Numerical computing with Chebfun

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chebfun

www.maths.ox.ac.uk/chebfun

“An Extension of MATLAB to continuous functions and operators”
Z. Battles & L.N. Trefethen, SIAM J. Sci. Comp. (2004)

Taming the combinatorial explosion:

$f(x) := \exp(-x^2)\cdot\sin(3\cdot\cos(4\cdot\mathrm{pi}\cdot(1-x)^2))\cdot(1-x)^3$

$$x \to e^{-x^2}\sin(3\cos(4\pi(1-x)^2))\,(1-x)^3$$

$simplify(diff(f(x), x\$2))$

$$-2\Big(-288\sin(3\cos(4\pi(-1+x)^2))\,\pi^2\cos(4\pi(-1+x)^2)^2 - 1440\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x + 2880\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^2 - 2880\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^3 + 1440\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^4 - 2\sin(3\cos(4\pi(-1+x)^2))$$
$$+ 13\sin(3\cos(4\pi(-1+x)^2))\,x^2 - \sin(3\cos(4\pi(-1+x)^2))\,x^3 - 6\sin(3\cos(4\pi(-1+x)^2))\,x^4 + 2\sin(3\cos(4\pi(-1+x)^2))\,x^5 + 288\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 + 1440\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x\cos(4\pi(-1+x)^2)^2$$
$$- 2880\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^2\cos(4\pi(-1+x)^2)^2 + 2880\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^3\cos(4\pi(-1+x)^2)^2$$
$$- 1440\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^4\cos(4\pi(-1+x)^2)^2 + 96\cos(3\cos(4\pi(-1+x)^2))\cos(4\pi(-1+x)^2)\,\pi^2$$
$$+ 84\cos(3\cos(4\pi(-1+x)^2))\sin(4\pi(-1+x)^2)\,\pi - 288\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^5 - 6\sin(3\cos(4\pi(-1+x)^2))\,x$$
$$- 96\cos(3\cos(4\pi(-1+x)^2))\cos(4\pi(-1+x)^2)\,\pi^2 x^5 + 204\cos(3\cos(4\pi(-1+x)^2))\sin(4\pi(-1+x)^2)\,\pi x^3$$
$$+ 288\sin(3\cos(4\pi(-1+x)^2))\,\pi^2 x^5\cos(4\pi(-1+x)^2)^2 - 480\cos(3\cos(4\pi(-1+x)^2))\cos(4\pi(-1+x)^2)\,\pi^2 x$$
$$+ 960\cos(3\cos(4\pi(-1+x)^2))\cos(4\pi(-1+x)^2)\,\pi^2 x^2 - 960\cos(3\cos(4\pi(-1+x)^2))\cos(4\pi(-1+x)^2)\,\pi^2 x^3$$
$$+ 480\cos(3\cos(4\pi(-1+x)^2))\cos(4\pi(-1+x)^2)\,\pi^2 x^4 - 204\cos(3\cos(4\pi(-1+x)^2))\sin(4\pi(-1+x)^2)\,\pi x$$
$$+ 60\cos(3\cos(4\pi(-1+x)^2))\sin(4\pi(-1+x)^2)\,\pi x^2 - 192\cos(3\cos(4\pi(-1+x)^2))\sin(4\pi(-1+x)^2)\,\pi x^4$$
$$+ 48\cos(3\cos(4\pi(-1+x)^2))\sin(4\pi(-1+x)^2)\,\pi x^5\Big)\,e^{-x^2}$$

Key point: *Finite precision does not require symbolic accuracy.*

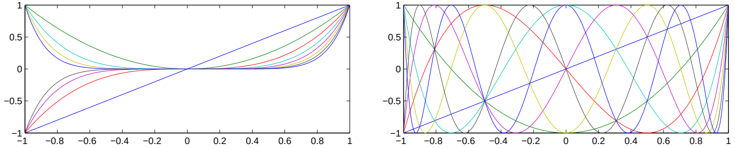
Instead,

- Represent functions using Chebyshev polynomials
- Round to 16 digits (IEEE double precision) at each step
- Further tasks follow naturally: integrals, derivatives, rootfinding, optimisation
- With some more work: linear and nonlinear differential equations.

```
>> f = chebfun('exp(-x.^2).*sin(3*cos(4*pi*(1-x).^2)).*(1-x).^3');
>> length(f)
      ans = 646
>> sum(f)
      ans = -0.001878416746701
```

Why Chebyshev polynomials?

Mathematically, they are close to optimal and well-conditioned



(a) Monomial basis (b) Chebyshev basis

Computationally, they are a good choice because of the one-to-one relationship between the coefficients a_k in the expansion

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

and the values $f_n(x_k)$ taken at *Chebyshev points* $x_k = -\cos(j\pi/n)$, $j = 0 \ldots n$.

An algorithm based on the FFT enables us to go back and forth between these data in $O(n \log n)$ time.

Beneath the surface, a lot is happening

- *Adaptive construction*: Decay of Chebyshev coeffcients
- *Evaluation*: Barycentric interpolation
- *Definite integration*: Clenshaw-Curtis quadrature
- *Rootfinding*: Eigenvalues of a colleague matrix
- *Piecewise representations:* Automatic edge detection
- *Divergent functions:* Exponent calculation
- *Linear DEs:* Adaptive spectral collocation
- *Nonlinear DEs:* Newton-Kantorovich iteration & automatic differentiation