Matrices II

CIS002-2 Computational Alegrba and Number Theory

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University of Bedfordshire

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Outline

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- A matrix whose determinant is zero is called a **singluar matrix**.

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Cofactors

- The determinant of a square matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$ is

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- The place sign is worked from $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

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Adjoint of a Square Matrix

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- adj $\mathbf{A} = \begin{bmatrix} -21 & 0 & 7 \\ 7 & 11 & -17 \\ 14 & -22 & -1 \end{bmatrix}$.

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- What is the inverse of the matrix we used in the last frame,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}?$$

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  $$\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}?$$
- $$\mathbf{A}^{-1} = \frac{1}{45}\begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix}.$$

Outline

Solution to a set of linear equations

- Consider a set of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
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- From our knowledge of matrix multiplication, this can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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- i.e. $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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- If we multiply both sides of this matrix equations by the inverse of $\mathbf{A}$, we have

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- Therefore, if we form the inverse of the matrix of coefficients and pre-multiply matrix $\mathbf{b}$ by it, we shall determine the matrix of the solutions of $\mathbf{x}$.

Class Exercises

Solver the following sets of equations

1.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 - 4x_2 - 2x_3 &= 2 \\ 5x_1 + 3x_2 + 5x_3 &= -1 \end{aligned}$$

2.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 - x_3 &= 11 \\ 2x_1 - 2x_2 + 5x_3 &= 3 \end{aligned}$$

Class Exercises - Answers

1. $x_1 = 2,\ x_2 = 3,\ x_3 = -4$
2. $x_1 = -1,\ x_2 = 5,\ x_3 = 3$

Outline

Gaussian elimination

Finding an inverse can be a time consuming task, and finding an inverse to large matrices is numerically inefficient. If we go back to our initial problem of finding solution to a set of linear, we could eliminate terms from successive equaions: Consider a set of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
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The **Guassian elimination** technique uses the first equation to eliminate the first unknown from the remaining equations. Then the new second equation is used to eliminate the second unknown from the third equation. In general we work down the equations, and then, with the last unknown determined, we wrok back up to solve for each of the other unknowns in succession.

Gaussian elimination - example

$$
\begin{aligned}
3x + 2y + z &= 11 \\
2x + 3y + z &= 13 \\
x + y + 4z &= 12
\end{aligned}
$$

- Divide each row by its initial coefficient

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\begin{aligned}
x + \frac{2}{3}y + \frac{1}{3}z &= \frac{11}{3} \\
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$$x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$$
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- Eliminate x from the second and thrid equations, using the first equation

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- Eliminate x from the second and thrid equations, using the first equation
- $$x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$$
  $$y + \frac{1}{5}z = \frac{17}{5}$$
  $$y + 11z = 25$$

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Gaussian elimination - example

$$\begin{aligned} x + \frac{2}{3}y + \frac{1}{3}z &= \frac{11}{3} \\ y + \frac{1}{5}z &= \frac{17}{5} \\ y + 11z &= 25 \end{aligned}$$

- Repeat the technique to eliminate y from the third equation, using the second equation

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Gaussian elimination - example

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- so we find $z = 2$, from this last equation.

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- so we find $z = 2$, from this last equation.
- Working back to the second equation we find $y + \frac{1}{5} \times 2 = \frac{17}{5}$ so $y = 3$

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Gaussian elimination - example

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- so we find $z = 2$, from this last equation.
- Working back to the second equation we find $y + \frac{1}{5} \times 2 = \frac{17}{5}$ so $y = 3$
- Finally, we work back to the first equation to find $x + \frac{3}{3} \times 3 + \frac{1}{3} \times 2 = \frac{11}{3}$ so $x = 1$.

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Gaussian elimination

The technique of Gaussian elimination may not seem so elegant as that of using an inverse of a matrix, but it is well adapted to modern computers and is far faster than the time spent with determinants.

The Gaussian technique may be used to convert a determinant into triangular form:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

For an n^{th}-order determinant the evaluation of the triangular form requires only $n-1$ multiplications compared with the n required for the general case.

Gauss-Jordan elimination - a variation of Gauss elimination

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- $$x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$$
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Gauss-Jordan elimination - a variation of Gauss elimination

- Then we would eliminate y from the first and third equations, using the second equation
- $$x + \frac{1}{5}z = \frac{7}{5}$$
  $$y + \frac{1}{5}z = \frac{17}{5}$$
  $$z = 2$$

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$$x = 1 \qquad y = 3 \qquad z = 2$$

Gauss elimination - the augmented matrix

- $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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- All the information for solving the set of equations is provided by the matrix of coefficients **A** and the column matrix **b**. If we write the elements of **b** within the matrix **A**, we obtain the **augmented matrix B** of the given set of equations.

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- All the information for solving the set of equations is provided by the matrix of coefficients **A** and the column matrix **b**. If we write the elements of **b** within the matrix **A**, we obtain the **augmented matrix B** of the given set of equations.
- $$\mathbf{B} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Gauss elimination - the augmented matrix

- $$\mathbf{B} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

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Gauss elimination - the augmented matrix

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- We then eliminate elements other than a_{11} from the first column by subtracting $\frac{a_{21}}{a_{11}}$ times the first row from the second row and $\frac{a_{31}}{a_{11}}$ times the first row from the third row, etc.

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Gauss elimination - the augmented matrix

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- We then eliminate elements other than a_{11} from the first column by subtracting $\frac{a_{21}}{a_{11}}$ times the first row from the second row and $\frac{a_{31}}{a_{11}}$ times the first row from the third row, etc.
- This gives a matrix of the form

$$\mathbf{B} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n2} & \cdots & c_{nn} & d_n \end{array}\right]$$

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Gauss elimination - the augmented matrix

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- We then eliminate elements other than a_{11} from the first column by subtracting $\frac{a_{21}}{a_{11}}$ times the first row from the second row and $\frac{a_{31}}{a_{11}}$ times the first row from the third row, etc.
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$$\mathbf{B} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n2} & \cdots & c_{nn} & d_n \end{array}\right]$$

- The process is then repeated to eliminate c_{i2} from the third and subsequent rows.

Augmented matrix - example

- Gven a set of linear equation, we can write them in matrix form, from a particular example we have

$$\begin{bmatrix} 1 & -4 & -2 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 3 \\ -2 \end{bmatrix}$$

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- The augmented matrix would be

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right]$$

Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right]$$

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Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right]$$
- We can now eliminate the x_1 coefficients from the second and third rows by subtracting 2 times the first row from the second row, and subtracting 3 times the first row from the third row.

Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right]$$
- We can now eliminate the x_1 coefficients from the second and third rows by subtracting 2 times the first row from the second row, and subtracting 3 times the first row from the third row.
- So the matrix becomes

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{array}\right]$$

Augmented matrix - example

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Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{array}\right]$$
- We can now eliminate the x_2 coefficients from the third row by subtracting $\frac{14}{9}$ times the second row from the third row.

Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{array}\right]$$
- We can now eliminate the x_2 coefficients from the third row by subtracting $\frac{14}{9}$ times the second row from the third row.
- So the matrix becomes

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 0 & -\frac{13}{3} & -\frac{13}{3} \end{array}\right]$$

Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 0 & -\frac{13}{3} & -\frac{13}{3} \end{array}\right]$$

Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 0 & -\frac{13}{3} & -\frac{13}{3} \end{array}\right]$$
- Re-forming the matrix equation

$$\begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & 6 \\ 0 & 0 & -\frac{13}{3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ -39 \\ -\frac{13}{3} \end{bmatrix}$$

Augmented matrix - example

- $$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 0 & -\frac{13}{3} & -\frac{13}{3} \end{array}\right]$$
- Re-forming the matrix equation

$$\begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & 6 \\ 0 & 0 & -\frac{13}{3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ -39 \\ -\frac{13}{3} \end{bmatrix}$$

- We can start from the bottom row to find the solution $x_3 = 1$, use this in the second row to find $x_2 = -5$, then use both of these in the first row to find $x_1 = 3$.

Outline

Coursework

Coursework assignment

Write a computer code in C++ that uses the General technique, Gauss elimination technique, Guass-Jordan elimination technique to solve a set of *n* linear equations with *n* unknowns. Compare the computing time of the three different methods and make a graph of the data. Please use at least 10 data points.

A decent book that could be useful for this coursework, and further scientific programming is:
Numerical Recipes 3rd Edition: The Art of Scientific Computing
W.H.Press, S.A.Teukolsky, W.T.Vetterling, B.P.Flannery; Cambridge University Press (2007)

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