Combinatorics & Discrete Probability Theory

CIS008-2 Logic and Foundations of Mathematics

David Goodwin

david.goodwin@perisic.com

University of Bedfordshire

11:00, Tuesday 28th February 2012

Outline

Outline

University of Bedfordshire

Introduction

Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. Mathematicians sometimes use the term “combinatorics” to refer to a larger subset of discrete mathematics that includes graph theory.

BASIC PRINCIPLES OF ENUMERATION

DEFINITION (MULTIPLICATION PRINCIPLE)

If one event can occur in m ways and a second can occur independently of the first in n ways, then the two events can occur in $m \cdot n$ ways.

DEFINITION (ADDITION PRINCIPLE)

The sum of a collection of pairwise disjoint sets is the size of the union of these sets. That is, if $S_1, S_2, \ldots, S_n$ are pairwise disjoint sets, then we have
$|S_1| + |S_2| + \cdots + |S_n| = |S_1 \cup S_2 \cup \cdots \cup S_n|$

Example

Example

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

1. How many ways can this be done?
2. How many ways can this be done if either Alice or Ben must be chairperson?
3. How many ways can this be done if Egbert must hold one of the offices?
4. How many ways can this be done if both Dolph and Francisco must hold office?

Example

Example

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

1. How many ways can this be done? $6 \cdot 5 \cdot 4 = 120$
2. How many ways can this be done if either Alice or Ben must be chairperson?
3. How many ways can this be done if Egbert must hold one of the offices?
4. How many ways can this be done if both Dolph and Francisco must hold office?

Example

Example

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

1. How many ways can this be done? $6 \cdot 5 \cdot 4 = 120$
2. How many ways can this be done if either Alice or Ben must be chairperson? $(5 \cdot 4) + (5 \cdot 4) = 40$
3. How many ways can this be done if Egbert must hold one of the offices?
4. How many ways can this be done if both Dolph and Francisco must hold office?

Example

Example

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

1. How many ways can this be done? $6 \cdot 5 \cdot 4 = 120$
2. How many ways can this be done if either Alice or Ben must be chairperson? $(5 \cdot 4) + (5 \cdot 4) = 40$
3. How many ways can this be done if Egbert must hold one of the offices? *a)* $(5 \cdot 4) + (5 \cdot 4) + (5 \cdot 4) = 40$ *b)* $3 \cdot 4 \cdot 5 = 60$
4. How many ways can this be done if both Dolph and Francisco must hold office? $3 \cdot 2 \cdot 4 = 24$

INCLUSION-EXCLUSION PRINCIPLE

The **Inclusion-Exclusion Principle** generalises the Addition principle by giving a formula to compute the numbetr of elements in the union without requiring the sets to be pairwise disjoint.

THEOREM

Let $|A|$ denote the cardinality of set A, and A and B are finite sets, then it follows immediately that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

Example

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. How many selections are there in which either Alice or Dolph or both are officers?

Example

Example

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. How many selections are there in which either Alice or Dolph or both are officers?

Let X denote the set of selections in which Alice is an officer and Y for Dolph. We must compute $|X \cup Y|$ (X and Y are not disjoint), so must use the inclusion-exclusion principle. From the previous example;
$|X| = |Y| = 3 \cdot 5 \cdot 4 = 60$ and $|X \cap Y| = 3 \cdot 2 \cdot 4 = 24$. So
$|X \cup Y| = |X| + |Y| - |X \cap Y| = 60 + 60 - 24 = 96$

Permutations

Definition

A permutation, also called an "arrangement number" or "order", is a rearrangement of the elements of an ordered list S into a one-to-one correspondence with S itself.

Theorem

The number of permutations on a set of n elements is given by $n!$ (n factorial).

The above theorem can be proved by use of the Multiplication principle.

R-PERMUTATIONS

We may wish to consider an ordering of r elements selected from n available elements, this is called an **r-permutation**.

DEFINITION

An r-permutation od n (disticnt) elements $x_1, \ldots, x_n$ is an ordering of an r-element subset of $[x_1, \ldots, x_n]$. The number of r-permutations of a set of n distinct elements is denoted $P(n, r)$ or ${}_nP_r$

THEOREM

The number of r-permutations of a set of n distinct objects is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Combinations

The selection of objects without regard to order is called a **combination**.

Definition

Given a set $X = [x_1, \ldots, x_n]$ containing n (distinct) elements,

1. An r-combination of X is an unordered selection of r-elements of x.
2. The number of r-combinations of a set of n distinct elements is denoted $C(n, r)$, ${}_nC_r$ or $\binom{n}{r}$.

Theorem

The number of r-combinations of a set of n distinct objects is

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!} $$

Theroem I

Theorem

Suppose that a sequence S of n items has n_1 identical objects of type 1, n_2 identical objects of type 2,..., and n_t identical objects of type t. Then the number of orderings of s is

$$\frac{n!}{n_1!n_2!\dots n_t!}$$

Theroem II

Theorem

If X is a set containing t elements, the number of unordered, k-element selections from X, repititions allaowed, is

$$_{k+t-1}C_{t-1}$$

Binaomial Theorem

We can relate some formulas to counting methods, particularly the formula $(a+b)^n$ can be related to the r-combinations of n objects. The **Binomial theorem** gives a formula for the coefficients in the expansion of $(a+b)^n$

Theorem

If a and b are real numbers and n is a positive integer, then

$$(a+b)^n = \sum_{r}^{n} = 0_n C_r a^{n-r} b^r$$

Lexicographic Order

Lexicographic order generalises ordinary dictionary order. Given two distinct words, to determine whether one preceeds the other in the dictionary, we compare the letters of the words. There are two possibilities:

1. The words have different lengths, and each letter in the shorter word is identical to the corresponding letter in the longer word.
2. The words have the same or different lengths, and at some position, the letters in the words differ.

Definition

Let $\alpha = s_1 s_2 \ldots s_p$ and $\beta = t_1 t_2 \ldots t_q$ be strings over $[1, 2, \ldots, n]$. We say that α is **lexicographically less then** β and write $\alpha < \beta$ if either

$$p < q \text{ and } s_i = t_i \quad \text{for } i = 1, \ldots, p \quad \text{or}$$

$$\text{for some } i, s_i \neq t_i \text{ and for the smallest } i, \text{ we have } s_i < t_i$$

Outline

1. Combinatorics
   - Enumeration
   - Permutations
   - Combinations
   - Generalisations
   - Binaomial Theorem
   - Algorithms

2. Discrete Probability Theory
   - Probability
   - Conditional Probability
   - Bayes' Theroem

Introduction

Probability was developed in the seventeenth century to analyse games, and in its earliest form diectly involved counting. An **experiment** is a process that yields an outcome. An **event** is an outcome or combination of outcomes from an experiment. The **sample space** is the event consisting of all possible outcomes.

Example

Experiment: Rolling a six-sided die.

Event: Obtaining a 4 when rolling a six-sided die.

Sample space: The numbers $1, 2, 3, 4, 5, 6$; all possible outcomes when a die is rolled.

PROBABILITY

DEFINITION

The **probability** $P(E)$ of an event E from the finite sample space S is

$$P(E) = \frac{|E|}{|S|}$$

(where $|X|$ denotes the number of elements in a finite set X.)

University of Bedfordshire

Probability Function

In general, events are not equally likely. To handle the case of outcomes that are not equally likely, we assign a probability $P(x)$ to each outcome x. The values $P(x)$ need not be the same. We call P a **probability function**

Definition

A **probability function** P assigns to each outcome x in the sample space S a number $P(x)$ so that

$$0 \leq P(x) \leq 1 \qquad \text{for all } x \in S$$

and

$$\sum_{x \in S} P(x) = 1$$

The first of the two conditions guarantees that the probability of an outcome is non-negative, and at most 1. The second condition guarantees that the sum of the probabilities of all possible outcomes is exactly equal to 1.

Probability of an Event

The **probability of an event** E is defined as the sum of the probabilities of the outcomes in E.

Definition

Let E be an event. The probability of E, $P(E)$, is

$$P(E) = \sum_{x \in E} P(E)$$

Theorem

Let E be an event. The probability of $\bar{E}$, the compliment of E, satisfies

$$P(E) + P(\bar{E}) = 1$$

Theorem

Let E_1 and E_2 be events. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 1$$

Mutually exclusive events

Events E_1 and E_2 are **mutually exclusive** if $E_1 \cap E_2 = \varnothing$.

Theorem

If E_1 and E_2 are mutually exclusive events,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Conditional Probability

A probability gioven that some event has occurred is called **conditional probability**.

Definition

Let E and F be events, and assume that $P(F) > 0$. The **conditional probability** os E given F is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Independent Events

If the probability of event E does not depend on event F in the sense that $P(E \mid F) = P(E)$, we say that E and F are **independent events**.

Definition

Events E and F are **independent** if

$$P(E \cap F) = P(E)P(F)$$

Pattern Recognition

Pattern recognition places items into various classes based on features of the items. For example, wine might be placed into the classes *premium*, *table wine*, *paint-stripper* etc, based on features such as acidity and bouquet. One way to perform such a classification uses probability theory. Given a set of features F, one computes the probability of a class given F for each class and places the item into the most probable class.

Bayes' Theorem

Theorem

Suppose that the possible classes are $C_1, \ldots .C_n$. Suppose futher that each pair of classes is mutually exclusive and each item to be classified belongs to one of the classes. For a feature set F, we have

$$P(C_j \mid F) = \frac{P(F \mid C_j)P(C_j)}{\sum_{i=1}^{n} P(F \mid C_j)P(C_j)}$$