CIS009-2, MECHATRONICS
ROBOT COORDINATION SYSTEMS II

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Outline

1. Mapping
    - Translation
    - Rotational
2. Operators
    - Translation
    - Rotation
3. Transformation
    - multiplication
    - Inverting
    - transformation equations

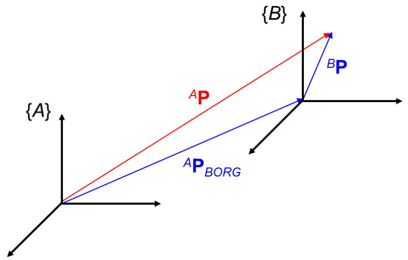
MAPPING

MAPPINGS

Mappings involving translated frames

- A translated frame shifts without rotation
- Translation takes place when ${}^{A}X \parallel {}^{B}X$, ${}^{A}Y \parallel {}^{B}Y$ and ${}^{A}Z \parallel {}^{B}Z$
- Mapping in this case means representing ${}^{B}P$ in $\{B\}$ in $\{A\}$ in the form of ${}^{A}P$

$$ {}^{A}P = {}^{B}P + {}^{A}P_{BORG} $$

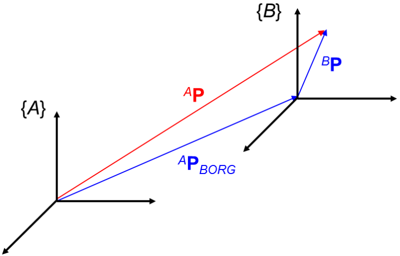


MAPPINGS

Mappings involving translated frames

- A translated frame shifts without rotation
- Translation takes place when $^AX \parallel {}^BX$, $^AY \parallel {}^BY$ and $^AZ \parallel {}^BZ$
- Mapping in this case means representing ^BP in $\{B\}$ in $\{A\}$ in the form of ^AP

$$^AP = {}^BP + {}^AP_{BORG}$$



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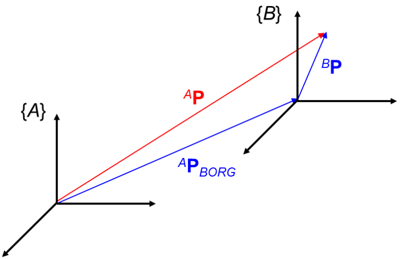

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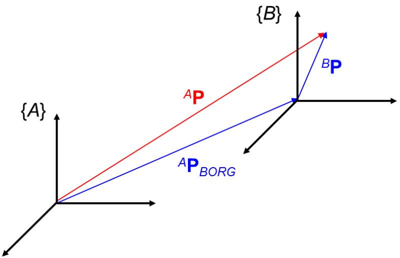
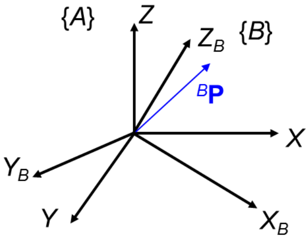


MAPPINGS

Mappings involving rotated frames

- Rotate a vector about an axis means the projection to that axis remains the same
- Mapping ^BP in $\{B\}$ to ^AP in $\{A\}$ is

$$^AP = {}^B_BR + {}^BP$$



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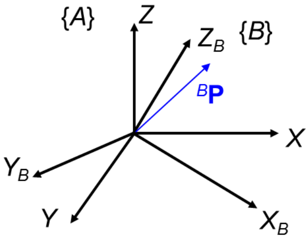

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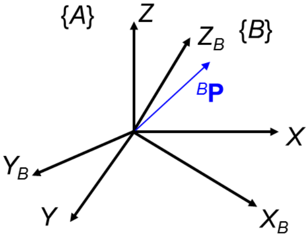
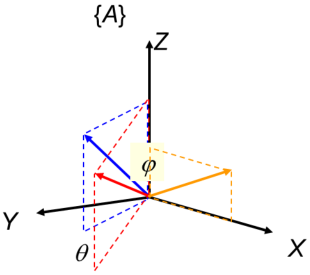


MAPPINGS

Example

- A vector ${}^{A}P$ is rotated about Z-axis by θ and is subsequently rotated about X-axis by ϕ. Give rotation matrix that accomplishes these rotations in the given order

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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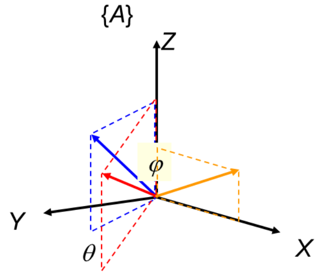


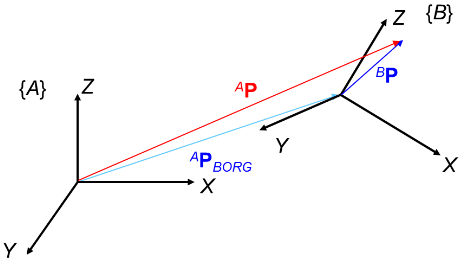
MAPPINGS

Mappings involving general frames

- These mappings involve both translation and rotation

$$^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$



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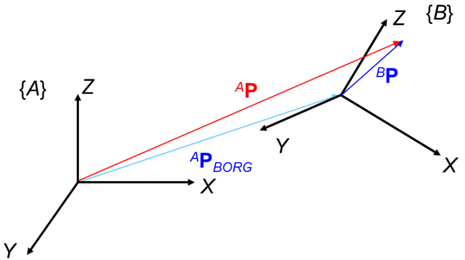
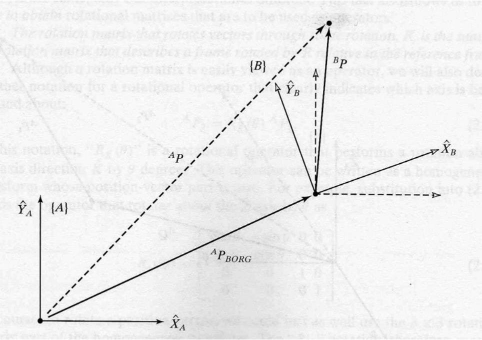


MAPPINGS

Example

- Given $\{A\}$, $\{B\}$, ${}^{B}P$ and ${}^{A}P_{BORG}$, calculate ${}^{A}P$, where $\{B\}$ is rotated relative to $\{A\}$ about Z_A-axis by 30 degrees, translated 10 units in X_A-axis and translated 5 units in Y_A-axis, and ${}^{B}P = \begin{bmatrix} 3.0 & 7.0 & 0.0 \end{bmatrix}$



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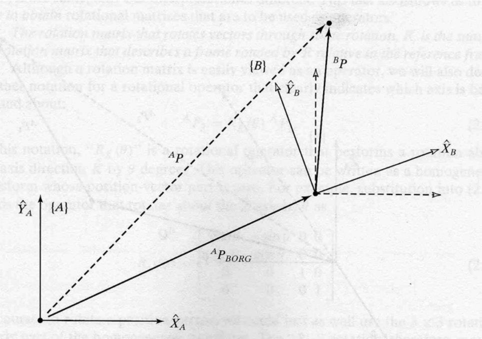

MAPPINGS

exercises

- Calculate rotation matrix

$$ {}^{A}_{B}\mathbf{R} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

- Calculate ${}^{A}P$

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P} + {}^{A}\mathbf{P}_{BORG} $$

$$ = \begin{bmatrix} {}^{A}_{B}\mathbf{R} & {}^{A}\mathbf{P}_{BORG} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\mathbf{P} \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} \cos 30 & -\sin 30 & 0 & 10 \\ \sin 30 & \cos 30 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} 3.0 \\ 7.0 \\ 0 \\ 1 \end{bmatrix} $$

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- Calculate ${}^A P$

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$$

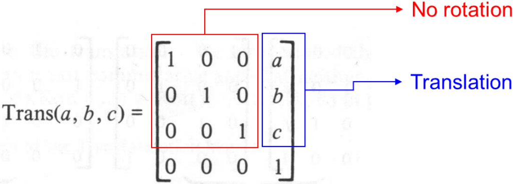
Operators

OPERATORS

- T matrix

$$\begin{bmatrix} {}^{A}_{B}\mathbf{R} & {}^{A}\mathbf{P}_{BORG} \\ \mathbf{0} & 1 \end{bmatrix} = {}^{A}_{B}\mathbf{T}$$

- Two special cases
  - Translation matrix of displacements a, b, and c along X, Y and Z axes



$$\mathrm{Trans}(a,b,c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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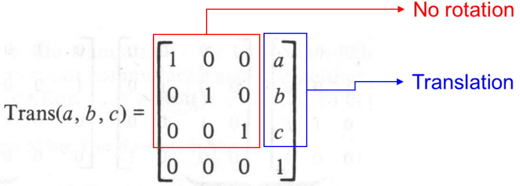

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No rotation

Translation

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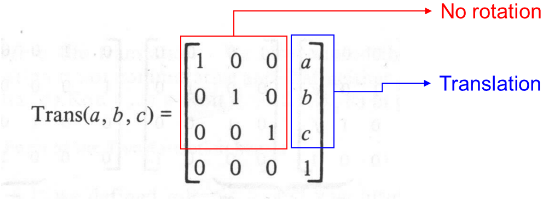

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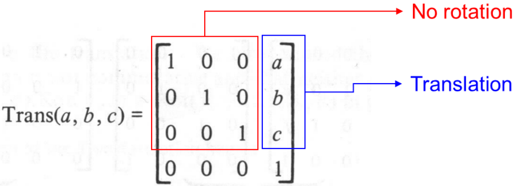

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No rotation

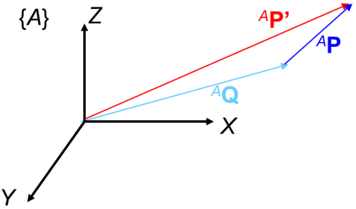
Translation

OPERATORS

$${}^{A}\mathbf{P}' = {}^{A}\mathbf{Q} + {}^{A}\mathbf{P}$$

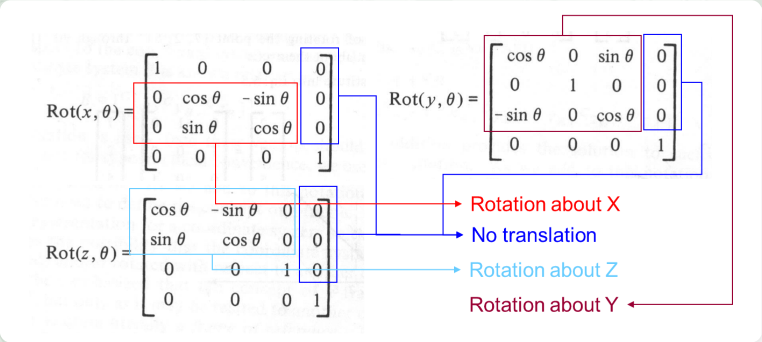
$$\begin{bmatrix} {}^{A}\mathbf{P}' \\ 1 \end{bmatrix} = Trans({}^{A}q_x, {}^{A}q_y, {}^{A}q_z) \begin{bmatrix} {}^{A}\mathbf{P} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & {}^{A}q_x \\ & 1 & & {}^{A}q_y \\ & & 1 & {}^{A}q_z \\ & & & 1 \end{bmatrix} \begin{bmatrix} {}^{A}\mathbf{P} \\ 1 \end{bmatrix}$$



OPERATORS

- Rotation
  - Rotation about X-axis
  - Rotation about Y-axis
  - Rotation about Z-axis



$$\mathrm{Rot}(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

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Rotation about X

No translation

Rotation about Z

Rotation about Y

OPERATORS

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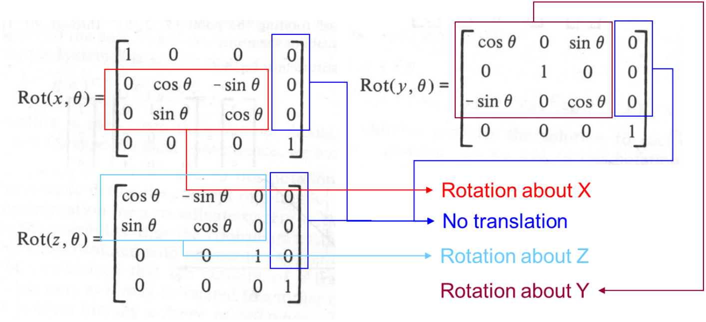

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Rotation about X

No translation

Rotation about Z

Rotation about Y

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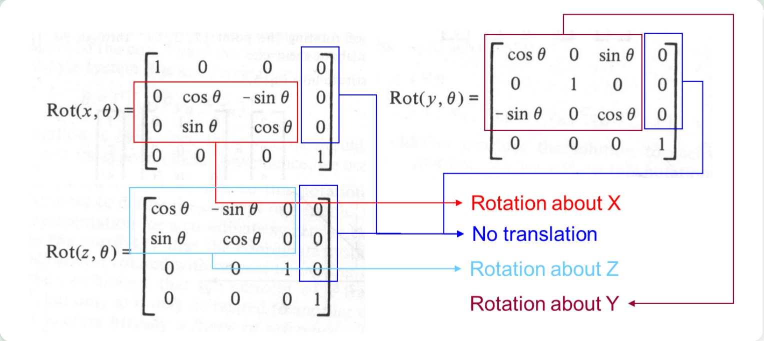

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Rotation about Y

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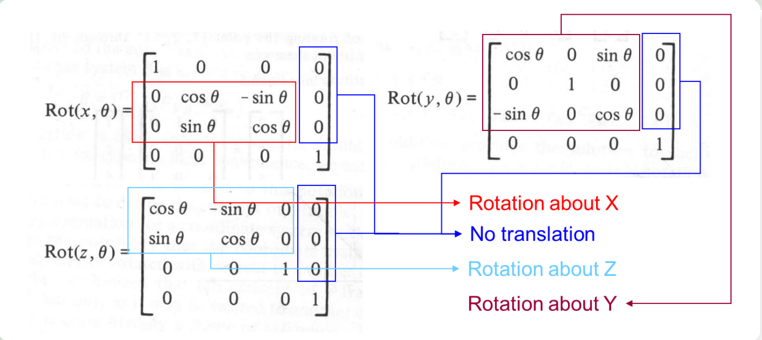

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No translation

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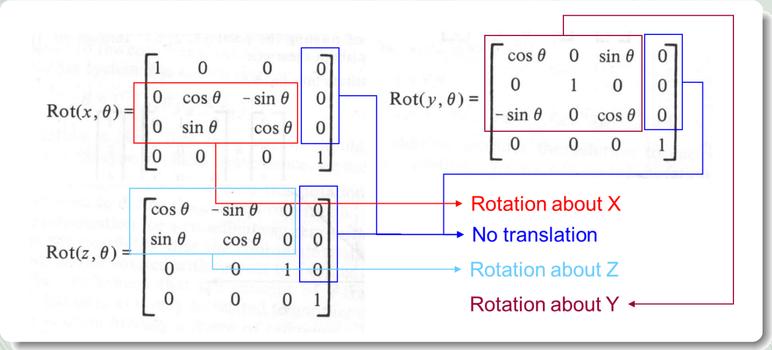

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Rotation about X

No translation

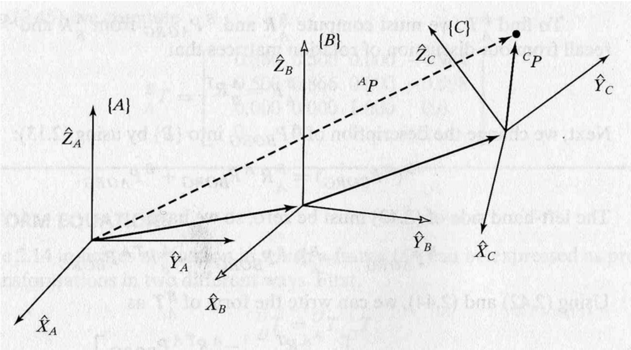
Rotation about Z

Rotation about Y

TRANSFORMATION

TRANSFORMATION

- Multiplication transformation
  - Known a point in $\{C\}$ and ${}^{B}_{C}T$ matrix from $\{C\}$ to $\{B\}$ and ${}^{A}_{B}T$ matrix from $\{B\}$ to $\{A\}$
  - Find its position and orientation in $\{A\}$



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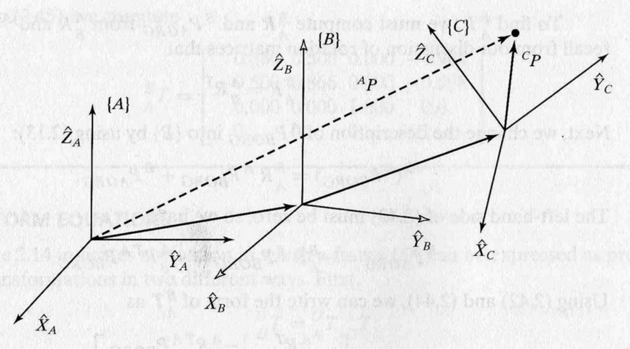

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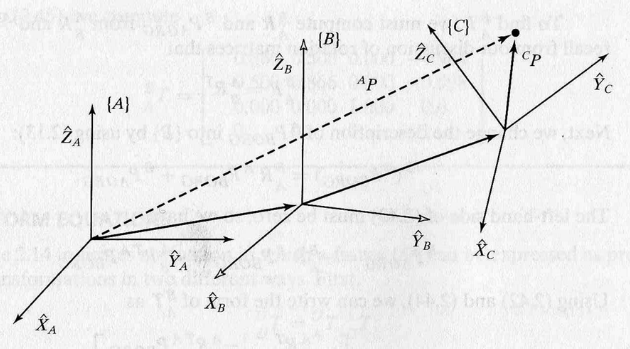

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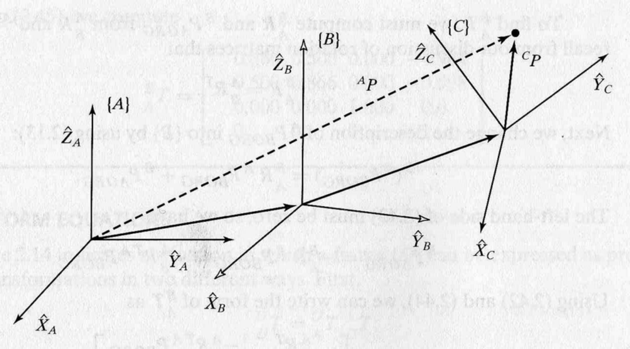


TRANSFORMATION

- ${}^{A}_{C}T$ matrix from $\{C\}$ to $\{A\}$ is the multiplication of ${}^{B}_{C}T$ matrix from $\{C\}$ to $\{B\}$ and ${}^{A}_{B}T$ matrix from $\{B\}$ to $\{A\}$

$${}^{B}\mathbf{P} = {}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \quad \text{and} \quad {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{T}\,{}^{B}\mathbf{P}$$

$${}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P}$$

$${}^{A}_{C}\mathbf{T} = {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}$$

$${}^{A}_{C}\mathbf{T} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{{}^{A}_{B}\mathbf{R}\,{}^{B}_{C}\mathbf{R}} & {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}_{CORG} + {}^{A}\mathbf{P}_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

TRANSFORMATION

- ${}^A_C T$ matrix from $\{C\}$ to $\{A\}$ is the multiplication of ${}^B_C T$ matrix from $\{C\}$ to $\{B\}$ and ${}^A_B T$ matrix from $\{B\}$ to $\{A\}$

$$ {}^B\mathbf{P} = {}^B_C\mathbf{T}\,{}^C\mathbf{P} \quad \text{and} \quad {}^A\mathbf{P} = {}^A_B\mathbf{T}\,{}^B\mathbf{P} $$

$$ {}^A\mathbf{P} = {}^A_B\mathbf{T}\,{}^B_C\mathbf{T}\,{}^C\mathbf{P} $$

$$ {}^A_C\mathbf{T} = {}^A_B\mathbf{T}\,{}^B_C\mathbf{T} $$

$$ {}^A_C\mathbf{T} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{{}^A_B\mathbf{R}\,{}^B_C\mathbf{R}} & {}^A_B\mathbf{R}\,{}^B\mathbf{P}_{CORG} + {}^A\mathbf{P}_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

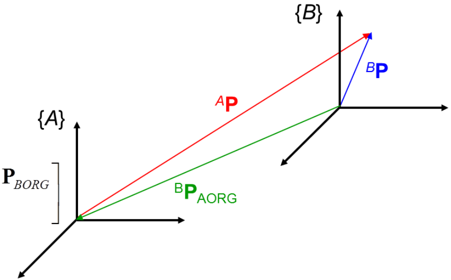
TRANSFORMATION

- Inverting a transformation
  - Known T matrix from $\{B\}$ to $\{A\}$
  - Find T matrix from $\{A\}$ to $\{B\}$

$$
\begin{aligned}
{}^{B}\mathbf{P} &= {}^{B}_{A}\mathbf{R}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG} \\
&= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG} \\
&= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} - {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG}
\end{aligned}
$$

$$
{}^{B}_{A}\mathbf{T} = \left[\begin{array}{cccc} \multicolumn{3}{c}{{}^{B}_{A}\mathbf{R}^{T}} & -{}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG} \\ 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
= {}^{A}_{B}\mathbf{T}^{-1}
$$



TRANSFORMATION

- Inverting a transformation
  - Known T matrix from $\{B\}$ to $\{A\}$
  - Find T matrix from $\{A\}$ to $\{B\}$

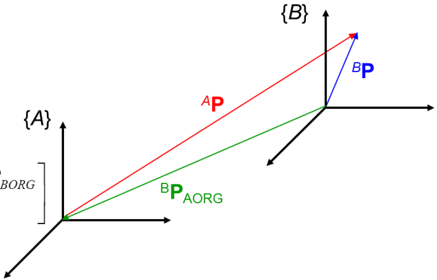
$$^{B}\mathbf{P} = {}^{B}_{A}\mathbf{R}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG}$$

$$= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG}$$

$$= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} - {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG}$$

$${}^{B}_{A}\mathbf{T} = \begin{bmatrix} & {}^{B}_{A}\mathbf{R}^{T} & & -{}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^{A}_{B}\mathbf{T}^{-1}$$



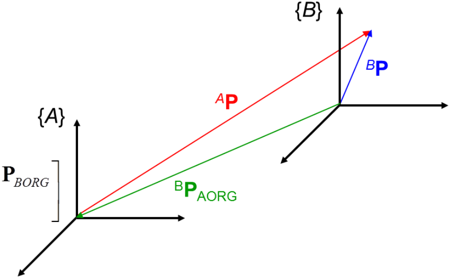
TRANSFORMATION

- Inverting a transformation
  - Known T matrix from $\{B\}$ to $\{A\}$
  - Find T matrix from $\{A\}$ to $\{B\}$

$$
\begin{aligned}
{}^{B}\mathbf{P} &= {}^{B}_{A}\mathbf{R}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG} \\
&= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG} \\
&= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} - {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG}
\end{aligned}
$$

$$
{}^{B}_{A}\mathbf{T} = \begin{bmatrix} & {}^{B}_{A}\mathbf{R}^{T} & & -{}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= {}^{A}_{B}\mathbf{T}^{-1}
$$



TRANSFORMATION

- Inverting a transformation
  - Known T matrix from $\{B\}$ to $\{A\}$
  - Find T matrix from $\{A\}$ to $\{B\}$

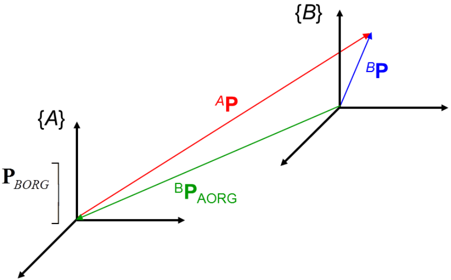
$$^{B}\mathbf{P} = {}^{B}_{A}\mathbf{R}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG}$$

$$= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} + {}^{B}\mathbf{P}_{AORG}$$

$$= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P} - {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG}$$

$$^{B}_{A}\mathbf{T} = \begin{bmatrix} & {}^{B}_{A}\mathbf{R}^{T} & & -{}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{P}_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^{A}_{B}\mathbf{T}^{-1}$$



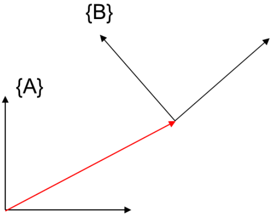
TRANSFORMATION

Example

- Frame $\{B\}$ is rotated relative to Frame $\{A\}$ about Z-axis by 30 degrees and translated 4 units in X-axis and 3 units in Y-axis. Find T matrix from $\{A\}$ to $\{B\}$.

$$
{}^{A}_{B}\mathbf{T} = \begin{bmatrix} 0.866 & -0.500 & 0 & 4 \\ 0.500 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{A}\mathbf{T} = {}^{A}_{B}\mathbf{T}^{-1} = \begin{bmatrix} 0.866 & 0.500 & 0 & -4.964 \\ -0.500 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



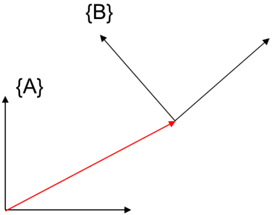
TRANSFORMATION

Example

- Frame $\{B\}$ is rotated relative to Frame $\{A\}$ about Z-axis by 30 degrees and translated 4 units in X-axis and 3 units in Y-axis. Find T matrix from $\{A\}$ to $\{B\}$.

$$ {}^{A}_{B}\mathbf{T} = \begin{bmatrix} 0.866 & -0.500 & 0 & 4 \\ 0.500 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}\mathbf{T} = {}^{A}_{B}\mathbf{T}^{-1} = \begin{bmatrix} 0.866 & 0.500 & 0 & -4.964 \\ -0.500 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



TRANSFORMATION

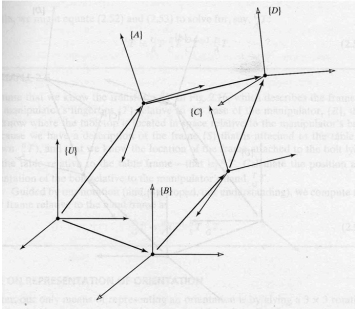
Transform equations

- Transform equation

$$\because\ {}^{U}_{D}\mathbf{T} = {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}$$

$$\because\ {}^{U}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$

$$\therefore\ {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



- Any unknown T matrix can then be calculated from the ones given

$${}^{U}_{B}\mathbf{T} = {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}\,{}^{C}_{D}\mathbf{T}^{-1}\,{}^{B}_{C}\mathbf{T}^{-1}$$

TRANSFORMATION

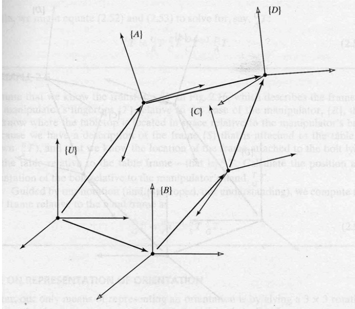
Transform equations

- Transform equation

$$\because \; {}^U_D\mathbf{T} = {}^U_A\mathbf{T}\,{}^A_D\mathbf{T}$$

$$\because \; {}^U_D\mathbf{T} = {}^U_B\mathbf{T}\,{}^B_C\mathbf{T}\,{}^C_D\mathbf{T}$$

$$\therefore \; {}^U_A\mathbf{T}\,{}^A_D\mathbf{T} = {}^U_B\mathbf{T}\,{}^B_C\mathbf{T}\,{}^C_D\mathbf{T}$$



- Any unknown T matrix can then be calculated from the ones given

$${}^U_B\mathbf{T} = {}^U_A\mathbf{T}\,{}^A_D\mathbf{T}\,{}^C_D\mathbf{T}^{-1}\,{}^B_C\mathbf{T}^{-1}$$

TRANSFORMATION

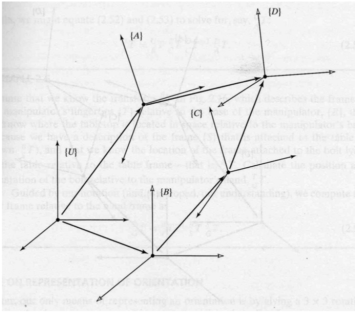
Transform equations

- Transform equation

$$\because \; {}^{U}_{D}\mathbf{T} = {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}$$

$$\because \; {}^{U}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$

$$\therefore \; {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



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TRANSFORMATION

Transform equations

- Transform equation

$$\because \; {}^{U}_{D}\mathbf{T} = {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}$$

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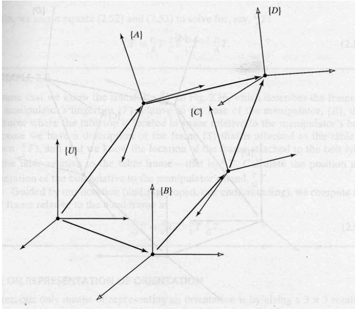
$$\therefore \; {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



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TRANSFORMATION

Transform equations

- Transform equation

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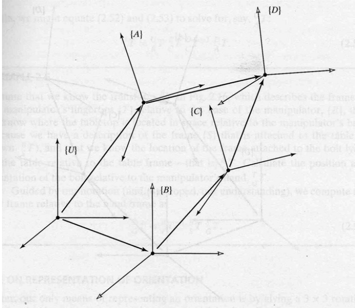
$$\therefore \; {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



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$${}^{U}_{B}\mathbf{T} = {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}\,{}^{C}_{D}\mathbf{T}^{-1}\,{}^{B}_{C}\mathbf{T}^{-1}$$

TRANSFORMATION

Transform equations

- Transform equation

$$\because\ {}^{U}_{D}\mathbf{T} = {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}$$

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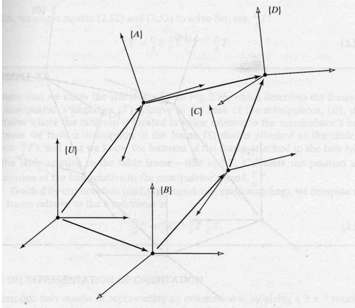
$$\therefore\ {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T} = {}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



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TRANSFORMATION

Transform equations

- Transform equation

$$\because\ {}^{U}_{D}\mathbf{T}={}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}$$

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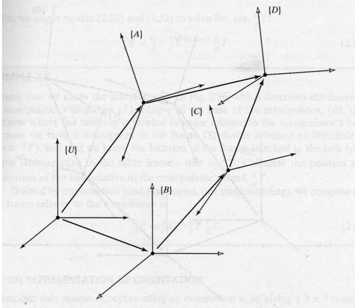
$$\therefore\ {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}={}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



- Any unknown T matrix can then be calculated from the ones given

$${}^{U}_{B}\mathbf{T}={}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}\,{}^{C}_{D}\mathbf{T}^{-1}\,{}^{B}_{C}\mathbf{T}^{-1}$$

TRANSFORMATION

Transform equations

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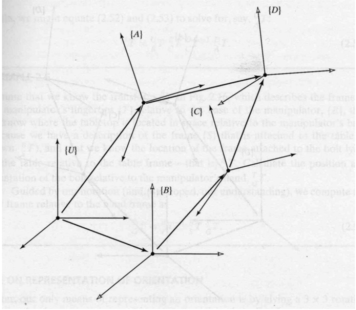
$$\therefore \; {}^{U}_{A}\mathbf{T}\,{}^{A}_{D}\mathbf{T}={}^{U}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}_{D}\mathbf{T}$$



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Department of Computer Science and Technology University of Bedfordshire