Relations

CIS008-2 Logic and Foundations of Mathematics

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12:00, Friday 02nd December 2011

Outline

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Relations

A (binary) **relation** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

A function is a special type of relation. A function f from X to Y is a relation from X to Y having the properties:

- The domain of f is equal to X.
- For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$

Relations - example

Let

$$X = \{2, 3, 4\} \quad \text{and} \quad Y = \{3, 4, 5, 6, 7\}$$

If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \mid y$$

we obtain

$$R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$$

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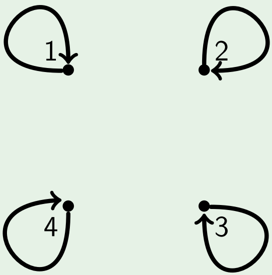
Relations - reflexive

A relation R on a set X is **reflexive** if $(x, x) \in X$, if $(x, y) \in R$ for all $x \in X$.

Example

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x = y$, with $x, y \in X$.



There must be a loop at each vertex.

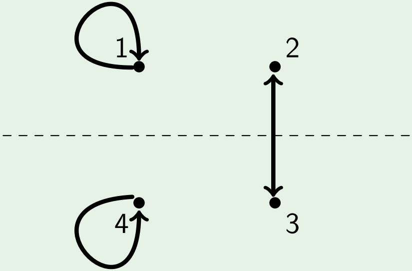
Relations - symmetry

A relation R on a set X is **symmetric** if for all $x, y \in X$, if $(x, y) \in R$ then $(y, x) \in R$. In symbols we can write

$$\forall x \forall y [(x, y) \in R] \rightarrow [(y, x) \in R]$$

Example

The relation $R = \{(1,1),(2,3),(3,2),(4,4)\}$ on $X = \{1,2,3,4\}$.



Notice the axis of symmetry.

Relations - antisymmetry

A relation R on a set X is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. In symbols we can write

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R] \rightarrow [x = y]$$

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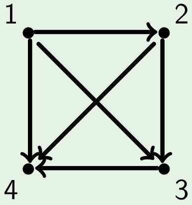
Example

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x < y$, with $x, y \in X$.



There is at most one directed edge between any two distinct vertices.

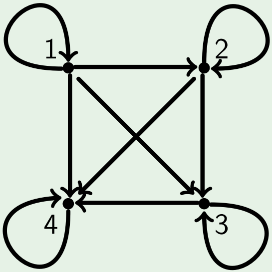
Relations - transitive

A relation R on a set X is **transitive** if for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. In symbols we can write

$$\forall x \forall y \forall z [(x, y) \in R \wedge (y, z) \in R] \rightarrow [(x, z) \in R]$$

Example

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, with $x, y \in X$.



Wherever there are a directed edges from x to y and from y to z, there is also a directed edge from x to z

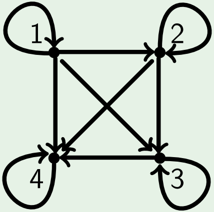
Relations - partial order

A relation R on a set X is a **partial order** if R is reflexive, antisymmetric, and transitive. If R is a partial order on a set X, the notation $x \preceq y$ is sometimes used to indicate that $(x, y) \in R$.

Suppose that R is a partial order on a set X. If $x, y \in X$ and either $x \preceq y$ or $y \preceq x$ we say x and y are **comparable**. If $x, y \in X$ and $x \npreceq y$ and $y \npreceq x$ we say x and y are **incomparable**. If every pair of elements in X is comparable, we call R a **total order**.

Example

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, with $x, y \in X$.



There are loops at every vertex, there is at most one directed edge between any two distinct vertices and Wherever there are a directed edges from x to y and from y to z, there is also a directed edge from x to z.

Relations - inverse

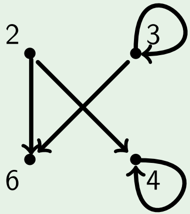
A R be a relation from X to Y. The **inverse** of R, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

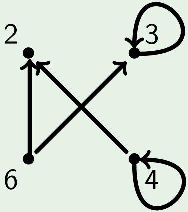
Example

The relation R from $X = \{2, 3, 4\}$ to $X = \{3, 4, 5, 6, 7\}$ is $(x, y) \in R$ if x divides y.

R



R^{-1}



Reverse the direction of the arrows.

Relations - composition

Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition** of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$$

Outline

Theorem

Let $\mathcal{S}$ be a partition of the set X. Define xRy to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, antisymmetric, and transitive.

A partition of a set X is a collection $\mathcal{S}$ of nonempty subsets of X such that every element in X belongs to exactly one member of $\mathcal{S}$.

Equivalence relation

A relation that is reflexive, symmetric, and transitive on a set X is called an **equivalence relation** on X.

Theorem

Let R be an equivalence realtion on a set X. For each $a \in X$, let

$$[a] = \{x \in X \mid xRa\}$$

(In words, $[a]$ is a set of all elements in X that are related to a.)
Then

$$\mathcal{S} = \{[a] \mid a \in X\}$$

is a partition of X.

Equivalence classes

Let R be an equivalence relation on a set X. The sets $[a]$ are called the **equivalence classes** of X given by the relation R.

Theorem

Let R be an equivalence realtion on a finite set X. If each equivalence class has r elements, there are $|X|/r$ equivalence classes.

Outline

Exercises

1. Determine whether the following relation defined on the set of positive integers is reflexive, symmetric, antisymmetric, transitive, and / or a partial order
   - A $(x, y) \in R$ if $2 \mid x + y$
   - B $(x, y) \in R$ if $3 \mid x + y$
2. Give an example of a relation on $\{1, 2, 3, 4\}$ that is reflexive, not antisymmetric, and not transitive.

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EXERCISES

3. Suppose that R is a relation on X that is symmetric and transitive but not reflexive. Suppose also that $|X| \geq 2$. Define the relation $\overline{R}$ on X by

$$\overline{R} = X \times X - R$$

Which of the following must be true? For each false statement, provide a counterexample

A $\overline{R}$ is reflexive
B $\overline{R}$ is symmetric
C $\overline{R}$ is not antisymmetric
D $\overline{R}$ is transitive

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Exercises

4. Is the relation

$$\{(1,1),(1,2),(2,2),(4,4),(2,1),(3,3)\}$$

an equivalence relation on $\{1,2,3,4\}$? Explain.

5. Given the relation

$$\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(3,4),(4,3)\}$$

is an equivalence relation on $\{1,2,3,4\}$, find $[3]$, the equivalence class containing 3. How many distinct equivalence classes are there?

Exercises

6. Find the equivalence relation (as a set of ordered pairs) on $\{a, b, c, d, e\}$, whose equivalence classes are $\{a\}$, $\{b, d, e\}$, $\{c\}$.
7. Let R be the relation defined on the set of eight-bit strings by $s_1 R s_2$ provided that s_1 and s_2 have the same number of zeros.
   - A Show that R is an equivalence relation.
   - B How many equivalence classes are there?
   - C List one member or each equivalence class.

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