cdot -5)\\ (4\cdot 1+1\cdot 0+0\cdot 0) & (4\cdot 1+1\cdot -1+0\cdot 0) & (4\cdot 1+1\cdot 0+0\cdot -5)\\ (9\cdot 1+6\cdot 0+1\cdot 0) & (9\cdot 1+6\cdot -1+1\cdot 0) & (9\cdot 1+6\cdot 0+1\cdot -5)\end{bmatrix}\\
&= \begin{bmatrix}1 & 1 & 1\\ 4 & 3 & 4\\ 9 & 3 & 4\end{bmatrix}
\end{aligned}$$

However, as previously mentioned, LU decomposition is not unique. For the previous example we could have decomposed to the following matrices:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4444 & 1 & 0 \\ 0.1111 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & 4 \\ 0 & 1.6667 & 2.2222 \\ 0 & 0 & -0.3333 \end{bmatrix}$$

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Class exercise

1. Form the augmented matrix and solve the following set of equations by Gaussian elimination

$$
\begin{aligned}
x_1 + 2x_2 - 3x_3 &= 3 \\
2x_1 - x_2 - x_3 &= 11 \\
3x_1 + 2x_2 + x_3 &= -5
\end{aligned}
$$

Class exercise

1. Form the augmented matrix and solve the following set of equations by Gaussian elimination

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 3 \\ 2x_1 - x_2 - x_3 &= 11 \\ 3x_1 + 2x_2 + x_3 &= -5 \end{aligned}$$

2. $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ is a lower triangular martix and $\mathbf{U}$ is an upper triangular matrix. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, what is $\mathbf{L}$?

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Singular Value Decomposition

If **A** is an $m \times n$ real matrix with $m > n$, then **A** can be written using a so-called singular value decomposition of the form

$$A = UDV^T$$

Where

$$U^T U = I$$

and

$$V^T V = I$$

(where the two identity matrices may have different dimensions), and D has entries only along the diagonal.

Consider the 4×5 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

A singular value decomposition of this matrix is given by UDV^T

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ \sqrt{0.8} & 0 & 0 & 0 & -\sqrt{0.2} \end{bmatrix}$$

Notice D contains only zeros outside of the diagonal. Furthermore, because the matrices U and V^T are unitary matrices, multiplying by their respective conjugate transposes yields the identity matrix, as shown below.

$$UU^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv I_4$$

and

$$VV^T = \begin{bmatrix} 0 & 0 & \sqrt{0.2} & 0 & \sqrt{0.8} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{0.8} & 0 & -\sqrt{0.2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ \sqrt{0.8} & 0 & 0 & 0 & -\sqrt{0.2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \equiv I_5$$

It should also be noted that this particular singular value decomposition is not unique. Choosing V such that

$$V^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ \sqrt{0.4} & 0 & 0 & \sqrt{0.5} & -\sqrt{0.1} \\ -\sqrt{0.4} & 0 & 0 & \sqrt{0.5} & \sqrt{0.1} \end{bmatrix}$$

is also a valid singular value decomposition.