CIS009-2, Mechatronics Control Systems & Robotics

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Outline

State-Space Controller

State-Space Controller

- Modern control theory solves many of the limitations by using a much "richer" description of the plant dynamics. The so-called state-space description provide the dynamics as a set of coupled first-order differential equations in a set of internal variables known as state variables, together with a set of algebraic equations that combine the state variables into physical output variables.

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System State

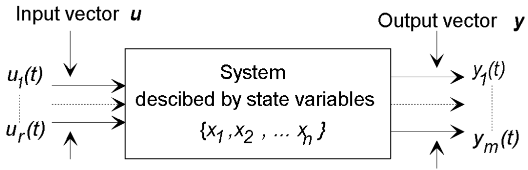
Definition

- a state-determined system model has the characteristic that:
  - A mathematical description of the system in terms of a minimum set of variables $x_i(t), i = 1, \ldots, n$, together with knowledge of those variables at an initial time t_0 and the system inputs for time t . t_0, are sufficient to predict the future system state and outputs for all time $t > t_0$.
- This definition asserts that the dynamic behavior of a state-determined system is completely characterized by the response of the set of n variables $x_i(t)$, where the number n is defined to be the order of the system.

Mathematical description

Vector form of state equations

- It is common to express the state equations in a vector form:



- set of n state variables is written as a state vector

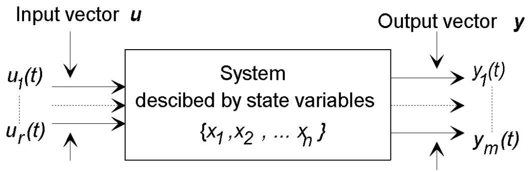
$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

- set of r inputs is written as an input vector

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

Mathematical description

Vector form of state equations



- For a system of order n, and with r inputs, a get set of n coupled differential equations with constant coefficients

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1r}u_r \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2r}u_r \\
\vdots &= \qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nr}u_r
\end{aligned}
$$

- coefficients a_{ij} and b_{ij} are constants describing the system.

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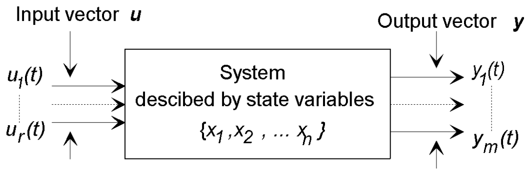
Mathematical description

Matrix form of state equations



- We can write these coupled differential equations in a compact matrix form:

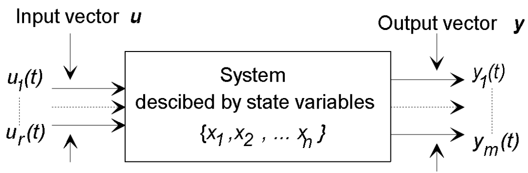
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{21} & b_{22} & \ldots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nr} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

- which may be summarised as:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Mathematical description

Matrix form of state equations



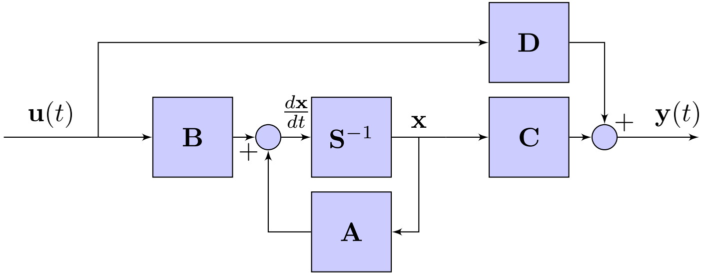
- An important property of the linear state equation description is that all system variables may be represented by a linear combination of the state variables x_i and the system inputs u_i

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1r} \\ d_{21} & d_{22} & \ldots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \ldots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

- which may be summarised as:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

State Space
Block Diagram



- This general block diagram shows the matrix operations from input to output in terms of the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ matrices.
- $\mathbf{S}^{-1}$ is the integrator block (system of order n has n integrators)

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Servomechanism Controller

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- A servo control is one of the most important and widely used forms of control systems.
- Any machine or piece of equipment that has rotating parts will contain one or more servo control systems
  - Maintaining the speed of a motor within certain limits (even when the load varies). This is called regulation
  - Vary the speed of a motor according to an external programme. This is called set point, or reference tracking)

Servomechanism Controller

- Inertial load, J.
- Friction in motor, b.
- Input voltage, $u(t)$.
- Torque, $T(t)$.
- Angular possition of the servo output shaft, x, Angular velocity x', and Angular acceleration x''.
- $u(t)$ is related to $T(t)$ throught the gain, K and the inertia divided by the friction.

$$Jx'' + bx' = T(t)$$

$$\frac{J}{b}x'' + x' = Ku(t)$$

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Servomechanism Controller

- Linear part of the servo system can be put in the transfer function:

$$y(s) = \frac{K}{s(\frac{J}{b}s + 1)}u(s)$$

- where $y(s)$ is the output shaft position and $u(s)$ is the motor input.

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Servomechanism Controller

- This can be decomposed into a transfer function from the motor input to the motor speed $v(s)$, and a transfer function from the motor speed to the output shaft position $y(s)$.

$$v(s) = \frac{1}{(\frac{J}{b}s + 1)} u(s)$$

$$y(s) = \frac{K}{s} v(s)$$

- state space form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & K \\ 0 & \frac{-b}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{J} \end{bmatrix} u$$

$$\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Resources

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Resources

- Sontag, Eduardo D. (1999). Mathematical Control Theory: Deterministic Finite Dimensional Systems (2nd ed.). http://www.math.rutgers.edu/~sontag/FTP_DIR/sontag_mathematical_control_theory_springer98.pdf
- Rowell, D (2002). State-Space Representation of LTI Systems. http://web.mit.edu/2.14/www/Handouts/StateSpace.pdf
- Hellerstein, J (2008). State-Space Models for LTI Systems. http://research.microsoft.com/en-us/um/people/liuj/cse590k2008winter/ct-lecture4.pdf
- http://www.uotechnology.edu.iq/dep-cse/lectures/4/computer/interface/ce110-servo%20trainer%203.pdf