Quadratic harnesses and Askey-Wilson polynomials

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Let $X = (X_t)_{t\geq 0}$ be a real square integrable stochastic process with

$$\mathbb{E}\,X_t = 0 \quad \text{oraz} \quad \mathbb{E}\,X_s X_t = s, \quad 0 \leq s \leq t.$$

Let $\mathcal{F}_{s,u} = \sigma(X_t,\, t \notin (s,u))$, $0 \leq s < u$. The process X is called a **quadratic harness** if for any $0 \leq s < t < u$

$$\mathbb{E}(X_t|\mathcal{F}_{s,u}) = aX_s + bX_u \tag{1}$$

and

$$\mathbb{E}(X_t^2|\mathcal{F}_{s,u}) = AX_s^2 + BX_sX_u + CX_u^2 + DX_s + EX_u + F \tag{2}$$

where $a, b, A, \ldots, F$ are nonrandom functions of s, t, and u.

Examples include: Lévy'-Meixner process (eg. Wiener, Poisson and gamma processes), free Brownian motion (classical version), q-Gaussian processes, free Poisson process, q-Meixner processes ([3]), and bi-Poisson processes ([2]).

Rather unexpectedly functions $A, ..., F$ are uniquely determined by five numerical parameters. Actually, these parameters uniquely determine the whole process! It appears that all known quadratic harnesses are Markov, [1].

Description and construction of quadratic harnesses is heavily based on properties of polynomials which are supposed to be orthogonal with respect to marginal and transition (conditional) distributions. Moreover, these polynomials are also martingales. Apparently, there is a deep algebraic relation (not fully understood yet) between quadratic harnesses and orthogonal polynomials. Of particular importnace are, so called, projection formulas and formulas for connection coefficients. Recently, we have understood that the leading role belongs here to the **Askey-Wilson system of orthogonal polynomials**. In a sense, conditions (1) and (2), through a q-commutation equation (to be explained in the talk), lead to a quadratic algebra of operators, which is also related to the Askey-Wilson system. In special cases the algebra of operators has an explicit representation among operators acting on formal power series.

The talk will be based on a recent paper [4], joint with Wlodek BRYC (Univ. of Cincinnati).

References

1. Bryc, W., Matysiak, W., Wesołowski, J. (2007) Quadratic harnesses, q-commutations and orthogonal martingale polynomials. *Trans. AMS* **359**, 5449-5483.

2. Bryc, W., Matysiak, W., Wesołowski, J. (2008) The Bi-Poisson process: a quadratic harness. *Ann. Probab.* **36**, 623-646.

3. Bryc, W., Wesołowski, J. (2005) Conditional moments of q-Meixner processes. *Probab. Th. Rel. Fields* **131**, 415-441.

4. Bryc, W., Wesołowski, J. (2010) Askey-Wilson polynomials, quadratic harnesses and martingales. *Ann. Probab.* **38(3)** (2010), 1221-1262.