AEA 2007 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. (a) The question requires us to expand an expression of the form $(a+b)^n$. According to binomial theorem, if the exponent n is a positive integer, then, for any a and b,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-1}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n. \quad (1)$$

More generally, we can write

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots, \quad (2)$$

where the right-hand side here converges whenever $n \geq 0$ is an integer, or when $|b/a| < 1$. Note that if $n \geq 0$ is an integer, then the series terminates after a finite number of terms (and when $n \neq 0$ is equal to the expression at (1)). Comparing $\frac{1}{(1-y)^2}$ to the left-hand side of equation (2), we have $n = -2$, $a = 1$ and $b = -y$. Substituting these values into the right-hand side, we obtain that, for $|y| < 1$,

$$\frac{1}{(1-y)^2} = 1 + 2y + 3y^2 + 4y^3 + \ldots. \quad (3)$$

(b) The word 'hence' immediately tells us that we should be trying to find a way to use the previous part of the question. To do this, we start by observing that we can write the left-hand side of the given equation as

$$\left(\frac{1}{2}\text{cosec}^2\left(\frac{\theta}{2}\right)\right)^2 = \frac{1}{(2\sin^2(\frac{\theta}{2}))^2} = \frac{1}{(1-\cos\theta)^2},$$

where the second equality is obtained by applying the double-angle formula $\cos(2\alpha) = 1 - 2\sin^2\alpha$ with $\alpha = \theta/2$. Now, we recognise that right-hand side above has the form $1/(1-y)^2$, with $y = \cos\theta$. Hence, by applying (3), it follows that

$$\frac{1}{4}\text{cosec}^4\left(\frac{\theta}{2}\right) = 1 + 2\cos\theta + 3\cos^2\theta + 4\cos^3\theta + \cdots + (r+1)\cos^r\theta + \ldots.$$

Finally, this series only converges when $|\cos\theta| < 1$. The values of θ for which this is not the case are $0, \pm\pi, \pm 2\pi, \ldots$.

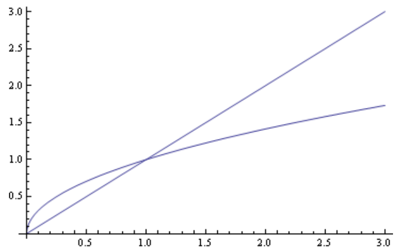
(c) The general term in (3) is given by $(r+1)y^r$. This is equal to $(r+1)/2^r$ when $y = \frac{1}{2}$. Thus, by applying (3) with this choice of y, we find that

$$1 + \frac{2}{2} + \frac{3}{2^2} + \cdots + \frac{(r+1)}{2^r} + \cdots = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$$

(d) Similarly to the previous part of the question, by applying (3) with $y = -\frac{1}{2}$, we obtain

$$1 - \frac{2}{2} + \frac{3}{2^2} + \cdots + (-1)^r\frac{(r+1)}{2^r} + \cdots = \frac{1}{\left(1+\frac{1}{2}\right)^2} = \frac{4}{9}.$$

2. (a) The two functions can be sketched as follows:

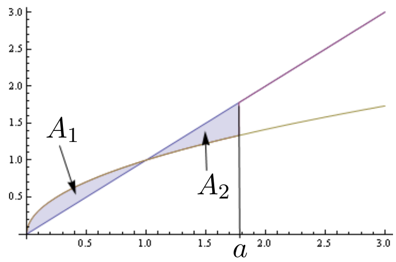


Note that the intersections occur at $(0, 0)$ and $(1, 1)$. To check this, we need to solve $x = \sqrt{x}$ for $x \geq 0$. Clearly $x = 0$ is a solution. If $x > 0$, then we can divide $x = \sqrt{x}$ by $\sqrt{x}$ to obtain $\sqrt{x} = 1$, which is solved by $x = 1$.

(b) We note from the sketch that for $0 < x < 1$ we have $x < \sqrt{x}$. This means that

$$\int_0^1 x dx = \int_0^1 \sqrt{x} dx - A_1,$$

where $A_1 > 0$ is the area shown here:



On the other hand for $x > 1$, we have $x > \sqrt{x}$. Thus

$$\int_1^a x dx = \int_1^a \sqrt{x} dx + A_2,$$

where A_2 is also shown on the above sketch. Clearly, as a increases from 1 to ∞, the area of A_2 increases continuously from 0 to ∞. Thus there exists a value a such that A_2 is identical to A_1. For this choice of a, we obtain

$$\int_0^a x dx = \int_0^1 x dx + \int_1^a x dx = \int_0^1 \sqrt{x} dx - A_1 + \int_1^a \sqrt{x} dx + A_2 = \int_0^a \sqrt{x} dx,$$

as desired.

(c) This part of the question requires usual integration. In particular, we have that

$$\int_0^a x dx = \left[\frac{x^2}{2}\right]_0^a = \frac{a^2}{2},$$

and also

$$\int_0^a \sqrt{x} dx = \left[\frac{2x^{3/2}}{3}\right]_0^a = \frac{2a^{3/2}}{3}.$$

Thus, for the two integrals to be equal, we require

$$\frac{a^2}{2} = \frac{2a^{\frac{3}{2}}}{3}.$$

Since $a > 1$, we can divide both sides by $a^{3/2}/2$ to obtain

$$\sqrt{a} = \frac{4}{3},$$

which implies

$$a = \frac{16}{9}.$$

(d) There are several approaches to this part of the question, but since we already know that $\int_0^{16/9} x dx = \int_0^{16/9} \sqrt{x} dx$, we will try to apply this. Firstly, by reflecting about the line $x = 0$, we find that

$$\int_{-16/9}^{0} (-x) dx = \int_0^{16/9} x dx = \int_0^{16/9} \sqrt{x} dx = \int_{-16/9}^{0} \sqrt{-x} dx.$$

Combining this with our previous result yields

$$\int_{-16/9}^{16/9} |x| dx = \int_{-16/9}^{16/9} \sqrt{|x|} dx,$$

where we note that $|x| = -x$ for $x < 0$ and $|x| = x$ otherwise. Thus the problem is solved by setting $f(x) = |x|$ and $b = 16/9$. An alternative method would be to use a translation along the x-axis so that the interval $(0, 16/9)$ is shifted to an interval of the form $(-b, b)$. In particular, letting $f(x) = x + \frac{8}{9}$ and $b = \frac{8}{9}$, we obtain

$$\begin{aligned}
\int_{-b}^{b} f(x) dx &= \int_{-8/9}^{8/9} \left(x + \frac{8}{9} \right) dx \\
&= \int_0^{16/9} x dx \\
&= \int_0^{16/9} \sqrt{x} dx \\
&= \int_{-8/9}^{8/9} \sqrt{\left(x + \frac{8}{9} \right)} dx \\
&= \int_{-b}^{b} \sqrt{f(x)} dx.
\end{aligned}$$

3. (a) We will start by rewriting the equation in such a way that the arguments of the trigonometric functions are the same. In particular, we will replace $\cos 2x$ by $2\cos^2 x - 1$ (which is one of the usual double-angle formulae), to obtain

$$0 = \cos x + \cos 2x = 2\cos^2 x + \cos x - 1.$$

This is simply a quadratic equation in the variable $\cos x$, i.e. by letting $\cos x = y$, we have $2y^2 + y - 1 = 0$. For this quadratic, we have the factorisation $2y^2 + y - 1 = (2y-1)(y+1)$, and so the equation has roots $\frac{1}{2}$ and -1. It follows that we need to find all the values of $x \in [0, 2\pi)$ such that either $\cos x = \frac{1}{2}$ or $\cos x = -1$. The solutions of the former equation are given by $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$, and the solution of the latter is $x = \pi$. As with many trigonometry problems, this is not the only approach. An alternative would be to rewrite $\cos x + \cos 2x$ as $2\cos(3x/2)\cos(x/2)$ using a factor formula, and find values of x such that either $\cos(3x/2) = 0$ or $\cos(x/2) = 0$.

(b) As we are told in the question, the function $\arccos(x)$ is the inverse of $\cos(x)$, so that $\cos(\arccos(x)) = x$. Thus, if we apply cos to both sides of the equation

$$\arccos(2x) = \frac{\pi}{2} - \arccos(x),$$

then we obtain

$$2x = \cos\left(\frac{\pi}{2} - \arccos(x)\right).$$

Although the right-hand side here looks complicated, we can simplify it using the addition formula

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

In particular, we deduce

$$2x = \cos\left(\frac{\pi}{2}\right)\cos\left(\arccos(x)\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\arccos(x)\right).$$

We know that $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$. This means we can simplify the above equation to

$$2x = \sin\left(\arccos(x)\right).$$

Now, setting $\arccos x = y$, i.e $x = \cos y$, we can rewrite this as $2\cos y = \sin y$, and squaring both sides yields

$$4\cos^2 y = \sin^2 y = 1 - \cos^2 y.$$

Hence $x = \cos y = \pm 1/\sqrt{5}$. The exact value of x, $x \geq 0$, for which the relevant equation holds must therefore be $x = 1/\sqrt{5}$. Again, alternative approaches to simplifying the equation are possible.

4. (a) This is not a standard equation, so we will proceed by plugging in the information we are given and seeing what happens. In particular, letting $h(x) = \left(\frac{dy}{dx}\right)^2$, we have

$$\begin{aligned}
\sqrt{\int h(x)dx} &= \int \sqrt{h(x)}dx \\
&= \int \sqrt{\left(\frac{dy}{dx}\right)^2} dx \\
&= \int \frac{dy}{dx} dx \\
&= \int dy \\
&= y + c,
\end{aligned}$$

and so, squaring both sides of this equation,

$$\int h(x)dx = (y + c)^2.$$

This is close to what we want, but we need to get rid of the integral. To do this, we will differentiate, recalling that, if we have a function F defined as an integral of f, i.e. $F(x) = \int f(x)dx$, then the derivative of F at x is given by $f(x)$. Applying this in our case yields

$$h(x) = \frac{d}{dx}\left((y + c)^2\right) = 2(y + c)\frac{dy}{dx},$$

where the second inequality is obtained using the chain rule. Substituting in the definition of $h(x)$, this is equivalent to

$$\left(\frac{dy}{dx}\right)^2 = 2(y+c)\frac{dy}{dx}.$$

Finally, we are told that $h(x) > 0$, and so it must also be the case that $\frac{dy}{dx} \neq 0$. This means we can divide the above equation by $\frac{dy}{dx}$ to obtain

$$\frac{dy}{dx} = y + c,$$

as desired.

(b) To find an expression for y, we will integrate the equation $\frac{dy}{dx} = 2(y+c)$. First, though, let us rearrange so that all the terms involving y are on the left-hand side:

$$\frac{1}{y+c}\frac{dy}{dx} = 2.$$

Now, integrating with respect to x,

$$\begin{aligned} \int \frac{1}{y+c}\frac{dy}{dx}dx &= \int 2dx \\ \int \frac{1}{y+c}dy &= \int 2dx \\ \ln(y+c) &= 2x + \alpha, \end{aligned}$$

where α is a constant of integration. Taking exponentials, this implies $y + c = Ae^{2x}$ for some constant $A(= e^{\alpha})$, or equivalently

$$y = Ae^{2x} - c.$$

(c) We know that $y = Ae^{2x} - c$ and hence $\frac{dy}{dx} = 2Ae^{2x}$. Substituting this into $h(x)$, we obtain

$$h(x) = \left(\frac{dy}{dx}\right)^2 = (2Ae^{2x})^2 = 4A^2e^{4x}.$$

Thus, finding A^2 will determine $h(x)$ completely. Since we are given that $h(0) = 1$, it must hold that $1 = 4A^2e^0 = 4A^2$, and so $h(x) = e^{4x}$.

5. (a) To the original square S_1, 4 squares of side $\frac{a}{3}$ were added to form S_2. To each of these, 3 squares of side $\frac{a}{9}$ were added to form S_3. To each of these, 3 squares of side $\frac{a}{27}$ need to be added to form S_4. Thus, in total $4 \times 3 \times 3 = 36$ squares of side $\frac{a}{27}$ need to be added to S_3 to form S_4.

(b) Denote by P_n the perimeter of S_n. The perimeter of S_1 is readily observed to be $P_1 = 4a$. The perimeter of S_2 is obtained from this by adding $2 \times \frac{a}{3}$ for each of the 4 squares of side $\frac{a}{3}$ added. Hence

$$P_2 = 4a + 4 \times 2 \times \frac{a}{3} = \frac{20a}{3}.$$

Similarly, the perimeter of S_3 is obtained from P_2 by adding $2 \times \frac{a}{9}$ for each of the $4 \times 3 = 12$ squares of side $\frac{a}{9}$ added. Hence

$$P_3 = \frac{20a}{3} + 12 \times 2 \times \frac{a}{9} = \frac{28a}{3}.$$

(c) By continuing the iteration described in part (a), we have that, in general, $4 \times 3^{n-1}$ squares of side $\frac{a}{3^n}$ need to be added to go from S_n to S_{n+1}. For each of these, we increase the perimeter of S_n by $2 \times \frac{a}{3^n}$. Hence we obtain the relationship

$$P_{n+1} = P_n + 4 \times 3^{n-1} \times 2 \times \frac{a}{3^n} = P_n + \frac{8a}{3}.$$

Thus, we obtain that $P_1, P_2, \dots$ is an arithmetic progression with common difference $\frac{8a}{3}$. Since $P_1 = 4a$, this means that

$$P_n = 4a + \frac{8a(n-1)}{3} = \frac{4a}{3} + \frac{8a}{3}n.$$

(d) From part (d), we see that as n increases, so does the perimeter P_n in a linear fashion. A particular consequence of this is that as $n \to \infty$, we have $P_n \to \infty$, i.e. for any constant C, we can always find a value of n such that the perimeter P_n is larger than C.

(e) Denote by A_n the area of S_n. Clearly $A_1 = a^2$. The area of S_2 is obtained from this by adding $\left(\frac{a}{3}\right)^2$ for each of the 4 squares of side $\frac{a}{3}$ added. Hence

$$A_2 = a^2 + 4 \times \left(\frac{a}{3}\right)^2 = \frac{13a^2}{9}.$$

Similarly, the area of S_3 is obtained from A_2 by adding $\left(\frac{a}{9}\right)^2$ for each of the 12 squares of side $\frac{a}{9}$ added. Hence

$$A_3 = \frac{13a^2}{9} + 12 \times \left(\frac{a}{9}\right)^2 = \frac{43a^2}{27}.$$

(f) As we have already noted, in general, $4 \times 3^{n-1}$ squares of side $\frac{a}{3^n}$ need to be added to go from S_n to S_{n+1}. For each of these we add $\left(\frac{a}{3^n}\right)^2$ to the area of A_n. Hence

$$A_{n+1} = A_n + 4 \times 3^{n-1} \times \left(\frac{a}{3^n}\right)^2 = A_n + \frac{4a^2}{3^{n+1}}.$$

Since $A_1 = a^2$, this implies that

$$A_n = A_1 + \frac{4a^2}{3} \sum_{m=1}^{n-1} 3^{-m} = a^2 + \frac{4a^2}{9} \sum_{m=0}^{n-2} 3^{-m}.$$

Now, for any geometric series, we have that

$$\sum_{m=0}^{N} r^m = \frac{1 - r^{N+1}}{1 - r},$$

(assuming $r \neq 1$), and so

$$A_n = a^2 + \frac{4a^2}{9} \times \frac{1 - 3^{-(n-1)}}{1 - 3^{-1}}.$$

This value is strictly smaller than

$$S = a^2 + \frac{4a^2}{9} \times \frac{1}{1 - 3^{-1}} = \frac{5a^2}{3}.$$

Moreover, by letting $n \to \infty$, the area A_n can be made as close as we like to this value, and so this is the smallest value of a constant with this property.

6. (a) The area of the rectangle R in Figure 2 of the exam paper is obtained by multiplying its width, $\frac{\pi}{2}-x$, by its height, $\tan\frac{x}{2}$ (since point P has coordinates $(x, \tan\frac{x}{2})$). Hence its area is

$$A = \left(\frac{\pi}{2} - x\right)\tan\frac{x}{2}.$$

(b) The area A is a product of two functions, $f(x) = \frac{\pi}{2} - x$ and $g(x) = \tan(x/2)$. As a consequence, we can apply the product rule when differentiating it:

$$\begin{aligned}\frac{dA}{dx} &= \frac{df}{dx}g(x) + f(x)\frac{dg}{dx} \\ &= -\tan\frac{x}{2} + \left(\frac{\pi}{2} - x\right)\frac{d(\tan\frac{x}{2})}{dx}.\end{aligned}$$

We now recall that the derivative of $\tan x$ is given by $\sec^2 x$, and so

$$\frac{dA}{dx} = -\tan\frac{x}{2} + \frac{1}{2}\left(\frac{\pi}{2} - x\right)\sec^2\frac{x}{2}.$$

Moreover, because

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \cos\frac{x}{2}\sin\frac{x}{2}\sec^2\frac{x}{2} = \frac{1}{2}\sin x\sec^2\frac{x}{2},$$

where we have applied the identity $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, it follows that

$$\begin{aligned}\frac{dA}{dx} &= -\frac{1}{2}\sin x\sec^2\frac{x}{2} + \frac{1}{2}\left(\frac{\pi}{2} - x\right)\sec^2\frac{x}{2} \\ &= \frac{1}{4}(\pi - 2x - 2\sin x)\sec^2\frac{x}{2}\end{aligned}$$

as required.

(c) The quantity $\frac{dA}{dx}$ is a measure of rate of change in the area A in terms of x, and to find a maximum it would be helpful to find where it takes the value of 0. However, the function is a bit too complicated to do this directly. Instead, we start by observing that A is increasing if $\frac{dA}{dx}$ is positive and decreasing if $\frac{dA}{dx}$ is negative. Hence, because $\frac{dA}{dx}$ is a continuous function, to show that A has a maximum in the interval $(\frac{\pi}{4}, \frac{\pi}{3})$, it will be enough to check that $\frac{dA}{dx} > 0$ for $x \le \pi/4$ and $\frac{dA}{dx} < 0$ for $x \ge \pi/3$. To do this, first note that if $0 \le x \le \pi/4$, then $\sin x \le \sin\frac{\pi}{4} = 1/\sqrt{2}$. Hence, for $x \in [0, \frac{\pi}{4}]$, we have

$$\frac{dA}{dx} \ge \frac{1}{4}\left(\pi - \frac{\pi}{2} - \frac{2}{\sqrt{2}}\right)\sec^2\frac{x}{2} \ge \frac{1}{4}\left(\frac{\pi}{2} - \sqrt{2}\right)\sec^2\frac{x}{2} > 0.$$

(Note that $\sec^2 x$ is always strictly positive.) Similarly, if $x \in [\frac{\pi}{3}, \frac{\pi}{2}]$, we have $\sin x \ge \sin\frac{\pi}{3} = \sqrt{3}/2$, and so

$$\frac{dA}{dx} \le \frac{1}{4}\left(\pi - \frac{2\pi}{3} - \sqrt{3}\right)\sec^2\frac{x}{2} = \frac{1}{4}\left(\frac{\pi}{3} - \sqrt{3}\right)\sec^2\frac{x}{2} < 0.$$

This confirms that A does indeed reach its maximum at a value of x that is strictly between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$.

(d) To solve this part of the question, we start by recalling that $\tan\frac{\pi}{4} = 1$. Applying this fact in conjunction with the double-angle formula

$$\tan x = \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}},$$

we obtain

$$1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}.$$

Rearranging, this implies that

$$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0.$$

Now, the solutions of the quadratic equation $x^2+2x-1=0$ are $-1\pm\sqrt{2}$. Hence, because $\tan\frac{\pi}{8}$ is positive, we must have that

$$\tan\frac{\pi}{8} = \sqrt{2}-1.$$

(e) From 6(c) we know that at $x=\frac{\pi}{4}$ the area A of the rectangle R is smaller than its maximum value. Moreover, by applying the conclusion of part (d) we know that the area of A when $x=\frac{\pi}{4}$ is given by

$$A\big|_{x=\frac{\pi}{4}} = \left(\frac{\pi}{2}-x\right)\tan\frac{x}{2}\bigg|_{x=\frac{\pi}{4}} = \left(\frac{\pi}{2}-\frac{\pi}{4}\right)\tan\frac{\pi}{8} = \frac{\pi}{4}(\sqrt{2}-1).$$

Thus the maximum value of A is strictly greater than $\frac{\pi}{4}(\sqrt{2}-1)$.

7. (a) The key to solving this part of the question is the observation that the angles formed by drawing lines from the ends of the diameter of a circle to its circumference form a right angle. In particular, the angle $\angle OPQ$ is a right-angle, and so

$$\overrightarrow{PO}\cdot\overrightarrow{PQ} = |\overrightarrow{PO}||\overrightarrow{PQ}|\cos(\angle OPQ) = 0.$$

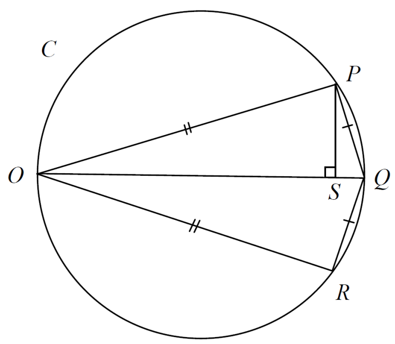
Now, we can write that $\overrightarrow{PO} = -\mathbf{p}$ and $\overrightarrow{PQ} = \mathbf{q}-\mathbf{p}$, which means the above inequality can be written as

$$-\mathbf{p}\cdot(\mathbf{q}-\mathbf{p}) = \mathbf{0}.$$

Since the dot product is distributive (i.e. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$), this implies that

$$\mathbf{p}\cdot\mathbf{q} = \mathbf{p}\cdot\mathbf{p} = |\mathbf{p}|^{\mathbf{2}}.$$

(b) A quick sketch using the facts about S given in the question helps us to clarify where the point lies:



Since $\overrightarrow{PS} = \lambda\mathbf{q}-\mathbf{p}$ is perpendicular to $\overrightarrow{OQ} = \mathbf{q}$, it must be the case that

$$(\lambda\mathbf{q}-\mathbf{p})\cdot\mathbf{q} = \mathbf{0}.$$

Rearranging, again using the distributivity of the dot product, we obtain

$$\lambda\mathbf{q}\cdot\mathbf{q} = \mathbf{p}\cdot\mathbf{q} = |\mathbf{p}|^2,$$

where we have applied the conclusion of part (a) to deduce the second inequality. Now, $\mathbf{q}\cdot\mathbf{q} = |\mathbf{q}|^2 = 2^2+1^2+(-2)^2 = 9$, while $|\mathbf{p}|^2 = 1^2+2^2+(-1)^2 = 6$. Substituting these values into the above equation, we find that

$$\lambda = \frac{2}{3}.$$

(c) Since $OPQR$ is a kite, the triangle OQR is just a mirror image of OQP (with respect to the line of reflection OQ). It follows that

$$\overrightarrow{SR} = -\overrightarrow{SP} = \overrightarrow{PS},$$

and subsequently

$$\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR} = 2\overrightarrow{PS}.$$

Therefore, the position of R relative to O is given by

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + 2\overrightarrow{PS} = \overrightarrow{OP} + 2\left(\overrightarrow{OS} - \overrightarrow{OP}\right) = 2\overrightarrow{OS} - \overrightarrow{OP} = 2\lambda\mathbf{q} - \mathbf{p}.$$

Substituting $\mathbf{p} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{q} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\lambda = 2/3$, we obtain

$$\overrightarrow{OR} = \frac{5}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{5}{3}\mathbf{k}.$$

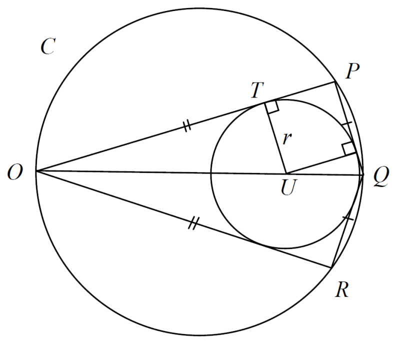
(d) The area of K is equal to the sum of the areas of the triangles OPQ and OQR. Since by symmetry both triangles have the same area, we must therefore have the area of K is equal to twice the area of OPQ. Thus,

$$\begin{aligned}\text{Area of } K &= 2 \times \text{Area of } OPQ \\ &= 2 \times \frac{1}{2}|\overrightarrow{OQ}||\overrightarrow{PS}| \\ &= \sqrt{18},\end{aligned}$$

since $|\overrightarrow{OQ}| = |\mathbf{q}| = 3$ and

$$|\overrightarrow{PS}| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{18}}{3}.$$

(e) Since OQP and OQR are mirror images, it suffices just to look at one triangle OQP. We are told that the circle C_1 is tangent to the sides of the kite, which includes the lines OP and PQ. This means that if we draw a radius from the centre of the circle, U say, to the point where it touches OP, T say, then this will be perpendicular to the line OP. A similar observation may be made about the radius that touches PQ. Hence, we obtain the square shown in the following figure:



As a consequence, we are able to deduce that

$$\frac{|\overrightarrow{PQ}|}{|\overrightarrow{OP}|} = \tan(\angle POQ) = \tan(\angle TOU) = \frac{|\overrightarrow{TU}|}{|\overrightarrow{OT}|} = \frac{r}{|\overrightarrow{OP}| - r}.$$

Substituting in the values $|\overrightarrow{PQ}| = \sqrt{3}$ and $|\overrightarrow{OP}| = \sqrt{6}$, this implies that

$$\frac{\sqrt{3}}{\sqrt{6}} = \frac{r}{\sqrt{6} - r},$$

and rearranging gives

$$r = \frac{\sqrt{18}}{\sqrt{6} + \sqrt{3}} = \sqrt{6}(\sqrt{2} - 1).$$

(f) We are given that K_1 is to C_1 as K is to C. Moreover, we know that the radius of C is given by $\frac{1}{2}|\mathbf{q}| = \frac{3}{2}$ and the radius of C_1 is given by $\sqrt{6}(\sqrt{2}-1)$. This means that the ratio between the side lengths of the kite K_1 to those of K is equal to $\frac{2}{3}\sqrt{6}(\sqrt{2} - 1)$. (Note that similarity means the angles of the two kites will be the same.) Since areas scale like length squared, it follows that

$$\begin{aligned}
\text{Area of } K_1 &= \text{Area of } K \times \left(\frac{2}{3}\sqrt{6}(\sqrt{2} - 1)\right)^2 \\
&= \sqrt{18} \times \left(\frac{2}{3}\sqrt{6}(\sqrt{2} - 1)\right)^2 \\
&= 8\sqrt{2}\left(\sqrt{2} - 1\right)^2.
\end{aligned}$$

Notice that we did not have to do any complicated calculations to compute the area directly.