Modified Gaussian likelihood estimators for ARMA models on $\mathcal{Z}^d$

Chrysoula Dimitriou-Fakalou
University College London
Department of Statistical Science
Email: chrysoula@stats.ucl.ac.uk

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The problem

- The analysis of spatial and SPATIO-TEMPORAL statistics has often replaced the classical time series analysis.
- The inclusion of more than one dimension (i.e. the time axis) in the statistical analysis, has made it necessary to study the stationary processes that take place on $\mathcal{Z}^d$, where d can be ANY positive integer (and $\mathcal{Z} = \{0, \pm 1, \cdots\}$).
- However, when trying to establish the statistical properties of the standard (maximum Gaussian likelihood) estimators for the parameters of interest, obstacles of MATHEMATICAL NATURE arise (known as the 'edge-effects').

Main reference paper:

- GUYON, X. (1982). Parameter Estimation for a Stationary Process on a d-Dimensional Lattice. *Biometrika* **69** 95-105.

Other related papers:

- DAHLHAUS, R. and KUNSCH, H. (1987). Edge Effects and Efficient Parameter Estimation for Stationary Random Fields. *Biometrika* **74** 877-882.
- YAO, Q. and BROCKWELL, P. J. (2006). Gaussian Maximum Likelihood Estimation for ARMA models II: Spatial Processes. *Bernoulli* **12** 403-429.

The purpose of this work:

- To propose a MODIFIED GAUSSIAN LIKELIHOOD, which if maximized, produces consistent and asymptotically normal estimators for ANY DIMENSIONALITY d (like Guyon (1982)).
- To use the ARMA model as a tool to represent almost ANY stationary process on $\mathcal{Z}^d$ (unlike Guyon (1982)). Thus, to use the TIME DOMAIN and propose a SIMPLE quantity to maximize.
- To take advantage of the smart characteristics of an ARMA process and defeat the edge-effects. Thus, to establish the ASYMPTOTIC NORMALITY of the estimators proposed.

What is the 'edge-effect'?

- It is a problem of MATHEMATICAL nature; it is not related to what the dimensions represent but to how many dimensions are allowed in the analysis (implying number of data recordings for these dimensions tending to infinity for inference).

- The standard maximum Gaussian likelihood estimators ARE asymptotically unbiased. However, the speed with which the bias tends to 0 is equal ($d=2$) or slower ($d>2$) than the speed of the standard error. Thus, the ASYMPTOTIC NORMALITY of the estimators CANNOT be established.

What is the modified likelihood to be maximized?

For observations from the process $\{Z(\mathbf{v}),\ \mathbf{v}^{\tau} \in \mathcal{Z}^d\}$, which satisfies the ARMA equation

$$Z(\mathbf{v}) - \sum_{n=1}^{p} b_{\mathbf{i}_n} Z(\mathbf{v}-\mathbf{i}_n) = \varepsilon(\mathbf{v}) + \sum_{m=1}^{q} a_{\mathbf{j}_m} \varepsilon(\mathbf{v}-\mathbf{j}_m)$$

where $\{\varepsilon(\mathbf{v})\}$ are zero-mean uncorrelated random variables with variance σ^2, we work as follows:

- For the polynomial $b(\mathbf{z}) = 1 - \sum_{n=1}^{p} b_{\mathbf{i}_n} \mathbf{z}^{\mathbf{i}_n}$, we create the MOVING-AVERAGE PROCESS

  $$M(\mathbf{v}) = b(\mathbf{B}) b(\mathbf{B}^{-1}) Z(\mathbf{v})$$

  as a function of the unknown auto-regressive parameters (we write $\mathbf{B}$ for the vector backshift operator).

- For the polynomial $a(\mathbf{z}) = 1 + \sum_{m=1}^{q} a_{\mathbf{j}_m} \mathbf{z}^{\mathbf{j}_m}$, we create the polynomial

  $$d(\mathbf{z}) = \{a(\mathbf{z}) a(\mathbf{z}^{-1}) b(\mathbf{z}) b(\mathbf{z}^{-1})\}^{-1}$$

  as a function of both the unknown auto-regressive and moving-average parameters.

- We minimize the quantity

  $$\sum_{\mathbf{v}} M(\mathbf{v}) \sum_{\mathbf{j}} d_{\mathbf{j}}\ M(\mathbf{v}-\mathbf{j}).$$

Results

1. Using standard arguments on variables transformations, it can be shown that minimizing the proposed quantity is the same as MAXIMIZING A GAUSSIAN LIKELIHOOD OF THE OBSERVATIONS.

2. The quantity is a (-) MODIFIED Gaussian log-likelihood, since the summations $\sum_{\mathbf{v}}$ and $\sum_{\mathbf{j}}$ extend in a 'convenient' way to tackle the 'edge-effects'. The trick is that not only $M(\mathbf{v})$ is a moving-average sequence but so are its derivatives with respect to any $b_{\mathbf{i}_n},\ n=1,\cdots,p$. That would not be true if we had used the moving-average sequences $b(\mathbf{B})Z(\mathbf{v})$ or $b(\mathbf{B}^{-1})Z(\mathbf{v})$ originally...

3. The estimators proposed are not only consistent but also ASYMPTOTICALLY NORMAL. Their variance matrix reproduces Hannan's asymptotic result (1973) for the standard Gaussian likelihood estimators of ARMA models on $\mathcal{Z}$. We have not had such a result before for ANY dimensionality d.

4. A finite FOURTH moment for the process is required; this is the price we pay for increasing the dimensionality from $d=1$ (only a finite second moment is required then).

Causality - Invertibility - Unilaterality

We start by defining a CAUSAL POLYNOMIAL

$$b(\mathbf{z}) = 1 - \sum_{n=1}^{p} b_{\mathbf{i}_n} \, \mathbf{z}^{\mathbf{i}_n}, \;\; \mathbf{0} < \mathbf{i}_1 < \cdots < \mathbf{i}_p,$$

to be such that

$$b(\mathbf{z})^{-1} = 1 + \sum_{\mathbf{j}>\mathbf{0}} \Phi_{\mathbf{j}} \, \mathbf{z}^{\mathbf{j}}, \;\; \sum_{\mathbf{j}>\mathbf{0}} |\Phi_{\mathbf{j}}| < \infty.$$

The '$\mathbf{j} > \mathbf{0}$' refers to the UNILATERAL ORDERING of locations, as this was given by Whittle (1954) for $d=2$ and generalized by Guyon (1982) for higher dimensionalities.

Consequently, causal and invertible or unilateral ARMA models can be defined based on causal auto-regressive and moving-average polynomials.

However, it is often claimed that the unilateral ordering of locations is UNNATURAL and MEANINGLESS (eg. spatial processes). A BILATERAL ARMA model might be preferred to represent the second-order dependence over space then.

For bilateral schemes, we remember the following:

- Their parameters to be estimated ARE NOT best linear prediction coefficients (bilateral auto-regressions).

- The deterministic part of the Gaussian likelihood needs to be fixed twice (following the route of Whittle (1954), who fixed it for bilateral auto-regressions only), in order to secure the fact that the estimators derived are not inconsistent. Thus, OUR RESULTS STILL APPLY FOR BILATERAL ARMA MODELS. In the contrary, Guyon's (1982) abstract expression of a stationary process does not allow for the simple generalization of Whittle's (1954) result.