QCD is a key part of the Standard Model but quark confinement complicates things.

QCD only tested to 5-10% level at high energies from comparison of e.g. jet phenomena to pert.th.

But properties of hadrons calculable from QCD if fully nonperturbative calc. is done - can test QCD and determine parameters very accurately (1%).
Rates for simple weak or em quark processes inside hadrons also calculable, but not multi-hadron final states.

\[ \bar{B}_s \rightarrow D_s e^- \nu \]

\[ (D_s \rightarrow K^+ K^- \pi^+) \]

Compare to exptl rate gives \( V_{qq'} \) accurately.
Solving a path integral: quantum mechanical case

Solve Schrödinger’s eq. for eigenvalues/fns of H or:

- Discretise time and integrate over all paths possible weighted by $e^{iS}$
- Classical path is $m\ddot{x} = V'$
- QM path fluctuates about this.

In Euclidean time solve numerically, by making sets of $x(t_i)$

$$< x(t_2)x(t_1) > = \frac{\int Dx\, x(t_2)x(t_1)e^{-S}}{\int Dxe^{-S}} = \sum_n A_n e^{-(E_n - E_0)(t_2 - t_1)}$$

Fit as fn of time can extract excitation energies - amps are transition matrix elements

Further reading: G.P.Lepage, hep-lat/0506036
Solving a path integral: QCD

Now path integral over gluon and quark fields on a 4-d space-time lattice - quarks anticommute so do by hand.

\[ \mathcal{L}_{QCD} = \frac{1}{2} Tr F_{\mu\nu}^2 + \bar{\psi} (\gamma \cdot D + m) \psi \]

\[ \int \mathcal{D}U \mathcal{D}D \mathcal{D}\bar{\psi} O(\psi, \bar{\psi}) e^{-S_{QCD}} \rightarrow \]

Integral over gluon fields only

\[ \int \mathcal{D}U \mathcal{D}O (M^{-1}) e^{-(S_g - \ln(\det M))} \]

valence quarks inc. in operator

complicated prob, distn for gluons - inc. effects of sea quarks

ensemble average

\[ \langle O \rangle = \langle H(t) H^\dagger(0) \rangle = \sum_n A_n e^{-E_n t} \]

Fit as fn of t to get hadron mass and transn amp.

Thursday, 17 March 2011
Lattice QCD = fully nonperturbative QCD calculation

RECIPE

• Generate sets of gluon fields for Monte Carlo integrn of Path Integral (inc effect of u, d and s sea quarks)
• Calculate averaged “hadron correlators” from valence q props.
• Fit for masses and simple matrix elements
• Determine $a$ and fix $m_q$ to get results in physical units.
• extrapolate to $a = 0, m_{u,d} = \text{phys}$ for real world
Lattice results need to be extrapolated to the real world where $a=0$ and $m_{u/d} = \text{small}$.

To do this well needs:
- statistical precision
- small disc. errors and several values of $a$
- small $m_{u/d}$

Using HISQ charm quarks

E. Follana, CTHD et al, HPQCD, 0706.1726
Including u, d and s sea quarks is critical for accurate results, but numerically expensive - particularly light $m_{u,d}$.

**Latt./expt**

Quenched

- $f_\pi$
- $f_K$
- $m_\Omega$
- $m_N$
- $m_{D_s}$
- $m_D$
- $m_D^* - m_{D_s}$
- $m_\psi - m_{\eta_c}$
- $\psi(1P-1S)$
- $2m_{B_{s,av}} - m_Y$
- $m_{B_c}$
- $Y(3S-1S)$
- $Y(2P-1S)$
- $Y(1P-1S)$
- $Y(1D-1S)$

**Latt./expt**

with sea quarks

Multiple values of $a$, and of $m_{u,d}$. Extrapolate to physical point.

**HPQCD/MILC 2008 “ratio plot”**.

Update of:
C. Davies et al, hep-lat/0304004

Thursday, 17 March 2011
Example parameters for calculations now being done. Lots of different formalisms for handling quarks.

Volume of lattice also an issue - need $(2.5 \text{fm})^4$ or more

$m_{u,d} \approx m_s/5$

$m_{u,d} \approx m_s/10$

$m_{u,d} \approx m_s/27$

$m_{\tau, \text{min}} / \text{GeV}^2$

$m_{u,d} \approx m_s/10$

$m_{u,d} \approx m_s/27$

$m_{u,d} \approx m_s/5$

$m_{u,d} \approx m_s/10$

$m_{u,d} \approx m_s/27$

$m_{u,d} \approx m_s/5$

$m_{u,d} \approx m_s/10$

$m_{u,d} \approx m_s/27$

$m_{u,d} \approx m_s/5$

$m_{u,d} \approx m_s/10$

$m_{u,d} \approx m_s/27$

$m_{u,d} \approx m_s/5$

$m_{u,d} \approx m_s/10$

$m_{u,d} \approx m_s/27$
The gold-plated meson spectrum - HPQCD 2009

I. Allison et al, hep-lat/0411027, A. Gray et al, hep-lat/0507013
Determining quark masses

Lattice QCD has direct access to parameters in Lagrangian for accurate tuning
- issue is converting to continuum schemes such as $\overline{MS}$

Can now rule out some quark mass matrix models

Can now rule out some quark mass matrix models

$V_{us}$

$V_{cb}$

$V_{ub}$

$V_{cb} = \sqrt{m_d/m_b}$ vs PDG

C. McNeile, 1004.4985, model from Chkareuli+Froggatt, hep-ph/9812499

C. McNeile, CTHD, HPQCD, 1004.4285

$\begin{align*}
\overline{m}_b(\overline{m}_b) &= 4.165(23) \text{GeV} \\
\overline{m}_c(\overline{m}_c) &= 1.273(6) \text{GeV} \\
\overline{m}_s(2 \text{GeV}) &= 92.2(1.3) \text{MeV} \\
\overline{m}_d(2 \text{GeV}) &= 4.77(15) \text{MeV} \\
\overline{m}_u(2 \text{GeV}) &= 2.01(10) \text{MeV}
\end{align*}$
Strong convergence of lattice results for strange quark mass this year.

Lattice world average = $93.6(1.1)$ MeV
Determining $\alpha_s$

Lattice QCD now has several determinations of $\alpha_s$ to 1%.

Key points:
- high statistical precision
- high order pert. th. exists and can estimate higher orders
- higher twist not a significant issue
- approaches very different - good test

see 2011 Munich alphas workshop

Y decays
$\tau$ decays
DIS [$F_2$]
DIS [e,p -> jets]
e$^+e^-$[jets shps]
e$^+e^-$[jets shps]
HPQCD: wloops
HPQCD: heavy q corrs
JLQCD: light q. vac. poln
World average:
Bethke 0908.1135

CTHD et al, HPQCD 0807.1687; 1004.4285; JLQCD, 1002.0371.
Determining the Cabibbo-Kobayashi-Maskawa matrix

\[
\begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
\pi \to l\nu & K \to l\nu & B \to \pi l\nu \\
K \to \pi l\nu \\
V_{cd} & V_{cs} & V_{cb} \\
D \to l\nu & D_s \to l\nu & B \to D l\nu \\
D \to \pi l\nu & D \to Kl\nu \\
V_{td} & V_{ts} & V_{tb} \\
\langle B_d | \bar{B}_d \rangle & \langle B_s | \bar{B}_s \rangle
\end{pmatrix}
\]

If \( V_{ab} \) known, compare lattice to experiment to test QCD.

\( V_{ab} \) appears in trivial way in decay processes involving quarks \( a + b \). Calculating QCD effects is non-trivial - need precision lattice QCD to get accurate CKM results.

If \( V_{ab} \) known, compare lattice to experiment to test QCD.
Recent CKM highlights

K decay constant parameterises QCD effects in leptonic decay.- calculate in lattice QCD. Combine with expt.(KLOE)

\[ V_{us} = 0.2262(14) \] (current world’s best but theory still dominates error)

\[ 1 - V_{ud}^2 - V_{us}^2 - V_{ub}^2 = 0.0006(8) \] test of first row unitarity of CKM matrix

Follana, CTHD et al HPQCD, 0706.1726
HISQ action allows us to study $c$ meson decays also

\[ \mathcal{f}_{D_s} = 0.2480(25) \text{GeV} \]

1\% error!

update from 2007 value of 0.241(3) GeV because of recalibration of lattice spacing

Follana, CTHD et al, HPQCD, hep-lat/0610092, 0706.1726; CTHD et al, HPQCD, 1008.4018
New physics would give smaller decay constant than Standard Model - allows limits on charged Higgs.
Semileptonic form factors

\[ \frac{d\Gamma}{dq^2} \propto |V_{cs}|^2 f_+(q^2) \]

\[ D \to Kl\nu \]

\[ f_0(q^2) = 0.747(11)(15) \]

\[ f_+(q^2) = 2.5\% !! \]

With HISQ quarks, no renormalisation issues and good statistics possible.

\[ f_0(q^2 = 0) = 0.747(11)(15) \]

\[ f_+(q^2 = 0) = 2.5\% !! \]

\[ \Gamma dq^2 \propto |V_{cs}|^2 \]

\[ D \to Kl\nu \]

\[ V_{qq'} \]

CKM element

Na, Shigemitsu et al., HPQCD, 1008.4562
Comparison to experimental leptonic and semileptonic rates allows direct determination of $V_{cs}$

0.961(26) error dominated by theory (now 2.5%)

1.010(22) error dominated by expt (now 2%)

Aim: similar accuracy for CKM elements from B meson decay and mixing rates

Thursday, 17 March 2011
New work: calculating shape of $f_+(q^2)$

J. Koponen, CTHD, in progress

Comparison to expt will provide more detailed test of QCD. Note how form factor same for different processes all involving $c \rightarrow s$ decay.
New work: mapping shape of decay constants and form factors as a function of heavy quark mass from charm upwards - needs very fine lattices...

McNeile, CTHD, in progress

\[
a = 0.15 \, fm \quad a = 0.09 \, fm \\
0.12 \, fm \quad a = 0.06 \, fm \\
0.09 \, fm \quad a = 0.045 \, fm
\]
New physics sensitivity in neutral B mixing

Neutral B (and K) mix - gives rise to ‘oscillations’. Mixing determined by box diagram. Calculate in lattice QCD

\[ B^0 \quad \overline{B^0} = \]

Parameterise with \( f_B^2 B_B \) where \( f_B \) is decay constant.

Using ratio for \( B_s \) to \( B_d \)

\[ \xi = \frac{f_{B_s} \sqrt{B_{B_s}}}{f_B \sqrt{B_B}} \]

\[ \left| \frac{V_{td}}{V_{ts}} \right| = \xi \sqrt{\frac{\Delta M_d M_{B_s}}{\Delta M_s M_{B_d}}} \]

Lattice QCD results (HPQCD)

using NRQCD b quarks:

\[ \xi = 1.26(3) \]

7% normal error cancels in ratio.

E. Gamiz, CTHD et al, HPQCD, 0902.1815
Use this to provide SM rate for LHCb of:

\[ Br(B_s \rightarrow \mu^+ \mu^-) = 3.19(19) \times 10^{-9} \]

6% error from taking ratio to mixing rate

E. Gamiz et al, 0902.1815[hep-lat]

New LHCb limit

LanFranchi, LaThuile 2011

Now working to improve lattice QCD result ..
Lattice QCD calculations are key to constraining sides of CKM Unitarity triangle

Future with 1% lattice errors for 
$B_K$, $f_K$  
$\xi$, $f_{B_s} \sqrt{B_{B_s}}$  
$V_{ub}$, $V_{cb}$ from exclusive SL decay

J. Laiho et al, 0910.2928; www.latticeaverages.org
### Error Budget

<table>
<thead>
<tr>
<th>Source</th>
<th>$f_K/f_π$</th>
<th>$f_K$</th>
<th>$f_π$</th>
<th>$f_{D_s}/f_D$</th>
<th>$f_{D_s}$</th>
<th>$f_D$</th>
<th>$Δ_s/Δ_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$ uncertainty</td>
<td>0.3</td>
<td>1.1</td>
<td>1.4</td>
<td>0.4</td>
<td>1.0</td>
<td>1.4</td>
<td>0.7</td>
</tr>
<tr>
<td>$a^2$ extrapol.</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>Finite vol.</td>
<td>0.4</td>
<td>0.4</td>
<td>0.8</td>
<td>0.3</td>
<td>0.1</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>$m_{u/d}$ extrapol.</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>Stat. errors</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>$m_s$ evoln.</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>$m_d$, QED, etc.</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>Total %</td>
<td>0.6</td>
<td>1.3</td>
<td>1.7</td>
<td>0.9</td>
<td>1.3</td>
<td>1.8</td>
<td>1.2</td>
</tr>
</tbody>
</table>

The difference in binding energies for a heavy-heavy state (e.g., $D^0 \bar{D}^0$) and a heavy-light state (e.g., $D^0 \bar{K}^0$) is given by $Δ_q = 2m_{D_q} - m_{η_c}$.

### Summary

This table shows the error budget for various sources of uncertainty in our decay constant calculations. The total percentage error ranges from 0.5% to 1.8%, with some sources contributing more than others. This will tell you what is possible in future calculations, especially regarding discretization errors, heavy-light meson extrapolations, and statistical uncertainties.
Conclusion

• very accurate results are available now from lattice QCD for QCD parameters and for simple hadron masses and decay matrix elements important for flavour physics.

Future

• sets of ‘next generation’ gluon configs will have $m_{u,d}$ at physical value (so no extrapoln) or down to 0.03fm (so b quarks are ‘light’) or much higher statistics (for harder hadrons) also can include charm in the sea now.

• Pushing errors down to 1% level will mean em corrections and $m_u \neq m_d$ must be understood.

• some harder calculations (flavor singlet, excited states, nuclear physics) will also become possible.