Weierstrass Institute for
Applied Analysis and Stochastics

Quantification of noise in MR experiments

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Outline

- Noise distribution in single and multiple coil systems
- Diffusion MRI
- Problems in modeling low SNR data
- Localized noise quantification
- Examples

- complex signal in K-space (one coil):

$$s_c(k) \sim N(x_c(k), \sigma_K^2)$$

- FFT provides complex valued image

$$S_c(x) \sim N(\xi_c(x), \sigma_I^2)$$

- MR image: $S(x)$ usually obtained as magnitude image
  Notation: $S_i = |S(x_i)|$
- Signal distribution: scaled noncentral χ
  $S_i/\sigma_I \sim \chi_{2,\eta_i}$ with $\eta_i = |\xi_c(x_i)|/\sigma_I$
  (Rician distribution Gudbjatrtsson (1995))
- Problem:
  $\mathbf{E} S_i/\sigma_I > \eta_i$

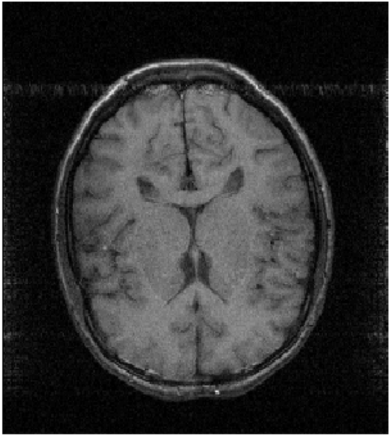


Image: F. Godtliebsen (Tromsœ)

Image reconstruction from multiple coils

- $8 - 32$ spherically arranged receiver coils
- inhomogeneous coil sensitivities or encoding matrices, correlation between receiver coils
- image reconstruction from coils $k = 1, \ldots, K$ as

  SOS:

  $$S_i = \sqrt{\sum_k S_k(x_i)^2}$$

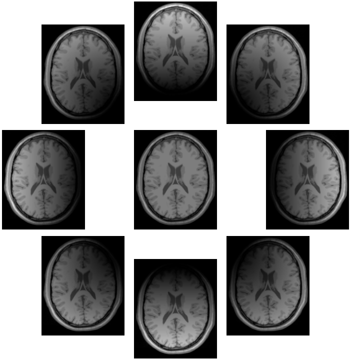
  inefficient, but known distribution, location dependent $\sigma_{I,i}$

  $$S_i / \sigma_{I,i} \sim \chi_{2K,\eta_i}$$

  with

  $$\eta_i = \sqrt{\sum_k \xi_k(x_i)\bar{\xi}_k(x_i) / \sigma_{I,i}}$$

  $\sigma_{I,i}$ depends on correlations



8-coil system (noiseless situation): Images from receiver coils and combined image

- $8 - 32$ spherically arranged receiver coils
- inhomogeneous coil sensitivities or encoding matrices, correlation between receiver coils
- image reconstruction from coils $k = 1, \ldots, K$ as

  GRAPPA: (Griswold 2002)

  E_k encoding matrices (coil sensitivity + FFT), $E = (E_1, \ldots, E_K), \Psi = cov(\epsilon)$

$$s_k = E_k S_k + \epsilon_k$$

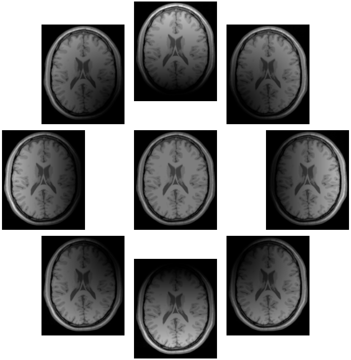
$$S = (E^H \Psi^{-1} E)^{-1} E^H \Psi^{-1} (s_1, \ldots, s_K)$$

efficient, but only approximatively

$$S_i / \sigma_{I,i} \sim \chi_{2L_i, \eta_i}$$

with location dependent L_i and $\sigma_{I,i}$

$\sigma_{I,i}$ depends on encoding matrices, correlations



8-coil system (noiseless situation): Images from receiver coils and combined image

- $8 - 32$ spherically arranged receiver coils
- inhomogeneous coil sensitivities or encoding matrices, correlation between receiver coils
- image reconstruction from coils $k = 1, \ldots, K$ as

  SENSE: (Pruessmann 1999), SENSE-1: (Sotiropoulos 2013)

$$S_i = |\sum c_{ik} S_k(x_i)|$$

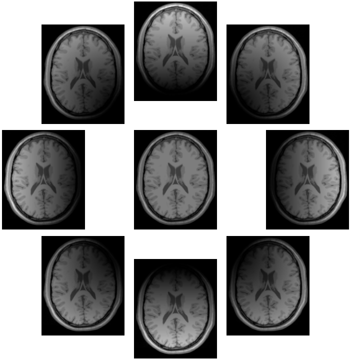
efficient, known distribution, location dependent $\sigma_{I,i}$, low df

$$S_i / \sigma_{I,i} \sim \chi_{2,\eta_i}$$

with

$$\eta_i = |\sum c_{ik} \xi_k(x_i)| / \sigma_{I,i}$$

$\sigma_{I,i}$ depends on coil sensitivities, correlations



8-coil system (noiseless situation): Images from receiver coils and combined image

Additional diffusion gradient:

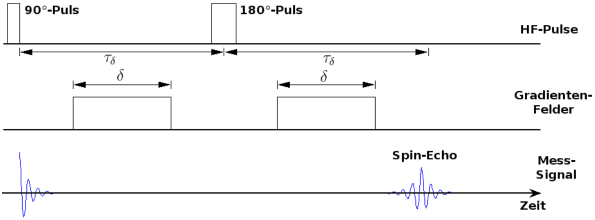


Figure: Thomas Schultz (Wikimedia)

Object of interest:

- Diffusion propagator: $P(\vec{r}, \vec{r}\,', \tau)$ - probability density for a particle (spin) to "travel" from position $\vec{r}\,'$ to $\vec{r}$ in time τ
- Aggregate over a voxel V (Ensemble Averaged Propagator, EAP):

$$P(\vec{R}, \tau) = \int_{\vec{r}\,' \in V,\ \vec{R} = \vec{r} - \vec{r}\,'} P(\vec{r}, \vec{r}\,', \tau)\, p(\vec{r}\,')\, d\vec{r}\,',$$

- $p(\vec{r}\,')$ - initial probability density of particle location

- Diffusion gradients lead to signal attenuation due to diffusion process - loss of phase coherence between precessing spins:

$$\zeta(\vec{q},\tau) = \zeta_0 \langle \exp(i\varphi) \rangle \qquad S(\vec{q},\tau)/\sigma_I \sim \chi_{2L,\zeta(\vec{q},\tau)/\sigma_I}$$

- Fourier relation

$$\zeta(\vec{q},\tau) = \zeta_0 \int_{\mathbb{R}^3} P(\vec{R},\tau)\, e^{i\vec{q}\,\vec{R}} d\vec{R}$$

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- Fourier relation

$$\zeta(\vec{q},\tau) = \zeta_0 \int_{I\!R^3} P(\vec{R},\tau)\, e^{i\vec{q}\,\vec{R}} d\vec{R}$$

- Experiment: Measure $S(\vec{q},\tau)$, $q = cb^{1/2}g$, (||g||=1) at N voxel locations (e.g. $N = 128 \cdot 128 \cdot 60$) for $6, \ldots, 200$ $\vec{g}$ vectors and (multiple) b
- Diffusion tensor model: Assumes Gaussian diffusion

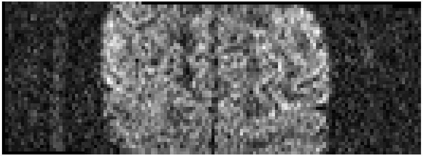
$$\zeta_i = \zeta_{0,i} \exp(-b\vec{g}^\top \mathcal{D}_i \vec{g})$$

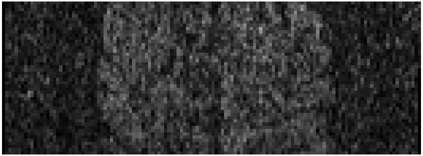
- Diffusion tensor: $\mathcal{D}_i$, Gradient direction $\vec{g}$, Gradient strength (b-value) b
- Characteristics: Fractional FA, eigenvectors

$$\mathrm{FA} = \sqrt{\frac{3}{2} \frac{(\lambda_1 - \bar{\lambda})^2 + (\lambda_2 - \bar{\lambda})^2 + (\lambda_3 - \bar{\lambda})^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

- Interest in high resolution dMRI
- Interest in more sophisticated models, e.g. Kurtosis tensor model, tensor mixtures –> need for multiple b-values

Both lead to low SNR situations



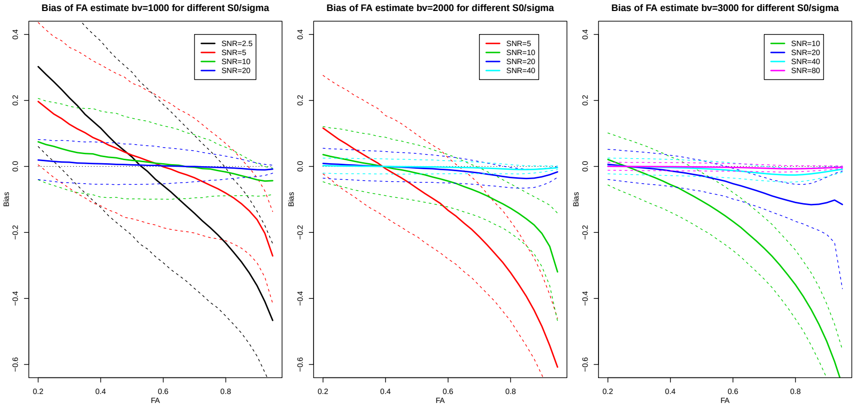


$1.2mm$ isotrope dMR images 3T (clinical scanner) , b-values 800 and 2000, Data: S. Mohammadi (UCL)

- can we quantify of low SNR on results
- can we handle low SNR situations –> here and K. Tabelows talk

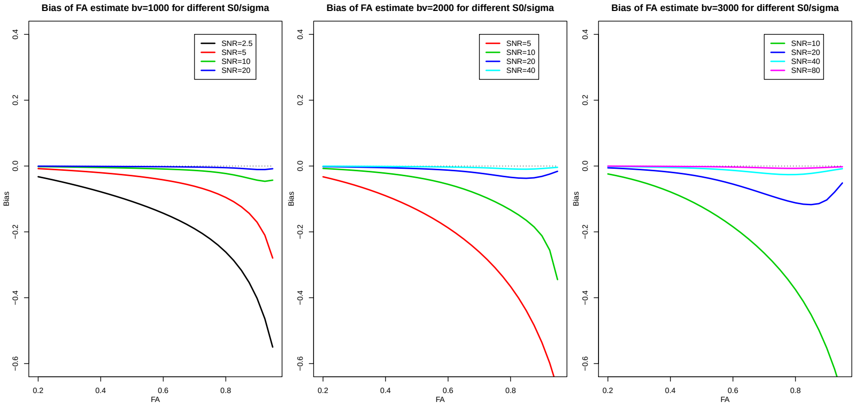
Estimation using (incorrect) nonlinear regression model:

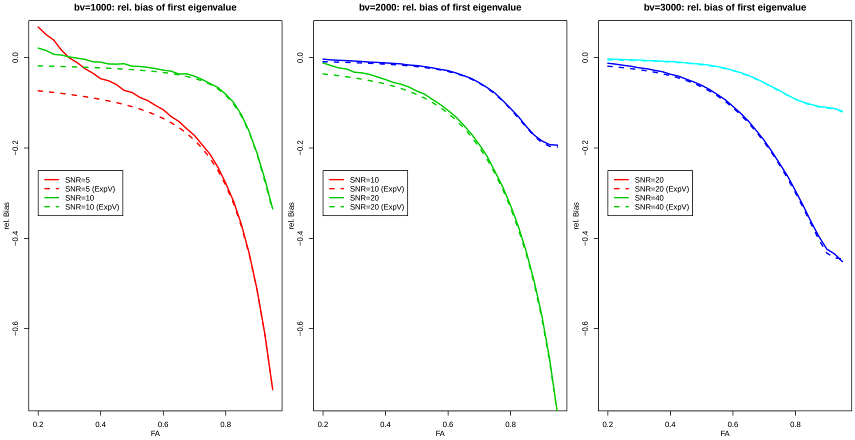
$$(\hat{\zeta}_{0,i}, \hat{\mathcal{D}}_i) = \underset{\zeta_{0,i}, \mathcal{D}_i}{\operatorname{argmin}} \sum_{b,g} (S_i(\vec{q}, \tau) - \zeta_{0,i} \exp(-b\vec{g}^\top \mathcal{D}_i \vec{g}))^2$$



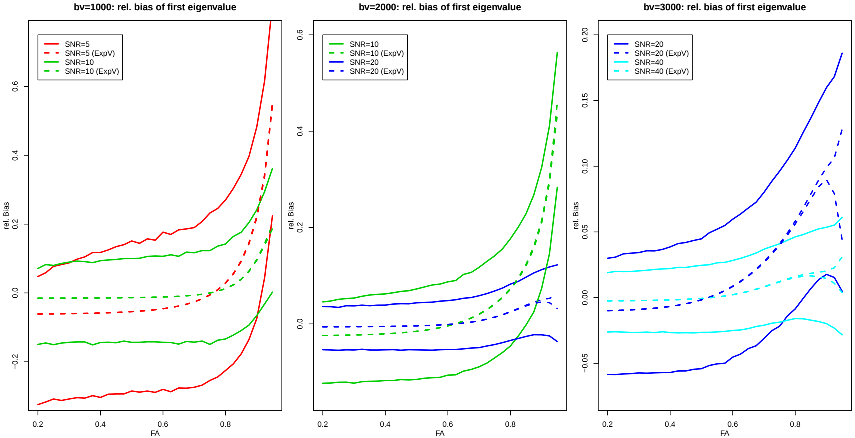
Bias due to model misspecification:

$$(\tilde{\zeta}_{0,i}, \hat{\mathcal{D}}_i) = \underset{\zeta_{0,i},\mathcal{D}_i}{\operatorname{argmin}} \sum_{b,g} (\mathbf{E}S_i(\vec{q},\tau) - \zeta_{0,i} \exp(-b\vec{g}^\top \mathcal{D}_i \vec{g}))^2$$





- Bias in eigenvalues is caused by model bias and data variability !
- need to address both !
- for both we need to quantify the data variability !



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Severe bias of estimates (and instability) for

- high resolution imaging (SNR proportional to voxel volume)
- high b-values

Correct models: $\mathcal{D}_i = R_i^T R_i,\ p = (\vec{g}, b),\ \ \zeta_{p,i} = \zeta_{p,0} exp(-b\vec{g}^T \mathcal{D}_i \vec{g})$

- Likelihood:

$$l(\{S_{p,i}\}_p; \{\sigma_{p,i}\}_p, L_i; \zeta_{0,i}, R_i) =$$

$$\sum_p \left[\log \left(\frac{S_{p,i}^{L_i} \zeta_{p,i}^{(1-L_i)}}{\sigma_{p,i}^2} \right) - \frac{1}{2} \left(\frac{S_{p,i}^2 + \zeta_{p,i}^2}{\sigma_{p,i}^2} \right) + \log \left(I_{L_i - 1} \left(\frac{\zeta_{p,i} S_{p,i}}{\sigma_{p,i}^2} \right) \right) \right],$$

- Quasi-likelihood: μ, ν expectation and variance of $\chi_{2L_i, \zeta_{p,i}/\sigma_{p,i}}$

$$\mathcal{R}(\{S_{p,i}\}_p; \{\sigma_{p,i}\}_p, L_i; \zeta_{0,i}, R_i) = \sum_p \left[\frac{(S_{p,i} - \mu(\zeta_{p,i}/\sigma_{p,i}, \sigma_{p,i}, L_i))^2}{\nu(\zeta_{p,i}/\sigma_{p,i}, \sigma_{p,i}, L_i)} \right]$$

Require correct assessment of $\sigma_{p,i}$ and L_i

Assumptions:

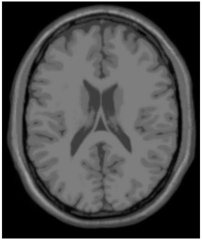
- $S_i/\sigma_i \sim \chi_{2L,\zeta_i/\sigma_i}$
- ζ_i local constant
- local homogeneity of tissue and fiber direction (diffusivity)
- σ_i slowly varying in space
- smooth variation of coil sensitivities

Sequential multi-scale algorithm

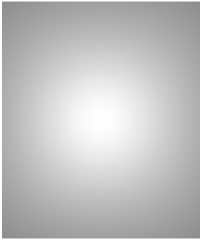
- Using local weighted likelihood estimates for ζ_i and σ_i
- Robust (median) smoothing for estimated σ_i
- Weighting schemes by localization in image space and adaptation in parameter space

Alternatives

- Global estimates from background
- Methods from Aja-Fernandez (201x), Landman (2009)



local variation of ζ_0



local variation of σ for artificial 8-coil system and SENSE-1

- Initialization: , $\tilde{\sigma}_i^{(0)} = \bar{\sigma}$ (global), $\hat{\zeta}_i^{(0)} = 1$, $N_i^{(0)} = 1, \forall i$, $k := 1$, Sequence of increasing bandwidths $h^{(k)}$.
- Compute adaptive weights

$$w_{ij}^{(k)} = K_{\text{loc}}\left(\frac{\|i-j\|}{h^{(k)}}\right) K_{\text{st}}\left(\frac{N_i^{(k-1)} \mathcal{KL}_{P_S}\left((\hat{\zeta}_i^{(k-1)}, \tilde{\sigma}_i^{(k-1)}), (\hat{\zeta}_j^{(k-1)}, \tilde{\sigma}_i^{(k-1)})\right)}{\lambda}\right)$$

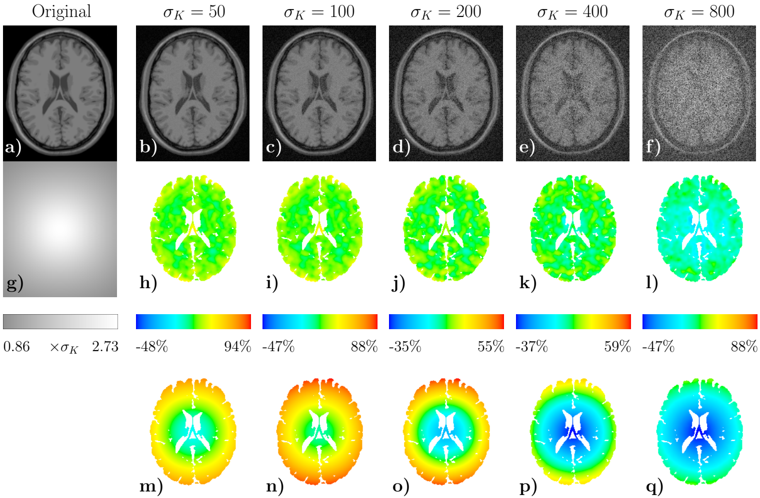
and $N_i^{(k)} = \sum_j w_{ij}^{(k)}$

- Estimation: If $N_i^{(k)} := \sum_j w_{ij}^{(k)} > N_0$ obtain estimates for ζ_i and σ_i by weighted log-likelihood

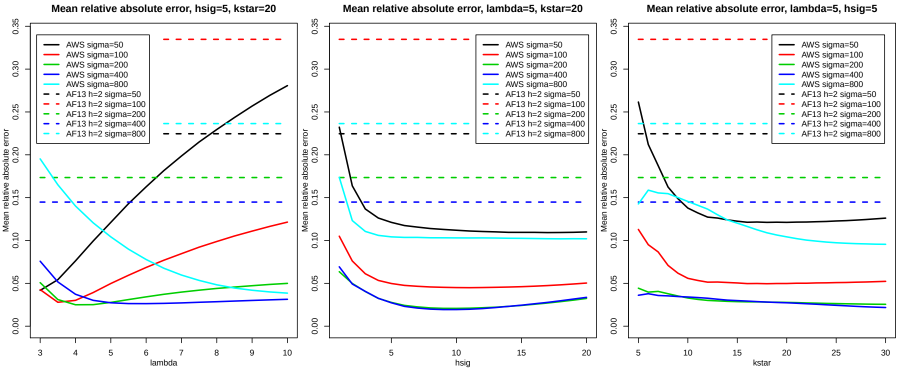
$$\left(\hat{\sigma}_i^{(k)}, \hat{\zeta}_i^{(k)}\right) = \underset{(\zeta,\sigma)}{\operatorname{argmax}} \sum_j w_{ij}^{(k)} \log p_S(S_j; \zeta, \sigma).$$

Otherwise set $\tilde{\sigma}_i^{(k)} := \tilde{\sigma}_i^{(k-1)}$ and $\hat{\zeta}_i^{(k)} := \sqrt{\left(\sum_j w_{ij}^{(k)} S_j^2 / \sum_j w_{ij}^{(k)} - 2L_i(\tilde{\sigma}_i^{(k-1)})^2\right)_+}$

- Median smoothing: If $N_i^{(k)} > N_0$ set $\tilde{\sigma}_i^{(k)} = \text{median}_{j:\|x_i - x_j\|_1 < h_{\text{med}}} \hat{\sigma}_j^{(k)}$
- Stopping: If $k = k^*$ stop, else increase k by 1 and recompute weights.



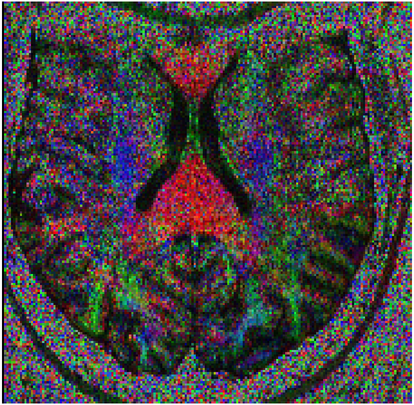
a) Original slice. b)-f) noisy SENSE1 reconstructions. g) locally varying effective σ_i parameters, h)-l) relative error of our local estimates $\hat{\sigma}_i$ m)-q) relative error of local estimates $\hat{\sigma}_i$ (Aja-Fernandez 2013)



Mean relative absolute error as a function of a) λ b) h_{med}, c) k^* for varying SNR. Results for Aja-Fernandez 2013 given for comparison.

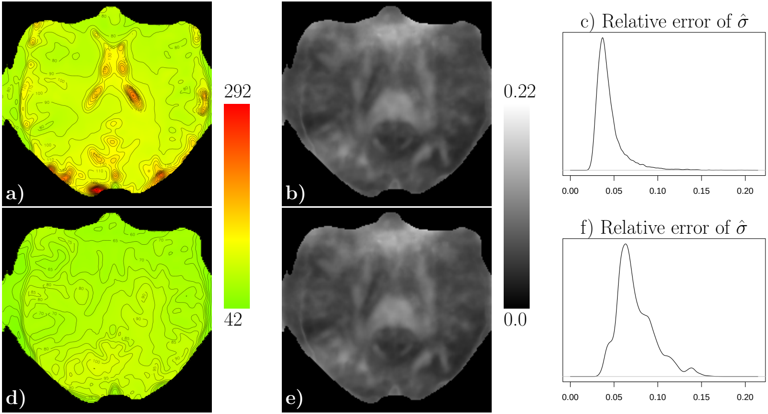
- $\zeta : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$
- $S_i/\sigma_i \sim \chi_{2L,\zeta_i/\sigma_i}$
- Reduction of variability by multi-shell Position Orientation Adaptive Smoothing (msPOAS)
- Requires local (or global) estimates $\hat{\sigma}_i$
- Approximately preserves $\mathbf{E}S_i$
- Enables estimation by Quasi-likelihood

Color coded FA map:

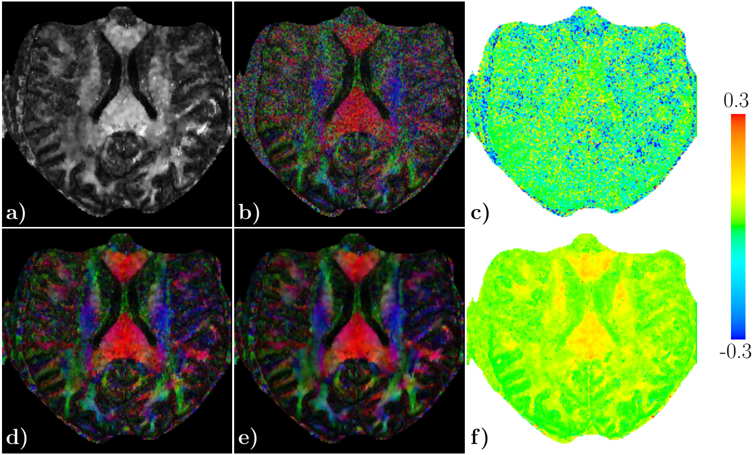


A. Anwander, R. Heidemann (MPI CBS Leipzig, Siemens)

High variability (and bias) in FA and main direction of diffusivity (color)



Mean estimated σ over all a) non-diffusion weighted (28) and d) diffusion weighted images (240). Relative error $\frac{\text{sd } \hat{\sigma}}{\text{mean } \hat{\sigma}}$ over all b) non-diffusion weighted and e) diffusion weighted images. c) and f) corresponding densities.



a) FA grayscale image, and e) corresponding color-coded FA image (msPOAS with local $\hat{\sigma}_i$). d) Color-coded FA (msPOAS with global $\hat{\sigma_i}$. b) Color-coded FA (unsmoothed) data. a), b), d), and e) using the quasi-likelihood. c) and f) FA differences between quasi-likelihood and non-linear regression for b) and d).

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