AEA 2010 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. (a) Due to the presence of the unknown quantity x under the square root, the first thing we have to determine is for which values of x the equation makes sense. Recall that for a square root to be defined, the argument has be greater or equal than zero. In particular,

$$\begin{cases} 3x+16 \geq 0 & \Longrightarrow \quad x \geq -\frac{16}{3}, \\ x+1 \geq 0 & \Longrightarrow \quad x \geq -1. \end{cases}$$

It follows that the solutions have to be greater or equal to -1. We now start solving the equation. To get rid of the square roots, we will square both of its sides:

$$\begin{aligned} \sqrt{3x+16} &= 3+\sqrt{x+1} \\ \left(\sqrt{3x+16}\right)^2 &= \left(3+\sqrt{x+1}\right)^2 \\ 3x+16 &= 9+x+1+6\sqrt{x+1} \\ 2x+6 &= 6\sqrt{x+1} \\ \frac{1}{3}x+1 &= \sqrt{x+1}. \end{aligned}$$

To remove the remaining square root, we need to square both sides of the equation again. This gives

$$\frac{1}{9}x^2+1+\frac{2}{3}x = x+1,$$

which is easily rearranged to give

$$x\left(\frac{1}{9}x-\frac{1}{3}\right) = 0.$$

Thus, the two possible solutions are $x=0$ and $\frac{1}{9}x-\frac{1}{3}=0 \Longrightarrow x=3$. Since they are both bigger than -1 they are potentially valid solutions of the equation. Note that, to check that we did not create extra solutions when squaring, we should verify both of these values actually solve the original equation (which they do).

(b) For this second equation, again the domain has to be computed. For logarithmic functions, the argument has to be greater than zero, and so we require

$$\begin{cases} x-7>0 & \Longrightarrow \quad x>7, \\ x>0. & \end{cases}$$

Thus, in this case, valid solutions will be bigger than 7. Recalling the property that a constant p times a logarithm of a number is equal to the logarithm of the p-th power of the same number (i.e. $p\log(x)=\log(x^p)$), we can rewrite $\frac{1}{2}\log_3(x)$ as $\log_3(\sqrt{x})$. We can also rewrite the number 1 in a logarithmic form as $\log_3(3)$, which simply follows from the definition of logarithm. Thus, the initial equation can be written as

$$\log_3(x-7)-\log_3(\sqrt{x}) = \log_3(3)-\log_3(2).$$

Now, since the difference of two logarithms with the same base is equal to the logarithm of the ratio (i.e. $\log(a) - \log(b) = \log(a/b)$), we have that

$$\log_3\left(\frac{x-7}{\sqrt{x}}\right) = \log_3\left(\frac{3}{2}\right).$$

At this point, the logarithm can be cancelled out (by simply applying on both sides the inverse function of the relevant logarithm, 3^x), so that

$$\begin{aligned}\frac{x-7}{\sqrt{x}} &= \frac{3}{2}\\ 2x - 14 &= 3\sqrt{x}.\end{aligned}$$

Then one can either square both sides of the equation to get rid of the square root, or, alternatively, define $\sqrt{x} = t$. In this way the previous equation can be written as

$$2t^2 - 3t - 14 = 0,$$

and solved as a standard quadratic equation. Thus

$$t = \frac{3 \pm \sqrt{121}}{4} \Longrightarrow t = \frac{7}{2} \text{ or } -2.$$

Since $\sqrt{x} = t$, then $x = t^2$, and the two possible solutions are therefore $49/4$ and 4. However, since the domain is $x > 7$, then the only valid solution is $x = 49/4$.

2. (a) The terms of an arithmetic series $u_1, u_2, u_3, \dots$ can be written in the form

$$u_n = u_1 + d(n-1),$$

where $d = u_n - u_{n-1}$ is the common difference. Moreover, the sum of the first n numbers S_n is recalled to be

$$\frac{n(2u_1 + d(n-1))}{2}.$$

Thus, the information given at the beginning of the question can be written as

$$q = \frac{p(2u_1 + d(p-1))}{2}, \tag{1}$$

$$p = \frac{q(2u_1 + d(q-1))}{2}. \tag{2}$$

In order to determine d in terms of p and q, we need to eliminate u_1 from both the previous two equations. This can be done in several ways. One possible approach is to isolate $2u_1$ in one of the two equations (say (1)) and then plug in the resulting expression at (2). This can be done in the following way.

$$\frac{2q}{p} = 2u_1 + d(p-1) \qquad \Longrightarrow \qquad 2u_1 = \frac{2q}{p} - d(p-1). \tag{3}$$

We now substitute the above expression in (2) to obtain

$$\begin{aligned} p &= \frac{q}{2}\left(\frac{2q}{p} - d(p-1) + d(q-1)\right)\\ p &= \frac{q^2}{p} + \frac{dq}{2}(q-1-p+1)\\ \Longrightarrow \quad p - \frac{q^2}{p} &= \frac{dq(q-p)}{2}\\ \frac{p^2 - q^2}{p} &= \frac{dq(q-p)}{2}.\end{aligned}$$

From the previous equation we can then isolate d, to conclude that

$$d = \frac{2(p^2 - q^2)}{pq(q-p)} = \frac{2(p-q)(p+q)}{pq(q-p)} = -\frac{2(p+q)}{pq} \tag{4}$$

(b) In order to determine the first term of the sequence, we can use equation (3), in which u_1 is a function of p, q and d, and substitute d with the result obtained at (4). We have that,

$$\begin{aligned} 2u_1 &= \frac{2q}{p} - (p-1)\left(-\frac{2(p+q)}{pq}\right) \\ &= \frac{2q}{p} + \frac{2(p-1)(p+q)}{pq} \\ &= \frac{2q^2 + 2p^2 + 2pq - 2p - 2q}{pq}. \end{aligned}$$

Thus,

$$u_1 = \frac{q^2 + p^2 + pq - p - q}{pq} = \frac{q^2 + (p-1)(p+q)}{pq}. \tag{5}$$

(Of course, there are other ways to write this final solution.)

(c) By the formula stated in the solution to part (a) for summing terms in an arithmetic progression, we can immediately write

$$S_{p+q} = \frac{(p+q)(2u_1 + (p+q-1)d)}{2}.$$

We can use the results in equations (4) and (5) to substitute d and u_1 and obtain an expression in p and q only. Indeed

$$\begin{aligned} S_{p+q} &= \frac{p+q}{2}\left(\frac{2(q^2 + p^2 + pq - p - q)}{pq} + (p+q-1)\left(-\frac{2(p+q)}{pq}\right)\right) \\ &= (p+q)\left(\frac{q^2 + p^2 + pq - p - q - p^2 - 2pq + q^2 + p + q}{pq}\right) \\ &= (p+q)(-\frac{pq}{pq}) \\ &= -(p+q). \end{aligned}$$

Notice that this part of the question used very little knowledge about arithmetic progressions – just some very careful algebraic manipulation. To reduce the chance of mistakes in doing this, it is a good idea to simplify expressions whenever possible!

3. (a) To compute the gradient of the curve C, one has to consider y as a function of x, and calculate its first derivative with respect to x. To do this, we start by differentiating the entire equation with respect to x, using the chain and product rule where needed. This gives:

$$2x + 2y(x)y'(x) + fy(x) + fxy'(x) = 0.$$

Rearranging the terms in the previous equation in order to isolate $y'(x)$, one obtains from this that

$$y'(x) = -\frac{2x + fy(x)}{2y(x) + fx}.$$

Finally, since we are looking for the gradient of C at the point (α, β), we have to substitute x and $y(x)$ with α and β respectively, which yields the relevant gradient is

$$m = -\frac{2\alpha + f\beta}{2\beta + f\alpha}.$$

(b) This part requires us to derive an expression for α and β in terms of f and g. For this purpose, since we have two unknowns, we need to consider a system of at least two equations. The equation defining the curve C in which x and y are substituted with α and β respectively can be used for this task. Moreover, since $m = 1$, one can use the equation of the gradient as well. In particular, these are:

$$\begin{cases} \alpha^2 + \beta^2 + f\alpha\beta - g^2 = 0 \\ 2\beta + f\alpha = -2\alpha - f\beta. \end{cases}$$

The second equation looks slightly more manageable, since it does not involve squares. By moving all the terms to one side and factorising, we can rewrite it as follows:

$$(\alpha + \beta)(f + 2) = 0,$$

which implies $\alpha = -\beta$ or $f = -2$. Since it is an assumption of the question that $f \neq -2$, the only valid solution is $\alpha = -\beta$. We now substitute this result into the equation of the curve C, to obtain

$$\alpha^2 + (-\alpha^2) + f\alpha(-\alpha) - g^2 = 0,$$

or equivalently,

$$\alpha^2(2 - f) = g^2.$$

Clearly this has solutions

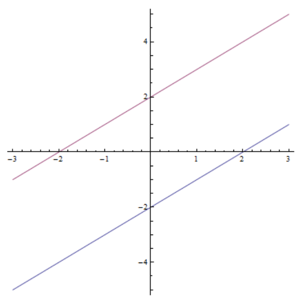
$$\alpha = \frac{\pm g}{\sqrt{2 - f}},$$

as required. (Note that here we use $f < 2$, else the square-root does not make sense.)

(c) Given $f = -2$, the equation of C is $x^2 + y^2 - 2xy = g^2$. The right-hand side of this is readily seen to be equal to $(x - y)^2$. It follows that $x - y = \pm g$. Thus C consists of the two lines

$$y = x + g,$$
$$y = x - g,$$

which are easily sketched. Note that the two lines are certainly distinct, since $g \neq 0$. The following figure shows C when $g = 2$. In general the y-intercepts of the two lines are at g and $-g$, respectively.



4. (a) The cosine of the angle $\angle CAF$ can be found recalling the geometric interpretation of scalar product. Indeed, the scalar product between two vectors a and b is equal to

$$a \cdot b = ||a||\ ||b|| \cos\theta, \tag{6}$$

where $||a||$ represents the length of a, $||b||$ the length of b, and θ is the angle between a and b. Thus, to answer this question, we first need to determine the vectors $\overrightarrow{AC}$ and $\overrightarrow{AF}$, then compute their length and the scalar product between them. We are told that the point A has coordinates $(5, 0, 0)$, C is at $(0, 10, 0)$, and for F the coordinates are $(5, 10, 20)$. It follows that

$$\overrightarrow{AC} = \begin{pmatrix} -5 \\ 10 \\ 0 \end{pmatrix}, \qquad \overrightarrow{AF} = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}.$$

The length of a vector is defined as the square root of the sum of the squares of its components. Applying this definition, we can compute the length of $\overrightarrow{AC}$ and $\overrightarrow{AF}$ as

$$||\overrightarrow{AC}|| = \sqrt{(-5)^2 + 10^2 + 0^2} = \sqrt{125} = 5\sqrt{5},$$
$$||\overrightarrow{AF}|| = \sqrt{0^2 + 10^2 + 20^2} = \sqrt{500} = 10\sqrt{5}.$$

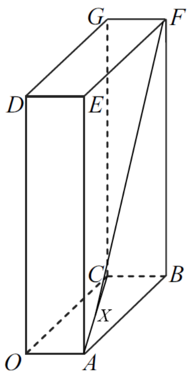
The scalar product of two products is defined as the sum of the products of corresponding entries. Thus,

$$\overrightarrow{AC} \cdot \overrightarrow{AF} = -5 \times 0 + 10 \times 10 + 0 \times 20 = 100.$$

We can now use equation (6) to compute the cosine of $\angle CAF$, which is

$$\cos\angle CAF = \frac{\overrightarrow{AC} \cdot \overrightarrow{AF}}{||\overrightarrow{AC}||\ \ ||\overrightarrow{AF}||} = \frac{100}{(5\sqrt{5})(10\sqrt{5})} = \frac{100}{250} = \frac{2}{5}.$$

(b) The following sketch shows the point X:



To answer the question, we will first define the vectors $\overrightarrow{OX}$, representing the position of X, and $\overrightarrow{FX}$, of which the length has to computed. We can think of $\overrightarrow{OX}$ as the composition of a movement from O to A and then a further part from A to a generic point X along the vector $\overrightarrow{AC}$, parameterised by t. This can be done saying

$$\overrightarrow{OX} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 - 5t \\ 10t \\ 0 \end{pmatrix}.$$

The same reasoning can be developed for $\overrightarrow{FX}$, thinking of it as composed by $\overrightarrow{FO}$ and $\overrightarrow{OX}$, which has been already derived parametrically. Thus,

$$\overrightarrow{FX} = \begin{pmatrix} -5 \\ -10 \\ -20 \end{pmatrix} + \begin{pmatrix} 5-5t \\ 10t \\ 0 \end{pmatrix} = \begin{pmatrix} -5t \\ 10t-10 \\ -20 \end{pmatrix}.$$

We now use the information that $\overrightarrow{AC}$ and $\overrightarrow{FX}$ are perpendicular to derive the value of t. Recall that two vectors are perpendicular if their scalar product is equal to zero. So, we may assume that $\overrightarrow{AC} \cdot \overrightarrow{FX} = 0$, and evaluating the left-hand side of this explicitly gives

$$-5 \times (-5t) + 10 \times (10t - 10) + 0 \times (-20) = 0.$$

A rearrangement of this yields $125t = 100$, i.e. $t = 4/5$. So the position vector of X and the vector $\overrightarrow{FX}$ are

$$\overrightarrow{OX} = \begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix}, \qquad \overrightarrow{FX} = \begin{pmatrix} -4 \\ -2 \\ -20 \end{pmatrix}.$$

From the second of these expressions, we can compute the length of $\overrightarrow{FX}$, which is

$$||\overrightarrow{FX}|| = \sqrt{(-4)^2 + (-2)^4 + (-20)^2} = \sqrt{420}.$$

(c) To write the two vector equations of l_1 and l_2 we need, first of all, to determine the midpoint of the face $ABFE$ and the midpoint of the edge FG. The former is also the midpoint of the line segment AF, which is given by

$$\frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OF}\right) = \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix}.$$

Similarly, the midpoint of the edge FG is given by

$$\frac{1}{2}\left(\overrightarrow{OF} + \overrightarrow{OG}\right) = \begin{pmatrix} 5/2 \\ 5 \\ 10 \end{pmatrix}.$$

Thus, the vector equations for the two lines are

$$l_1 : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix} \qquad \text{and} \qquad l_2 : \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -5/2 \\ 10 \\ 20 \end{pmatrix}.$$

Now, to find the intersection point, we need to solve the following system of equations.

$$\begin{cases} 5\lambda = 5 - \frac{5}{2}\mu \\ 5\lambda = 10\mu \\ 10\lambda = 20\mu. \end{cases}$$

It is straightforward to see that the latter two equations imply the same solution $\lambda = 2\mu$. Substituting into the first one, then we can conclude that $\mu = 2/5$, and consequently $\lambda = 4/5$. Finally, plugging the value of λ into the vector equation of l_1, we can find the point of intersection, which is $(4, 4, 8)$.

5. (a) Using the substitution suggested in the text, we start by rewriting the integrand:

$$\frac{1}{(x-1)\sqrt{x^2-1}} = \frac{1}{u^{-1}\sqrt{(1+u^{-1})^2-1}} = \frac{u}{\sqrt{u^{-2}+2u^{-1}}}.$$

We also compute

$$\frac{dx}{du} = -\frac{1}{u^2}.$$

Hence the integral can be rewritten as

$$I = \int \frac{u}{\sqrt{u^{-2}+2u^{-1}}}(-u^{-2})du.$$

We now use algebra to simplify this:

$$\begin{aligned} I &= -\int \left(\frac{1}{u^2}+\frac{2}{u}\right)^{-1/2}\left(\frac{1}{u}\right)du \\ &= -\int \left(\frac{1+2u}{u^2}\right)^{-1/2}\left(\frac{1}{u}\right)du \\ &= -\int (1+2u)^{-1/2}du. \end{aligned}$$

This is an integral we can solve to give

$$\begin{aligned} I &= -\frac{(1+2u)^{1/2}}{1/2}\frac{1}{2}+c \\ &= -(1+2u)^{1/2}+c, \end{aligned}$$

where c is the constant of integration. Finally, using that $u = 1/(x-1)$, we derive that

$$I = -\left(1+2\frac{1}{x-1}\right)^{1/2}+c = -\left(\frac{x+1}{x-1}\right)^{1/2}+c.$$

(b) To answer this question, it is clear we need to use the result obtained in the previous part. Indeed, by applying the limits of integration, we obtain

$$I_1 := \int_{\sec\alpha}^{\sec\beta} \frac{1}{(x-1)\sqrt{(x^2-1)}}dx = -\left(\frac{\sec\beta+1}{\sec\beta-1}\right)^{1/2} + \left(\frac{\sec\alpha+1}{\sec\alpha-1}\right)^{1/2}.$$

The difficulty is converting this to the desired form. Observing that the above expression involves the angles α and β, but the solution is in terms of $\alpha/2$ and $\beta/2$, our first guess is to somehow use a half-angle formula for trigonometric functions. To start with, using the definition of the secant of an angle α, that is $\sec\alpha = 1/\cos\alpha$, we rewrite the previous expression as

$$I_1 = -\left(\frac{1+\cos\beta}{1-\cos\beta}\right)^{1/2} + \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^{1/2}.$$

Now we can use the half-angle trigonometric formulae, $2\cos^2\left(\frac{\alpha}{2}\right) = 1+\cos\alpha$ and $2\sin^2\left(\frac{\alpha}{2}\right) = 1-\cos\alpha$, to obtain

$$\begin{aligned} I_1 &= -\left(\frac{\cos^2\left(\frac{\beta}{2}\right)}{\sin^2\left(\frac{\beta}{2}\right)}\right)^{1/2} + \left(\frac{\cos^2\left(\frac{\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)}\right)^{1/2} \\ &= -\frac{\cos\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}. \end{aligned}$$

Finally, using the definition of cotangent of an angle, $\cot(\alpha) = \cos(\alpha)/\sin(\alpha)$, we conclude that

$$I_1 = \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right).$$

6. (a) To show that $A = x^2 + y^2$ is maximised when $x = \pm y$, we will start by using the fact that $x^4 + y^4 = 1$ to reduce the expression for A to one variable. In particular, applying $x^4 + y^4 = 1$, or equivalently $y^4 = 1 - x^4$, we can compute:

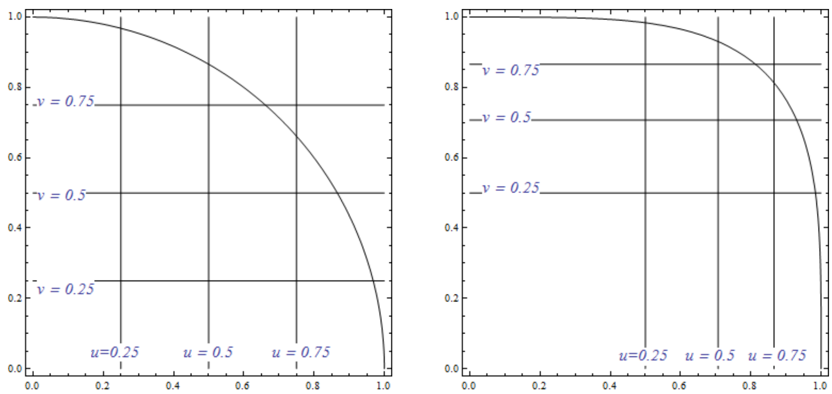
$$A = x^2 + y^2 = x^2 + \sqrt{1 - x^4}.$$

Now to find the stationary points of this expression, we will calculate its first derivative with respect to x, and equate it to zero. Since

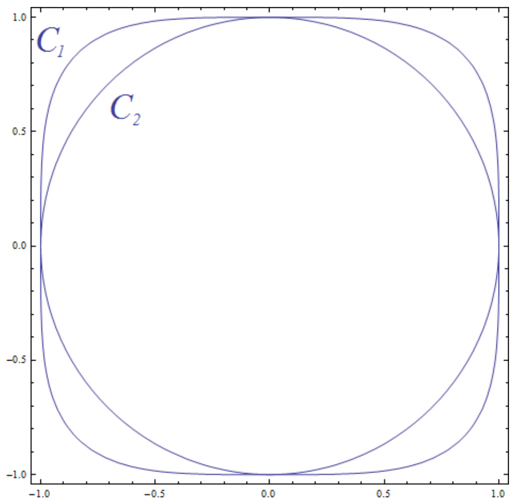
$$\begin{aligned} \frac{dA}{dx} &= 2x - \frac{1}{2\sqrt{1-x^4}}4x^3 \\ &= 2x\left(1 - \frac{x^2}{\sqrt{1-x^4}}\right), \end{aligned}$$

we find that there are stationary points at $x = 0$ and where $x^2/\sqrt{1-x^4} = 1$. When $x = 0$, we have that $A = 1$. For the second solution, note that it corresponds to $x^2 = \sqrt{1-x^4}$, which consequently implies $x^2 = y^2$ (recall that $y^4 = 1 - x^4$), and thus $x = \pm y$. Now, when $x = \pm y$, recalling the initial constraint $x^4 + y^4 = 1$, it follows that $x^4 = 1/2$ and consequently $x^2 = y^2 = 1/\sqrt{2}$, since $x = \pm y$. In this case, we therefore have $A = \sqrt{2}$. Thus, in summary, the minimum of A is 1 (and is obtained at $x = 0$, $y = \pm 1$), while the maximum is $\sqrt{2}$, which is attained when $x = \pm y$.

(b) We immediately recognise that C_2 is the equation of a circle of radius 1 centred at the origin. For C_1 one should expect a behaviour similar to that of C_2, resembling a circle, but drawing it does take some thought. First, let us set $u = x^2$ and $v = y^2$, then we can rewrite the equation $x^4 + y^4 = 1$ as $u^2 + v^2 = 1$, which is clearly the unit circle centred at the origin in the (u, v)-plane (the figure on the left-hand side below shows the curve in the first quadrant of this plane). We now need to consider how the curve is transformed when we revert back to the (x, y)-plane. To do this, we note that lines of the form $u = c$ appear in the (x, y)-plane at $x^2 = c$, that is $x = \sqrt{c}$. Since for $0 \le c \le 1$ we have that $\sqrt{c} \ge c$ (with equality only when $c = 0$ or $c = 1$), one can sketch how the relevant part of the (u, v)-plane is deformed away from the origin (the right-hand figure shows the (x, y)-plane):



In particular, the curve C_1 passes through the points $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$ but is otherwise further from the origin than C_2. Hence we can sketch these curves as follows:



(c) The equation of a generic circle, centred at the origin is $x^2 + y^2 = c^2$. To derive the value c of the radius of the circle C_3 touching C_1 at the points where $x = \pm y$, we have to consider a system of equations with the equations of C_1 and C_3 and the constraints $x = \pm y$, i.e.

$$\begin{cases} x^4 + y^4 = 1, \\ x^2 + y^2 = c^2, \\ x = \pm y, \end{cases}$$

which are equivalent to

$$\begin{cases} 2x^4 = 1, \\ 2x^2 = c^2, \\ x = \pm y. \end{cases}$$

Consequently, from the equation for C_1, one can derive $x^4 = 1/2$, and thus $x^2 = \frac{1}{\sqrt{2}}$. It follows that, $c^2 = \sqrt{2}$, and so the equation of C_3 is $x^2 + y^2 = \sqrt{2}$.

7. (a) Recalling the trigonometric addition formulae

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b, \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b, \end{aligned}$$

and that $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, we can rewrite $f(x)$ as

$$\left[1 + \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right]\left[1 + \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right].$$

By multiplying out the terms in the previous expression, we have that

$$f(x) = 1 + \frac{2}{\sqrt{2}}\cos(x) + \frac{1}{2}\cos^2(x) - \frac{1}{2}\sin^2(x).$$

Since the question requires to write $f(x)$ only as a function of the cosine, we eliminate the sine using that $\sin^2(x) = 1 - \cos^2(x)$. By simple rearrangements, it holds that

$$f(x) = \frac{1}{2} + \frac{2}{\sqrt{2}}\cos(x) + \cos^2(x) = \left(\frac{1}{\sqrt{2}} + \cos x\right)^2,$$

as required.

(b) To determine the range of $f(x)$, we need to find its maximum and its minimum. Since $-1 \le \cos x \le 1$, the maximum value of $f(x)$ is attained when $\cos x = 1$ and corresponds to

$$\left(\frac{1}{\sqrt{2}} + 1\right)^2 = \frac{1}{2} + 1 + \frac{2}{\sqrt{2}} = \frac{3}{2} + \sqrt{2}.$$

For the minimum, we just have to note that $1/\sqrt{2} + \cos x$ can be equal to zero (e.g. at $x = 3\pi/4$), and, because of the presence of the square, this corresponds to the minimum. Thus, the range of $f(x)$ is $[0, \frac{3}{2} + \sqrt{2}]$.

(c) In the previous part of the question we deduced that global maxima are obtained when $\cos x = 1$, which implies that $x = 0$ or $x = 2\pi$. Thus, the coordinates of the two global maxima are $(0, \frac{3}{2} + \sqrt{2})$ and $(2\pi, \frac{3}{2} + \sqrt{2})$. For the global minima, we have to find the solutions of $\frac{1}{\sqrt{2}} + \cos(x) = 0$, which are $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$. To find the x-coordinate of the local maxima of the curve, we will compute the first derivative and find where it is equal to zero. In particular, we need to solve

$$f'(x) = -2\sin x\left(\frac{1}{\sqrt{2}} + \cos x\right) = 0.$$

The solutions are given by solutions to $\sin x = 0$ or $\frac{1}{\sqrt{2}} + \cos x = 0$. The only solution that does not correspond to a point we have already identified is $x = \pi$. At this value of x, we have that $\cos x = -1$, and so

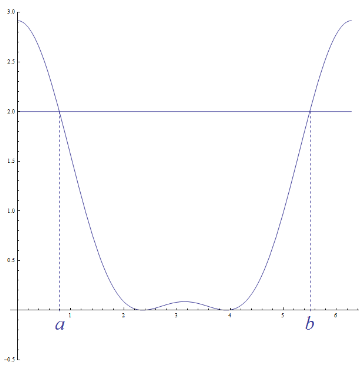
$$f(x) = \left(\frac{1}{\sqrt{2}} + \cos(x)\right)^2 = \frac{3}{2} - \sqrt{2}.$$

Thus, the local maximum has coordinates $(\pi, \frac{3}{2} - \sqrt{2})$.

(d) The region R is enclosed by the horizontal line $y = 2$ and by $f(x)$. Its area can be computed by solving the integral

$$A := \int_a^b (2 - f(x))\, dx,$$

where a and b correspond to the x-coordinate of the points of intersection between these two lines (see the following figure).



To compute a and b, we need to solve

$$\left(\frac{1}{\sqrt{2}} + \cos x\right)^2 = 2.$$

Taking square roots of both sides of this equation gives

$$\frac{1}{\sqrt{2}} + \cos x = \sqrt{2},$$

which can be rewritten as $\cos x = \frac{1}{\sqrt{2}}$. This has solutions $x = \frac{\pi}{4}$ and $x = \frac{7\pi}{4}$. (Note that we have not considered the equation $1/\sqrt{2} + \cos x = -\sqrt{2}$, since it would have had no solutions.)

Now, to compute the value of the area of R, we start deriving the indefinite integral of $2 - f(x)$:

$$\int (2 - f(x))dx = \int \left(2 - \frac{1}{2} - \cos^2 x - \sqrt{2}\cos x\right) dx.$$

All the terms here are straightforward to deal with, apart from the $\cos^2 x$. For this, we will use the double-angle trigonometric formula to substitute $\cos^2 x$ with $\frac{1}{2}(\cos(2x) + 1)$, so that

$$\begin{aligned} \int (2 - f(x))dx &= \int \left(1 - \frac{1}{2}\cos(2x) - \sqrt{2}\cos x\right) dx \\ &= x - \frac{1}{4}\sin(2x) - \sqrt{2}\sin x + c, \end{aligned}$$

where c is a constant of integration. At this point we can consider the definite version of the integral just computed, with limits $\pi/4$ and $7\pi/4$, obtaining

$$A = \left(\frac{7\pi}{4} - \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{4}(-1)\right) - \left(\frac{\pi}{4} - \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{4}(1)\right).$$

After some simplification, this gives that

$$A = \frac{3\pi}{2} + \frac{5}{2}.$$