Adjusting mis-specified likelihood functions

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Overview of talk

1. **Problem statement** (setup and notation; log-likelihoods; potential applications)
2. **Standard asymptotics for mis-specified likelihoods** (definition of estimator; large-sample properties of estimator and log likelihood ratio)
3. **Adjusting the working log-likelihood** (motivation; options for adjustment; geometry of adjustment in 1-D; multiparameter example; comparing nested models; other applications)
4. **Open questions**

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- Interested in / tractable model available for low-dimensional margins of joint distributions.
- Conditionally upon $\mathcal{C}_j$, low-dimensional margins are fully determined by $\theta \rightarrow \alpha$ is nuisance parameter for high-dimensional joint structure.

Log-likelihoods

- Full log-likelihood function is $\ell_{\text{FULL}}(\theta, \alpha) = \sum_{j=1}^{k} \log f_j\left(\mathbf{y}_j | \mathcal{C}_j; \theta, \alpha\right)$, but joint distributions usually difficult to model.

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  - **(Weighted) log pairwise likelihood**: $\ell_{\text{PAIR}}(\theta) = \sum_{j=1}^{k} w_j \sum_{i_1=1}^{n_j-1} \sum_{i_2=i_1+1}^{n_j} \log f_{i_1,i_2,j}(y_{i_1 j}, y_{i_2 j}|\mathcal{C}_j;\theta)$ so that $\log \tilde{f}_j(y_j|\mathcal{C}_j;\theta) = w_j \sum_{i_1=1}^{n_j-1} \sum_{i_2=i_1+1}^{n_j} \log f_{i_1,i_2,j}(y_{i_1 j}, y_{i_2 j}|\mathcal{C}_j;\theta)$.

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- **Space-time data (multiple time series):**
  - 'Clusters' are observations made at same time instant
  - Temporal autocorrelation may be present — can be handled by including previous observations into conditioning sets $\{\mathcal{C}_j\}$

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- Estimator $\hat{\theta}_{\mathrm{WORK}}$ satisfies $\mathbf{U}\left(\hat{\theta}_{\mathrm{WORK}}\right) = \sum_{j=1}^{k} \mathbf{U}_j\left(\hat{\theta}_{\mathrm{WORK}}\right) = \mathbf{0}$.

Large-sample properties of $\hat{\theta}_{\mathrm{WORK}}$

- Usual asymptotics hold e.g. for large k, $\hat{\theta}_{\mathrm{WORK}} \sim N\left(\theta_0, \mathbf{H}\mathbf{V}^{-1}\mathbf{H}\right)$ where

$$\mathbf{H} = \mathrm{E}\left(\left.\frac{\partial^2 \ell_{\mathrm{WORK}}}{\partial\theta\partial\theta'}\right|_{\theta=\theta_0}\right), \quad \mathbf{V} = \mathrm{Var}\left[\sum_{j=1}^{k} \mathbf{U}_j(\theta_0)\right] = \sum_{j=1}^{k} \mathrm{E}\left[\mathbf{U}_j(\theta_0)\,\mathbf{U}_j(\theta_0)'\right]$$

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- **NB** other techniques required for small k — application-dependent

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- Can be used for profile-based inference on components of θ.

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  - Borrow profile from $\ell_{\mathrm{WORK}}(\theta)$ — hopefully informative.

Options for adjustment

Horizontal scaling: define $\ell_{\mathrm{ADJ}}(\theta) = \ell_{\mathrm{WORK}}(\theta^*)$, where

$$\theta^* = \hat{\theta}_{\mathrm{WORK}} + \mathbf{M}^{-1}\mathbf{M}_{\mathrm{ADJ}}\left(\theta - \hat{\theta}\right)$$

with $\mathbf{M}'\mathbf{M} = \hat{\mathbf{H}}$, $\mathbf{M}'_{\mathrm{ADJ}}\mathbf{M}_{\mathrm{ADJ}} = \hat{\mathbf{H}}_{\mathrm{ADJ}}$. Possible choices for $\mathbf{M}$, $\mathbf{M}_{\mathrm{ADJ}}$:

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- Options asymptotically equivalent (and identical in quadratic case)
- Vertical scaling has practical (and theoretical) advantages

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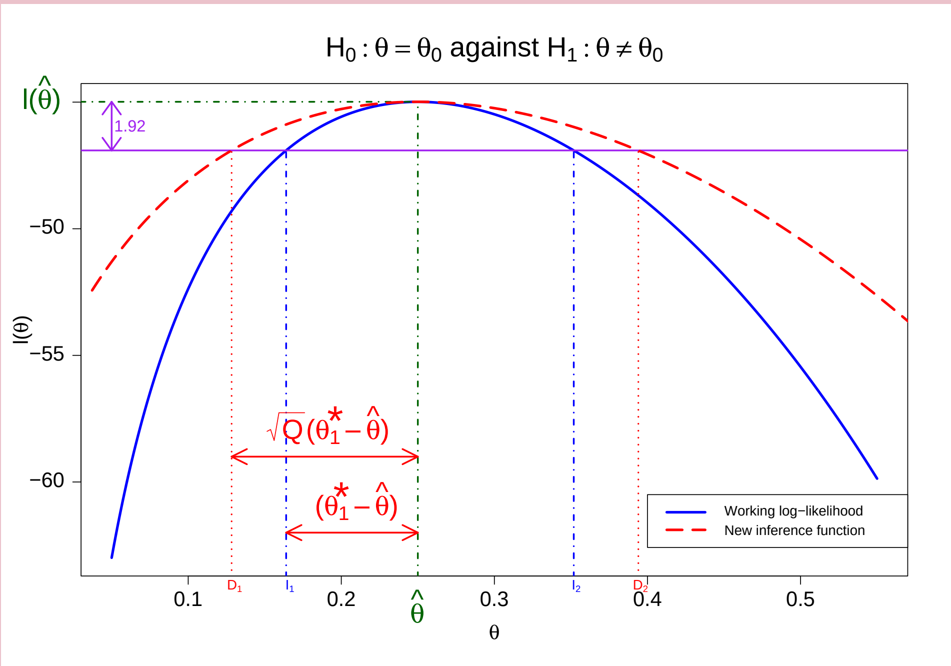

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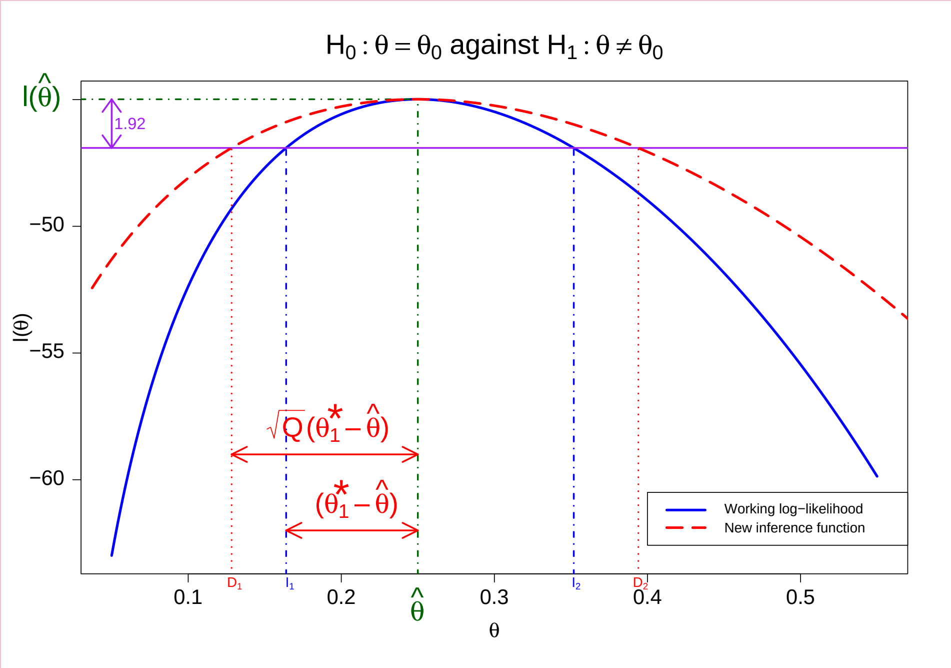

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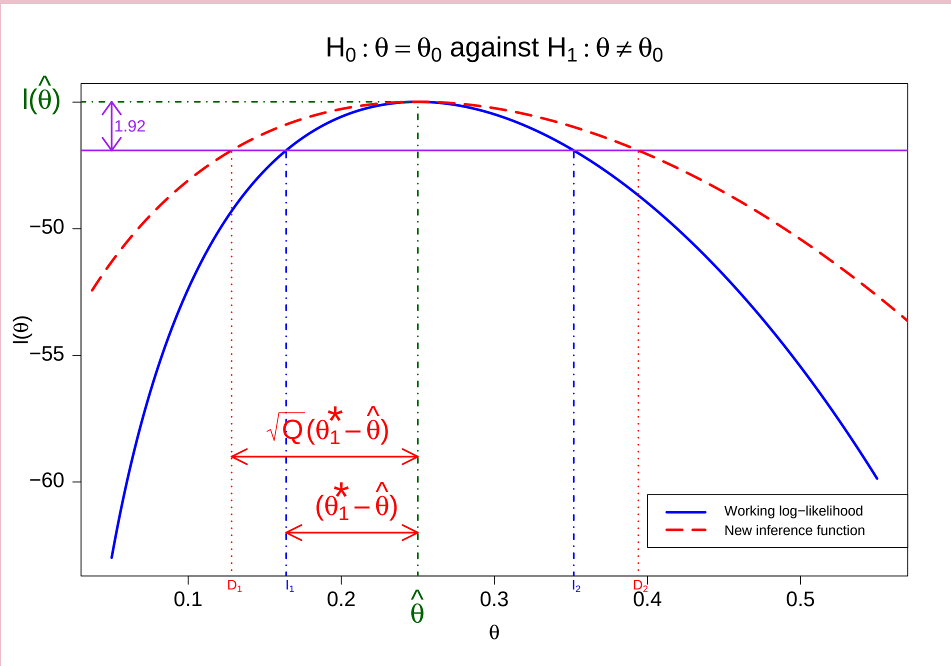

- Horizontal scaling is by ratio of robust to naïve standard errors.
- Vertical scaling is by ratio of robust to naïve variances (same as adjusting critical value)

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- **NB** contours of ℓ_{IND} are always circular — hence classical approach of adjusting critical value is sub-optimal.

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- Adjustment preserves χ^2 asymptotics by construction $\Rightarrow$ to test $H_0 : \Delta\theta = \delta_0$, use statistic $\Lambda_{\mathrm{ADJ}} = 2\left\{\ell_{\mathrm{ADJ}}\left(\hat{\theta}_{\mathrm{WORK}}\right) - \ell_{\mathrm{ADJ}}\left(\widetilde{\theta}_{\mathrm{ADJ}}\right)\right\}$, where $\widetilde{\theta}_{\mathrm{ADJ}}$ maximises ℓ_{ADJ} under H_0.

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- **Details:** Chandler & Bate, *Biometrika*, 2007.

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- **Example:** generalised method of moments — $\hat{\theta} = \arg\min_\theta S(\theta; \mathbf{y})$, where:
  - $S(\theta; \mathbf{y}) = \sum_{r=1}^{p} w_r \left[T_r(\mathbf{y}) - \tau_r(\theta)\right]^2$
  - $\{T_r(\mathbf{y}) : r = 1, \ldots, p\}$ are statistics (e.g. sample moments)
  - $\tau_r(\theta) = \mathrm{E}_\theta\left[T_r(\mathbf{y})\right]$ $(r = 1, \ldots, p)$.
  - $\{w_r : r = 1, \ldots, p\}$ are weights (independent of θ).

1. Problem statement
2. Standard asymptotics for mis-specified likelihoods
3. Adjusting the working log-likelihood
4. Open questions

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  - Requirement seems to be existence of joint densities $\left\{f_j\left(\mathbf{y}_j | \mathcal{C}_j; \theta, \alpha\right)\right\}$ for which adjustment recovers profile log-likelihood for θ (asymptotically?)

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ANY QUESTIONS / SUGGESTIONS?