An Introduction to Graph Theory

CIS008-2 Logic and Foundations of Mathematics

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12:00, Friday 17th February 2012

Outline

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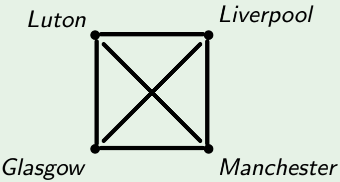
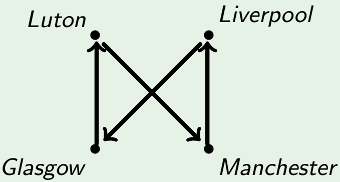
- A graph can refer to a function graph or graph of a function, i.e. a plot.
- A graph can also be a collection of points and lines connecting some subset of them.
- Points of a graph are most commonly known as graph **vertices**, but may also be called **nodes** or simply **points**.
- Lines connecting the vertices of a graph are most commonly known as graph **edges**, but may also be called **arcs** or **lines**.
- The study of graphs is known as **graph theory**, and was first systematically investigated by D. König in the 1930s.

- A normal graph in which edges have no direction is said to be **undirected**.
- When arrows are placed on one or both endpoints of the edges of a graph to indicate direction, the graph is said to be **directed**.
- A directed graph in which each edge is given a unique direction (one arrow on each edge) is called an **oriented graph**.
- A graph with numbers on the edges is called a **weighted graph**, in a weighted graph the length of a path is the sum of the weights of the edges in the path.
- A graph or directed graph together with a function which assigns a positive real number to each edge is known as a **network**.

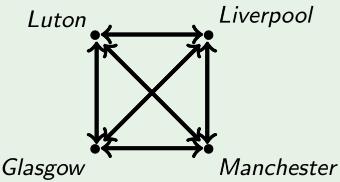
Example (Undirected graph)



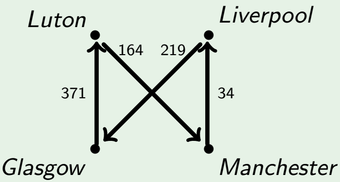
Example (Oriented graph)



Example (Directed graph)



Example (Weighted graph)



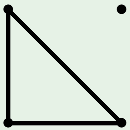
Graphs

- A graph (or **undirected graph**) G consists of a set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.
- If there is a unique edge e associated with the vertices v and w, we write $e = (v, w)$ or $e = (w, v)$. In this context, (v, w) denotes an edge between v and w in an undirected graph (not an ordered pair).
- A **directed graph** (or digraph) G consists of a set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an ordered pair of vertices.
- If there is a unique edge e associated with the ordered pair (v, w) of the vertices, we write $e = (v, w)$, which denotes an edge from v to w.
- An edge e of a graph (directed or undirected) that is associated with the pair of vertices v and w is said to be **incident** on v and w, and v and w are said to be incident on e and to be adjacent vertices.
- The **degree of a vertex** v, $\delta(v)$, is the number of edges incident on v.

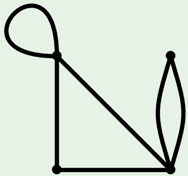
GRAPHS

- An edge incident on a single vertex is called a **loop**.
- A vertex that has no incident edges is called and **isolated vertex**.
- Distinct edges associated with the same pair of vertices are called **parallel edges**.
- A graph with neither loops nor parallel edges is called a **simple graph**.

Example (Simple Graph)



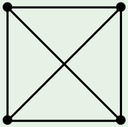
Example (Non-simple graph)



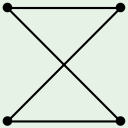
Graphs

- A **complete graph** on n vertices, denoted K_n, is a simple graph with n vertices in which there is an edge between every pair of distinct vertices.
- A graph $G = (V, E)$ is **bipartite** if there exist subsets V_1 and V_2 of v such that $V_1 \cap V_2 = \varnothing$, $V_1 \cup V_2 = V$, and each edge in E is incident on one vertex in V_1 and one vertex in V_2.
- A **complete bipartite graph** on m and n vertices, denotes $K_{m,n}$, is the simplest graph whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which the edge set consists of all edges of the form (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$

Example (Complete Graph)



Example (Complete Bipartite graph)



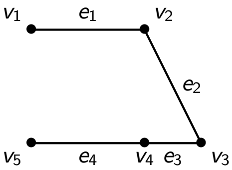
Outline

Paths

Let v_0 and v_n be vertices on a graph. A **path** from v_0 to v_n of length n is an alternating sequence of $n+1$ vertices and n edges beginning with vertex v_0 and ending with vertex v_n.

$$(v_0, e_1, v_1, e_2, v_v, \ldots, v_{n-1}, e_n, v_n)$$

in which edge e_i is incident on vertices v_{i-1} and v_i, for $i = 1, \ldots, n$.



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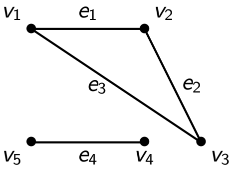
Connected graphs

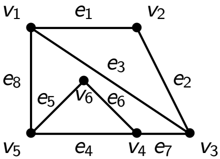
A graph G is **connected** if given any vertices v and w in G, there is a path from v to w. Below is an example of a graph that is not connected.



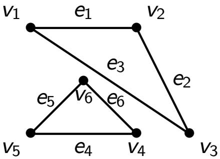
Subgraphs

Let $G = (V, E)$ be a graph. We call (V', E') a **subgraph** of G if:

- $V' \subseteq V$ and $E' \subseteq E$
- For every edge $e' \in E'$, if e' is incident on v' and w', then $v', w' \in V'$



A graph G



A graph G', a subgraph of G

Let G be a graph and let v be a vertex in G. The subgraph G' of G consisting of all edges and vertices in G that are contained in some path beginning at v is called the **component** of G containing v.

CYCLES

Let v and w be vertices in a graph G.

- A **simple path** from v to w is a path from v to w with no repeated vertices.
- A **cycle** (or **circuit**) is a path of nonzero length from v to v with no repeated edges.
- A **simple cycle** is a cycle from v to v in which, except for the beginning and ending vertices that are both equal to v, there are no repeated vertices.
- A cycle in a graph G that includes all of the edges and all of the vertices of G is called an **Euler cycle**.

Euler cycle

Theorem

If a graph G has an Euler cycle, then G is connected and every vertex has an even degree.

Theorem

If G is a connected graph and every vertex has even degree, the G has an Euler cycle.

Theorem

If G is a graph with m edges and vertices $\{v_1, v_2, \ldots, v_n\}$, then

$$\sum_{i=1}^{n} \delta(v_i) = 2m$$

In particular, the sum over the degrees of all the vertices in a graph is even. Also, in any graph, the number of vertices of odd degree is even.

Euler cycle

Theorem

A graph has a path with no repeated edges from v to w ($v \neq w$) containing all the egdes and vertices if and only if it is connected and v and w are the only vertices having odd degree.

Theorem

If a graph G contains a cycle from v to v, G contains a simple cycle from v to v.

Hamiltonian cycle

A cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice, is called a **Hamiltonian cycle**.

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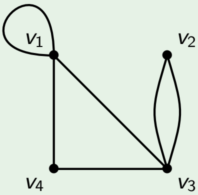
The adjacency matrix

- The adjacency matrix of a graph is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i, v_j) according to whether v_i and v_j are adjacent or not.
- For a simple graph with no self-loops, the adjacency matrix must have 0s on the diagonal.
- For an undirected graph, the adjacency matrix is symmetric.
- If $i = j$ the element is twice the number of the loops incident on the vertex.
- The eigenvalues of a graph are defined as the eigenvalues of its adjacency matrix.
- The set of eigenvalues of a graph is called a graph spectrum.

The adjacency matrix

Example (Adjacency matrix)



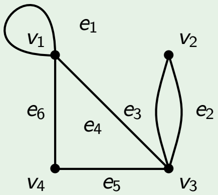
$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The incidence matrix

- The physicist Kirchhoff (1847) was the first to define the incidence matrix.
- To obtain the incidence matrix of a graph, we label the rows with the vertices and the columns with edges (in some arbitrary order).
- The entry for a row v and column e is 1 if e is incident on v and 0 otherwise.

The incidence matrix

Example (Incidence matrix)



$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercises

Draw a graph for the following adjacency matrices:

1. $$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

2. $$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Exercises

Draw a graph for the following incidence matrices:

1. $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

2. $$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Further Exercises

1. Wrtie a programme that determines whether a graph contains an Euler cycle, where inputs are given in the form of an adjaceny matrix or an incidence matrix.
2. Wrtie a programme that lists all simple paths between two given vertices.