Logic II

CIS008-2 Logic and Foundations of Mathematics

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Arguments

An **argument** is a sequence of propositions written

$$p_1, p_2, \ldots p_n / \therefore q$$

The symbol $\therefore$ is read "therefore", the propositions $p_1, p_2, \ldots p_n$ are called the *hypotheses* (or *premises*), and the proposition q is called the *conclusion*. The argument is **valid** provided that is the propositions $p_1, p_2, \ldots p_n$ are all true, then q must also be true; otherwise, the argument is **invalid** (or a **fallacy**). An argument is valid because of its form, not its content.

Rules of Inference

Rules of Inference, brief, valid arguments, are used within a larger argument

Modus ponens $p \to q, p / \therefore q$

Modus tollens $p \to q, \neg q / \therefore \neg p$

Addition $p / \therefore p \vee q$

Simplification $p \wedge q / \therefore q$

Conjunction $p, p / \therefore p \wedge q$

Hypothetical syllogism $p \to q, q \to r / \therefore p \to r$

Disjunctive syllogism $p \vee q, \neg q / \therefore q$

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Propositional function

Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a **propositional function** or **predicate** (with respect to D) if for each $x \in D$, $P(x)$ is a proposition. We call D the **domain of discourse**.

Universal quantifier

Let P be a propositional function with domain of discourse D. The **universally quantified statement** is written as

$$\forall x, P(x)$$

where the symbol $\forall$ is read "for all" and is called the **universal quantifier**. The above universally quantified statement is true if $P(x)$ is true for every x in D, and is false if $P(x)$ is false for at least one x in D.

Definition

Let P be a propositional function with domain of discourse D. The **existentially quantified statement** is written as

$$\exists x, P(x)$$

where the symbol $\exists$ means "there exists" and is called the **existential quantifier**. The above existentially quantified statement is true if $P(x)$ is true for at least one x in D, and is false if $P(x)$ is false for every x in D.

Generalised De Morgan's Laws for Logic

Theorem

If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values (i.e. either both are true or both are false).

A $\neg(\forall x P(x))\,;\,\exists x \neg P(x)$

B $\neg(\exists x P(x))\,;\,\forall x \neg P(x)$

Rules of Inference

Universal instantiation $\forall x P(x) / \therefore P(d)$ if $d \in D$

Universal generalisation $P(d)$ for every $d \in D / \therefore \forall x P(x)$

Existential instantiation $\exists x P(x) / \therefore P(d)$ for some $d \in D$

Existential generalisation

$P(d)$ for some $d \in D / \therefore \exists x P(x)$

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Nested Quantifiers

Consider writing the statement:

The sum of any two positive real numbers is positive.

symbolically. We need two variables, say x and y.

If $x > 0$ and $y > 0$, then $x + y > 0$.

$$(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$$

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Nested Quantifiers

Let $P(x)$ denote the following

$$(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$$

We need two universal quantifiers, the given statement can be written symbolically as

$$\forall x \forall y P(x, y)$$

We note the domain of discourse for this propositional function is $X \times Y$, or more specifically $\mathbb{R} \times \mathbb{R}$, which means that each variable x and y must belong to the set of real numbers. Multiple quantifiers such as $\forall x \forall y$ are said to be **nested quantifiers**.

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Examples

1. Restate $\forall m \exists n(m < n)$ in words. The domain of discourse is $\mathbb{Z} \times \mathbb{Z}$.
2. Write the assertion "Everybody loves somebody" symbolically, letting $L(x, y)$ be the statement "x loves y".
3. Consider the statement $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$, with the domain of discourse $\mathbb{R} \times \mathbb{R}$. Is this true or false?
4. Consider the statement $\forall x \exists y(x + y = 0)$, with the domain of discourse $\mathbb{R} \times \mathbb{R}$. Is this true or false?
5. Consider the statement $\exists x \forall y(x \geq y)$, with the domain of discourse $\mathbb{Z}^+ \times \mathbb{Z}^+$. Is this true or false?
6. Consider the statement $\exists x \exists y((x > 1) \wedge (y > 1) \wedge (xy = 6))$, with the domain of discourse $\mathbb{Z}^+ \times \mathbb{Z}^+$. Is this true or false?

Examples

1. Restate $\forall m \exists n(m < n)$ in words. The domain of discourse is $\mathbb{Z} \times \mathbb{Z}$. There is no greatest integer.
2. Write the assertion "Everybody loves somebody" symbolically, letting $L(x, y)$ be the statement "x loves y".
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3. Consider the statement $\forall x \forall y((x > 0) \wedge (y > 0) \to (x + y > 0))$, with the domain of discourse $\mathbb{R} \times \mathbb{R}$. Is this true or false? True.
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Exercises

Determine the truth values of each of the following, where the domain of discourse is $\mathbb{R} \times \mathbb{R}$:

1. $\forall x \forall y (x^2 < y + 1)$
2. $\forall x \exists y (x^2 < y + 1)$
3. $\exists x \forall y (x^2 < y + 1)$
4. $\exists x \exists y (x^2 < y + 1)$
5. $\exists y \forall x (x^2 < y + 1)$
6. $\forall y \exists x (x^2 < y + 1)$
7. $\forall x \forall y (x^2 + y^2 = 9)$
8. $\forall x \exists y (x^2 + y^2 = 9)$
9. $\exists x \forall y (x^2 + y^2 = 9)$
10. $\exists x \exists y (x^2 + y^2 = 9)$
11. $\forall x \forall y (x^2 + y^2 \geq 0)$
12. $\forall x \exists y (x^2 + y^2 \geq 0)$
13. $\exists x \forall y (x^2 + y^2 \geq 0)$
14. $\exists x \exists y (x^2 + y^2 \geq 0)$
15. $\forall x \forall y ((x < y) \rightarrow (x^2 < y^2))$
16. $\forall x \exists y ((x < y) \rightarrow (x^2 < y^2))$
17. $\exists x \forall y ((x < y) \rightarrow (x^2 < y^2))$
18. $\exists x \exists y ((x < y) \rightarrow (x^2 < y^2))$