AEA 2006 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. (a) In general, the binomial series expansion of $(1+x)^\alpha$ is given by

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots .$$

(The sum converges to a finite limit for any $|x| < 1$ and $\alpha \in \mathbb{R}$.) Thus, setting $x = -y$ and $\alpha = -2$, we obtain

$$(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \dots . \tag{1}$$

(When answering a question like this, remember to be careful about checking the signs of the terms are correct!)

(b) The key to this part of the question is in noticing that the numerical coefficients are increasing linearly, i.e. the first is 1, the second is 2, and so on. These match the coefficients of the binomial expansion for $(1-y)^{-2}$ obtained in part (a). In particular, if we set $y = x/(1+x)$, then we find that

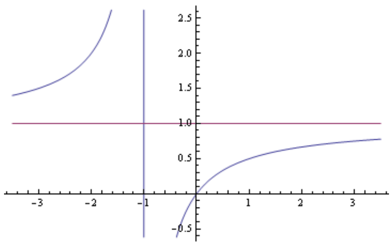
$$\begin{aligned} & 1 + \frac{2x}{1+x} + \frac{3x^2}{(1+x)^2} + \dots + \frac{rx^{r-1}}{(1+x)^{r-1}} + \dots \\ = \; & 1 + 2y + 3y^2 + 4y^3 + \dots \\ = \; & (1-y)^{-2} \\ = \; & \left(1 - \frac{x}{1+x}\right)^{-2} \\ = \; & (1+x)^2. \end{aligned}$$

Hence $a = 1$ and $n = 2$.

(c) Since the series in y at (1) converges if and only if $|y| < 1$, the series in x converges if and only if

$$\left|\frac{x}{1+x}\right| < 1.$$

A quick sketch of the function $x/(1+x) = 1 - 1/(1+x)$ shows what it looks like.



From this, it is clear that $|x/(1+x)| < 1$ if and only if $x > x_C$, where x_C satisfies $x_C/(1+x_C) = -1$. This is easily solved to give $x_C = -\frac{1}{2}$. In conclusion, the series in part (b) is convergent if and only if $x > -\frac{1}{2}$.

2. The first thing we observe about the given equation is that both sides contain the term $\sin 2\theta - \sqrt{3}\cos 2\theta$. This means that, after we move all the terms to the same side, we can factorise it:

$$\left(\sin 2\theta - \sqrt{3}\cos 2\theta\right)\left(\frac{2\cos 2\theta}{\sin\theta + \cos\theta} - \sqrt{6}\right) = 0.$$

This makes things much easier, because we now just need to check when either of the two terms is equal to zero.

For the first term, we need to solve

$$\sin 2\theta - \sqrt{3}\cos 2\theta = 0.$$

This is equivalent to $\tan 2\theta = \sqrt{3}$, which has solutions $2\theta = 60^\circ + n \times 180^\circ$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$. In particular, the solutions that satisfy $\theta \in [0^\circ, 360^\circ)$ are $\theta = 30^\circ, 120^\circ, 210^\circ, 300^\circ$.

For the second term, we need to solve

$$\frac{2\cos 2\theta}{\sin\theta + \cos\theta} - \sqrt{6} = 0.$$

So that we are left with only terms involving θ rather than 2θ, we will apply the double-angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$, to obtain

$$\frac{2(\cos^2\theta - \sin^2\theta)}{\sin\theta + \cos\theta} - \sqrt{6} = 0.$$

Now, since $\cos^2\theta - \sin^2\theta = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$ (and $\sin\theta + \cos\theta \neq 0$), we can simplify this to

$$2(\cos\theta - \sin\theta) = \sqrt{6}.$$

To reduce this, we will apply a second trigonometric identity,

$$\cos(\theta + 45^\circ) = \sin 45^\circ \cos\theta - \cos 45^\circ \sin\theta = \frac{1}{\sqrt{2}}\left(\cos\theta - \sin\theta\right).$$

This implies that

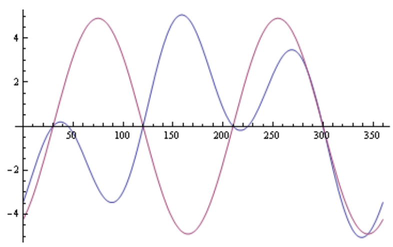
$$\cos(\theta + 45^\circ) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2},$$

which has solutions $\theta + 45^\circ = \ldots, 30^\circ, 330^\circ, 390^\circ, \ldots$, and so $\theta = 285^\circ, 345^\circ$ are the solutions in $[0^\circ, 360^\circ)$.

In conclusion, in the range $[0^\circ, 360^\circ)$, the equation is solved by

$$\theta = 30^\circ, 120^\circ, 210^\circ, 285^\circ, 300^\circ, 345^\circ.$$

Finally, it is worth noting that in questions like this involving trigonometric functions examiners are not usually so mean as to require you to go through extremely complicated algebra, and often what looks like a very awkward equation can be made simpler by a few clever manipulations. Although there is no way to know what these are in advance, it is typically a good idea to think about whether you can apply some of your background knowledge, such as double-angled formulae or addition formulae, in a useful way. Indeed, the above computations were not especially difficult, but enabled us to solve what started out as a seemingly-tricky problem of showing where the left-hand side of the given equation (shown in blue on the following figure) and right-hand side (shown in purple) are equal.



3. (a) By definition, the number $z = \log_y x$ solves $y^z = x$. Raising this equation to the power of $1/z$, we obtain $y = x^{1/z}$. This implies that

$$\log_x y = \frac{1}{z} = \frac{1}{\log_y x}.$$

(b) Combining $\log_x y = \log_y x$ with the conclusion of part (a), we obtain

$$\log_y x = \frac{1}{\log_y x},$$

which is equivalent to

$$(\log_y x)^2 = 1.$$

This is solved by $\log_y x = \pm 1$. If $\log_y x = 1$, then it holds that $x = y^1 = y$. However, we are told that $x > y$, and so this can not be the correct solution. Thus, it must the case that $x = y^{-1}$, which is of course the same as $y = x^{-1}$.

(c) From part (b), we already know that the first equation implies that $y = x^{-1}$. Substituting this into the second equation yields

$$\log_x \left(x - \frac{1}{x} \right) = \log_{1/x} \left(x + \frac{1}{x} \right). \tag{2}$$

To solve this equation, let us again apply the definition of a logarithm to deduce that

$$x^z = x - \frac{1}{x}, \qquad \left(\frac{1}{x} \right)^z = x + \frac{1}{x},$$

where z is defined to be equal to either side of (2). Multiplying these two equations together, we find that

$$\left(x - \frac{1}{x} \right) \left(x + \frac{1}{x} \right) = x^z \left(\frac{1}{x} \right)^z = 1,$$

and so

$$x^2 - \frac{1}{x^2} = 1.$$

Multiplying by x^2, this yields that $x^4 - x^2 - 1 = 0$, which is merely a quadratic equation in x^2. Solving this implies that

$$x^2 = \frac{1 + \sqrt{5}}{2},$$

where we note that we can ignore the root $(1 - \sqrt{5})/2$, because it is negative whereas x^2 is clearly positive. Since x is also positive, it follows that

$$x = \sqrt{\frac{1 + \sqrt{5}}{2}},$$

and also, recalling $y = x^{-1}$,

$$y = \sqrt{\frac{2}{1 + \sqrt{5}}}.$$

Note that, unlike the previous question, none of the steps here required a 'trick'; the key to getting to the end was simply a good grasp of the definition of a logarithm.

4. (a) If the line $y = mx$ is a tangent to the circle C_1, then there is exactly one point (x, y) that is on both the line and the circle. This point must satisfy both $y = mx$ and the equation of the circle C_1 that is given in the question. Substituting the first equation into the second and rearranging, we obtain that

$$(1 + m^2)x^2 + (8 - 14m)x + 52 = 0. \tag{3}$$

Now, because there is exactly one point (x, y) that is on both the line and the circle, it must also hold that this quadratic equation has exactly one solution. Hence, its discriminant "$b^2 - 4ac$" must be equal to zero, i.e.

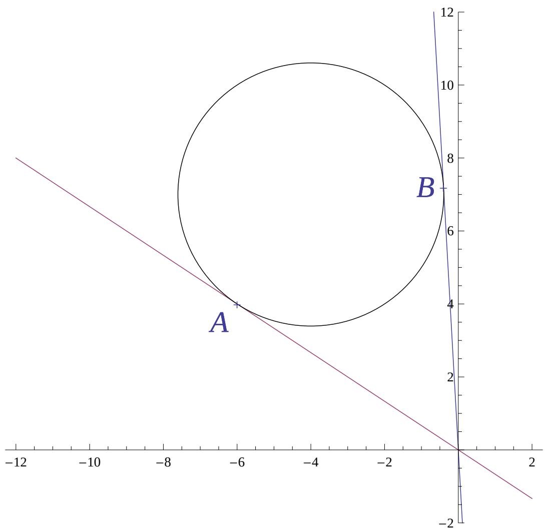
$$(8 - 14m)^2 - 208(1 + m^2) = 0.$$

A rearrangement yields

$$3m^2 + 56m + 36 = 0, \tag{4}$$

as desired.

(b) From the first part of the question, we know that the gradients m_A and m_B of the tangents that pass through A and B solve (4). It is straightforward to solve this equation to find $m_A = -2/3$, $m_B = -18$.



Writing $A = (x_A, y_A)$, we know from substituting m_A into (3) that

$$\frac{13}{9}x_A^2 + \frac{52}{3}x_A + 52 = 0.$$

Recalling that the discriminant of this quadratic equation has to be zero, this is easily solved using the quadratic formula:

$$x_A = \frac{-52/3}{2 \times 13/9} = -6.$$

Since $y_A = m_A x_A = 4$, we have proved that $A = (-6, 4)$.
An identical argument can be used to check that $B = (-2/5, 36/5)$.

(c) It would be possible to repeat the steps above with the new equations for the tangents and the circle C_2. However, this is a lot of work, and actually unnecessary. Instead, observing that the centre of the circle C_1 is at $(-4, 7)$, it is easy to see that the situation here is a translation of the earlier parts of the question, with the origin moved to $(4,-7)$. (Draw a quick sketch if you are not convinced!) In particular, it must be the case that

$$\begin{aligned} P &= A + (4,-7) = (-2,-3), \\ Q &= B + (4,-7) = (18/5, 1/5). \end{aligned}$$

5. *Note that there appears to be a mistake in the question. In particular, the line L_1 should be given by*

$$L_1 : \qquad \mathbf{r} = -2\mathbf{i} - 11.5\mathbf{j} + \lambda(3\mathbf{i} - 4\mathbf{j} - \mathbf{k}).$$

It is possible to solve the question with the given values, but this results in more awkward numerical solutions (see below).

(a) If the two lines intersect, then there must exist $\lambda, \mu \in \mathbb{R}$ such that

$$-2\mathbf{i} - 11.5\mathbf{j} + \lambda(3\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 11.5\mathbf{i} + 3\mathbf{j} + 8.5\mathbf{k} + \mu(7\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}).$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, this requires

$$\begin{aligned} 3\lambda - 7\mu &= 13.5, \\ 4\lambda + 8\mu &= -14.5, \\ \lambda - 11\mu &= -8.5. \end{aligned}$$

We now wish to show that no such λ and μ satisfy these equations. The general strategy is to solve any two of the equations, and then show that the solutions do not fit the remaining one. In particular, solving the first and third equations, we obtain that $\lambda = 8$ and $\mu = 1.5$. With these values, $4\lambda + 8\mu = 32 + 12 = 44$, which is inconsistent with the second equation. Hence, the lines L_1 and L_2 do not intersect.

(b) Observe that

$$(3\mathbf{i} - 4\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 3 \times 2 - 4 \times 1 - 1 \times 2 = 0,$$

$$(7\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 7 \times 2 + 8 \times 1 - 11 \times 2 = 0.$$

This implies that the vector $(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ is perpendicular to both L_1 and L_2.

(c) If A lies on L_1 and B lies on L_2, then

$$\begin{aligned} \overrightarrow{OA} &= -2\mathbf{i} - 11.5\mathbf{j} + \lambda(3\mathbf{i} - 4\mathbf{j} - \mathbf{k}), \\ \overrightarrow{OB} &= 11.5\mathbf{i} + 3\mathbf{j} + 8.5\mathbf{k} + \mu(7\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}), \end{aligned}$$

for some $\lambda, \mu \in \mathbb{R}$. Hence,

$$\overrightarrow{AB} = \begin{pmatrix} 13.5 + 7\mu - 3\lambda \\ 14.5 + 8\mu + 4\lambda \\ 8.5 - 11\mu + \lambda \end{pmatrix}.$$

The third fact we are given is that the line AB is perpendicular to both L_1 and L_2, which means that it is parallel to $(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, or in symbols:

$$\overrightarrow{AB} = \begin{pmatrix} 2\nu \\ \nu \\ 2\nu \end{pmatrix},$$

for some $\nu \in \mathbb{R}$. By comparing the various coordinates of the two expressions for $\overrightarrow{AB}$, we obtain that

$$13.5 + 7\mu - 3\lambda = 2\nu = 2(14.5 + 8\mu + 4\lambda),$$

$$13.5 + 7\mu - 3\lambda = 2\nu = 8.5 - 11\mu + \lambda.$$

These can be solved to give $\lambda = -1$ and $\mu = -1/2$. Hence we can conclude that

$$\begin{aligned} \overrightarrow{OA} &= -5\mathbf{i} - 7.5\mathbf{j} + \mathbf{k}, \\ \overrightarrow{OB} &= 8\mathbf{i} - \mathbf{j} + 14\mathbf{k}. \end{aligned}$$

If the question is solved with L_1 as given, then we find that $\lambda = 67/234$, $\mu = 33/13$.

6. (a) To show that $P = (1, 0)$ lies on the curve, we just need to note that if $x = 1$ and (x, y) lies on C, then $y = \sin(\ln 1) = \sin(0) = 0$.

   (b) To find the coordinates of Q, we need to find the first stationary point of the curve C with x-coordinate greater than or equal to 1. To do this, our first step will be to inspect when the derivative of $y(x) = \sin(\ln x)$ is equal to 0. In particular, by applying the chain rule,

$$y'(x) = \cos(\ln x)\frac{1}{x}.$$

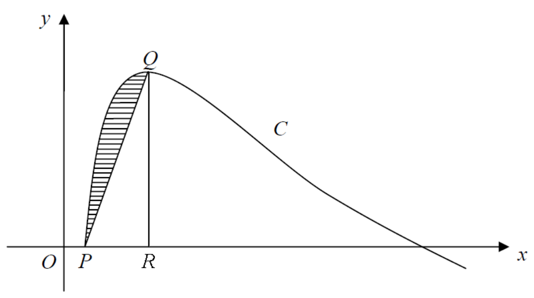
For this to be equal to zero, we require that $\cos(\ln x) = 0$. The first time this happens with $x \geq 1$ is when $\ln x = \pi/2$, i.e. $x = e^{\pi/2}$. The corresponding y-coordinate is given by $y = \sin(\ln e^{\pi/2}) = 1$. Hence

$$Q = \left(e^{\pi/2}, 1\right).$$

   (c) Rather than computing the area $\mathcal{A}$ of the shaded region directly, we will compute it as the difference

$$\mathcal{A} = \text{Area(under C)} - \text{Area}(PQR),$$

where: R is defined to be the point $(e^{\pi/2}, 0)$; the Area(under C) is that enclosed by C, the x-axis and the line QR; and the Area(PQR) is that of the triangle PQR.



Firstly,

$$\text{Area}(PQR) = \frac{1}{2}(\text{base}) \times (\text{height}) = \frac{1}{2}\left(e^{\pi/2} - 1\right). \quad (5)$$

Secondly, the Area(under C) can be obtained as the integral

$$\int_1^{e^{\pi/2}} y(x)dx = \int_1^{e^{\pi/2}} \sin(\ln x)dx.$$

The composition of two functions suggests that it will be helpful to make a substitution. Setting $u = \ln x$, we have that $du = (du/dx)dx = (1/x)dx$, or equivalently $dx = e^u du$, and so

$$\text{Area(under C)} = \int_0^{\pi/2} \sin(u)e^u du.$$

(Remember to change the limits of integration!) Since we can not integrate this directly, the product of two terms here leads us to try integrating by parts. Doing this yields

$$\begin{aligned} \text{Area(under C)} &= [e^u \sin u]_0^{\pi/2} - \int_0^{\pi/2} \cos(u)e^u du \\ &= e^{\pi/2} - \int_0^{\pi/2} \cos(u)e^u du \end{aligned}$$

Once again, we are left with an integral we can not solve directly, and so we will try integration by parts for a second time. This gives

$$\begin{aligned} \text{Area(under C)} &= e^{\pi/2} - [e^u \cos u]_0^{\pi/2} - \int_0^{\pi/2} \sin(u)e^u du \\ &= e^{\pi/2} + 1 - \int_0^{\pi/2} \sin(u)e^u du. \end{aligned}$$

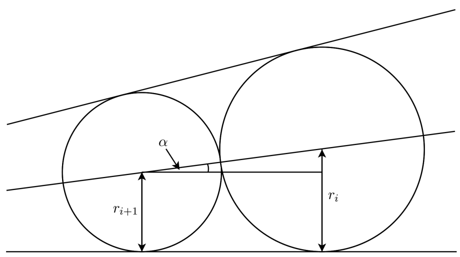
The final term is equal to the integral we started with, and from this observation it follows that

$$2 \times \text{Area(under C)} = e^{\pi/2} + 1. \tag{6}$$

Combining the results at (5) and (6), we obtain

$$\mathcal{A} = \frac{1}{2}\left(e^{\pi/2} + 1\right) - \frac{1}{2}\left(e^{\pi/2} - 1\right) = 1.$$

7. (a) Solving any question involving geometry is nearly always made easier by starting with a good sketch. The following shows the ith and $(i+1)$th circles in the sequence:



Here, r_i is the radius of the ith circle, and r_{i+1} the radius of the $(i+1)$th circle. We observe from this that the two radii and the line joining the centres of the two circles enclose a right-angled triangle, with the 'left-hand' angle being equal to α. Since the hypotenuse of this triangle is given by $r_i + r_{i+1}$, and the opposite side to the angle of size α is given by $r_i - r_{i+1}$, it follows that

$$\sin\alpha = \frac{r_i - r_{i+1}}{r_i + r_{i+1}}.$$

Rearranging this yields

$$\frac{r_{i+1}}{r_i} = \frac{1 - \sin\alpha}{1 + \sin\alpha}.$$

(b) The area enclosed by all the circles is given by

$$\pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \dots.$$

From the previous part of the question, we know that

$$r_i = Rs^{i-1},$$

where $s = (1 - \sin\alpha)/(1 + \sin\alpha)$. By applying the standard result for geometric series that $1 + r + r^2 + \dots = 1/(1 - r)$ with $r = s^2$, it follows that the area of interest is given by

$$\pi R^2 \left(1 + s^2 + s^4 + \dots\right) = \frac{\pi R^2}{1 - s^2}.$$

Substituting in the value of s, the area is thus equal to

$$\frac{\pi R^2 (1 + \sin\alpha)^2}{(1 + \sin\alpha)^2 - (1 - \sin\alpha)^2} = \frac{\pi R^2 (1 + \sin\alpha)^2}{4 \sin\alpha}. \tag{7}$$

(c) The first step here is to decompose the area into parts that we can compute. In particular, we can write the desired area as:

$$S = 2 \times \text{Area}(POA) + \text{Area}(\text{major sector } AOB) - \text{Area}(\text{enclosed by circles}),$$

where Area(POA) is the area of the triangle POA, Area(major sector AOB) is the larger sector of C_1 when decomposed by the radii AO and OB, and Area(enclosed by circles) was computed at (7).
To compute the area of POA, we use $\frac{1}{2} \times$ base $\times$ height. The height is given by R, and the base is given by $R \cot\alpha$, therefore Area(POA)$=\frac{1}{2}R^2 \cot\alpha$.
The area of a sector is given by $\frac{1}{2}R^2\theta$, where R is the radius of the circle and θ is the central angle. We know that $\angle POA = \pi/2 - \alpha$ (since the triangle has one right-angle). Thus, for the sector we are interested in, $\theta = 2\pi - 2(\pi/2 - \alpha) = \pi + 2\alpha$. It follows that Area(major sector AOB)$=\frac{1}{2}R^2\theta = (\frac{\pi}{2} + \alpha)R^2$.
Combining the various results, we obtain that

$$S = R^2 \left(\alpha + \cot\alpha + \frac{\pi}{2} - \frac{\pi(1 + \sin\alpha)^2}{4 \sin\alpha}\right).$$

By expanding the square and simplifying, this can be rewritten as

$$S = R^2 \left(\alpha + \cot\alpha - \frac{\pi}{4}\text{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right).$$

(d) Differentiating the trigonometric functions involved yields

$$\frac{dS}{d\alpha} = R^2 \left(1 - \text{cosec}^2\alpha + \frac{\pi}{4}\text{cosec}\alpha \cot\alpha - \frac{\pi}{4}\cos\alpha\right).$$

Now, observe that

$$\text{cosec}\alpha \cot\alpha = \frac{\cos\alpha}{\sin^2\alpha} = \cos\alpha\,\text{cosec}^2\alpha,$$

and so

$$\begin{aligned}
\frac{dS}{d\alpha} &= R^2 \left(1 - \text{cosec}^2\alpha + \frac{\pi}{4}\cos\alpha\,\text{cosec}^2\alpha - \frac{\pi}{4}\cos\alpha\right) \\
&= R^2 (1 - \text{cosec}^2\alpha) \left(1 - \frac{\pi}{4}\cos\alpha\right) \\
&= R^2 \cot^2\alpha \left(\frac{\pi}{4}\cos\alpha - 1\right),
\end{aligned}$$

where we have applied that $\text{cosec}^2\alpha - 1 = \cot^2\alpha$ to obtain the final equality.

(e) The usual approach to find a minimum of a functions is to start by looking at values where its derivative is zero, and this is what we will do here. In the range $\frac{\pi}{6} \le \alpha \le \frac{\pi}{4}$, we have that $\cot^2\alpha > 0$ and also

$$\frac{\pi}{4}\cos\alpha - 1 \le \frac{\pi}{4} - 1 < 0,$$

because $\cos x \le 1$ for any value of x. Therefore

$$\frac{dS}{d\alpha} < 0,$$

for every $\alpha \in [\frac{\pi}{6}, \frac{\pi}{4}]$. We learn from this that there are no values of $\alpha \in [\frac{\pi}{6}, \frac{\pi}{4}]$ where the derivative of S is zero. Nonetheless, because the above inequality means that the function S is decreasing in α, it is possible to conclude that its minimum is obtained when $\alpha = \pi/4$, i.e.

$$\begin{aligned} S &= R^2\left(\frac{\pi}{4} + \cot\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\text{cosec}\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)\right) \\ &= R^2\left(1 + \frac{\pi(\sqrt{2}-3)}{4\sqrt{2}}\right). \end{aligned}$$