

The Curious Incident of the Investment in the Market: Real Options and a Fair Gamble [†]

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“Is there any point to which you would wish to draw my attention?” “To the curious incident of the investment in the market.” “The agent did nothing in the market.” “That was the curious incident.” (with apologies to Sir Arthur Conan-Doyle.)

In this paper we study an optimal timing problem for the sale of a non-traded real asset. We solve this problem for a risk-averse manager under two scenarios: firstly when the manager has access to no other investment opportunities, and secondly when he may also invest in a continuously traded financial asset. We construct the model such that the financial asset has zero risk premium and thus represents a fair gamble, and such that it is uncorrelated with the real asset, so that it is not useful for hedging. In the absence of the real asset, the manager would not include the financial asset in his optimal portfolio.

Although the problem is designed such that naive intuition would imply that the optimal strategy is the same irrespective of whether the manager is allowed to invest in the financial asset or not, curiously we find that for certain parameter values this is not the case. Access to the fair gamble improves the manager’s expected utility in some situations, and reduces the probability that the real asset is ever sold. Our work has implications for modeling of portfolio choice problems since seemingly extraneous assets can impact on optimal behavior.

Keywords and Phrases: Real assets, Perpetual options, Optimal stopping, Incomplete markets, Portfolio choice, Real options, Portfolio constraints

JEL Classification Numbers: G11, G12, G31

In this paper we consider the problem facing a risk-averse manager who owns a real asset, and who wishes to choose the optimal time to sell this asset in order to maximize his expected utility of wealth. The real asset cannot be traded continuously, so the manager faces idiosyncratic risks in an incomplete market. Our aim is to take the simplest possible modeling formulation for this problem (CRRA utility, perfect information, constant parameter lognormal dynamics, zero interest rates, zero dividends and an infinite horizon) for which we can obtain explicit solutions for the manager's value function and the optimal selling rule. The reason for this choice is that our interest is in considering what happens when we embed this simple problem within a model in which there is a financial market in which the manager may make additional investments.

We design this market in such a way that it seems unnatural for the manager to be able to take advantage of any of these additional investment opportunities, either for hedging the real asset, or as investments in their own right. Unexpectedly, and most remarkably, we find that for some parameter values the solution of the optimal selling problem is not robust to embedding the problem in the more general setting. The fact that this phenomena exists in such a simple setting means that great care must be taken in interpreting the optimal solution to many classes of portfolio choice problems.

The general setting of our problem is in the field of real options, see the classic text of Dixit and Pindyck (1994) for an overview. The most natural interpretation is to consider a manager who has ownership of the real asset, for example a parcel of land, a patent right, or a family firm, and who wishes to choose the optimal time to sell this asset or disinvest. A key feature is that the manager is not able to trade the real asset in a market, although it has a well defined price which is the amount the manager would receive for selling it immediately. A second key feature is that the real asset is indivisible so that the manager faces a single, irreversible decision to sell.

In essence our problem is one of valuing real assets in an incomplete market. Here, following Henderson and Hobson (2002) and Kahl et al (2003) it is assumed that the real asset is not traded but there may be a second asset which is traded.¹ For simplicity, these assets are driven by exponential

¹Detemple and Sundaesan (1999) have a similar framework with a traded asset but use a discrete time trinomial scheme to value the option.

Brownian motions which may be correlated. Both papers consider only the European-style valuation of stock, but the problem here also contains an optimal timing problem and hence shares features with American option problems and especially perpetual American options.

Classical models of perpetual real options assume the real asset is spanned by the market so that the model is complete and risk preferences are irrelevant.² McDonald and Siegel (1986) generalized this setting to allow for an incomplete market. However they assumed that the agent needed compensating only for systematic or market risk, and that he was risk-neutral to any idiosyncratic risk. Recently, a strand of real options literature has asked the question of how incompleteness (via non-tradability of the real asset) impacts on valuation and investment and abandonment decisions. In these papers, the manager has access to a traded market asset and the comparison of interest is between the case where the real and market assets are perfectly correlated (representing a complete market), and the case of imperfect correlation and an incomplete market. Smith and Nau (1995) initially investigated incompleteness in a binomial model and for European options. Henderson (2004b) considered the option to invest in an incomplete market where the manager has exponential preferences. Her closed form solutions lead to the conclusion that incompleteness results in a lower option value and earlier exercise. She also finds that incompleteness causes fundamental qualitative differences in the manager's exercise or investment decision. When a complete model such as Dixit and Pindyck (1994) recommends waiting indefinitely, incompleteness may induce the manager to invest at some finite threshold. Miao and Wang (2004) ask a similar question in a consumption style model, but their analysis requires numerical solutions.

In contrast, in this paper we consider the impact of access to trading in a market on the valuation and selling decision of the risk-averse manager. We compare two incomplete market situations both with and without the market, where the manager faces incompleteness as the real asset is not traded. Our market is designed to represent a fair gamble and we investigate what effect, if any, such a gamble has on the manager's optimal behavior.

²The risk-neutral valuation of American perpetual options was first solved by McKean (1965) in an appendix to Samuelson (1965) and later considered by Merton (1973).

We now discuss our model and findings in more detail. Recall that in this paper we study the optimal time to sell a real asset. The key parameters of the problem are the coefficient of relative risk aversion and the risk premium and volatility of the real asset. In our first situation without a market, the manager maximizes expected utility of wealth where wealth consists of a fixed initial wealth plus the wealth realized from the sale of the real asset. This problem is entirely classical in structure and can be solved in many ways. It is also very closely related to the model of Kadam et al (2004). If the risk premium is negative so that the value of the real asset will decrease on average, then it is optimal for the manager to disinvest immediately. Conversely, if the risk premium is large, relative to other parameters, and positive, then it is never optimal to sell the real asset. The interesting case is when the risk premium is small and positive. In that case the expected growth in future value gives the manager an incentive to continue to hold the real asset but this is counterbalanced by an incentive to sell in order to reduce uncertainty and risk. It turns out there is a critical threshold, which can be characterized as the solution of a transcendental equation, and the manager should sell the real asset the first time the ratio of the real asset value to wealth exceeds this value.

We extend the above problem by embedding it within a larger model. In this model, in addition to the real asset there is a second financial asset, and we assume that the manager is free to invest dynamically in this second asset. The market for the financial asset is assumed to be complete and frictionless. There are two immediate reasons why the manager may choose to invest in the financial asset. The first reason is that, if this asset is correlated to the real asset, then the manager could use it to hedge her exposure to the risk implicit in continuing to hold the real asset. That is, the market risk could be hedged. The second reason is that if the financial asset has non-zero risk premium then it is optimal to hold some units of the asset in order to maximize the rate of growth of expected utility.

In order to remove both of these motivations we assume that the market asset is uncorrelated with the real asset and that it has zero Sharpe ratio. This market represents a fair gamble to the risk-averse manager, and it is natural to suppose that he will not accept it. We find, curiously, that this is not the case, and for some parameter values the manager can take advantage of the fair gamble

to generate greater utility. The region of parameter values over which this occurs is greatest when the manager has logarithmic utility.

In this region of parameter values where the manager can benefit from the market, the manager wants to hold onto the real asset for as long as possible, but sells when it becomes too large a proportion of his wealth. His risk-aversion to idiosyncratic risk can outweigh the benefit of holding onto the asset. However, the opportunity to invest in the fair gamble gives the manager a more varied strategy choice. If the investment in the gamble pays off, this allows him to hold onto the real asset for longer, as the real asset makes up a smaller proportion of his wealth. We find that access to a market reduces the probability that the real asset is ever sold.

To better understand the behavior in our model, it proves insightful to consider a one-period version of the continuous time problem. In this model, the manager facing a decision to sell (at time 0) or wait (till time 1) is offered a fair gamble. The one-period model allows us to see that the manager in such a situation is in fact locally risk-seeking if his wealth puts him near the indifference point between selling and waiting. At this point, he is better off accepting the fair gamble.

Our conclusions are robust within the class of CRRA preferences. The body of the paper considers a manager with logarithmic utility, and this is extended to CRRA utilities in Appendix B. The phenomena we find does not occur for exponential utility, see Henderson (2004). The reason for this is given in Section IV. In the limiting case of risk-neutrality, we recover the corresponding McDonald and Siegel (1986) solution. Again, the manager either sells immediately or waits indefinitely and the phenomena we find for CRRA does not occur.

As well as in the context of real options, non-tradability of stocks arises in models of executive stock options and restricted stock. Managers, executives and employees have undiversified portfolios often comprising a large proportion of their own company stock and options, which cannot be sold or hedged perfectly. The early literature (see Lambert et al (1991), Kulatilaka and Marcus (1994) and Hall and Murphy (2002)) considered the value of these options in isolation, where the executive simply faced trading restrictions and non-option wealth grew at the riskless rate of interest. These

were one-period models. Models were later extended to include a correlated market asset which was first shown by Detemple and Sundaresan (1999) (in a binomial setting) to be useful in hedging, see also the continuous time models of Kahl et al (2003), Henderson (2004a) and Ingersoll (2006). American exercise features are considered by a subset of these papers, namely Detemple and Sundaresan (1999) and Ingersoll (2006), although both treat approximations of the full problem. Our paper treats both the features of American exercise and non-tradability, and allows comparison between having and not having access to a market asset. The results of this paper imply executives faced with the decision of when to sell restricted stock are better off when they can trade a market asset, even if that asset represents a fair gamble and is uncorrelated with the stock. Executives with access to such a gamble will value their right to sell more highly than if they could not enter the gamble. Our model also has implications for the executive's optimal sale time. Our results imply executives with access to a fair gamble have a lower probability of ever selling their restricted stock than those without the gambling opportunities.

The optimal stopping problems we consider in this paper are sufficiently simple that it is possible to derive explicit solutions, including characterizations of the optimal strategies and expressions for the value functions. However, despite this apparent simplicity there are some counter-intuitive results. In particular, these results urge caution in modeling any optimal stopping problem in isolation. The simple act of omitting independent assets from a model may have a radical impact upon the form of the solution to a problem. Reiterating, our model formulation is completely standard in that the manager has CRRA preferences and faces an irreversible decision of when to sell a real asset. The model is incomplete since the real asset is not traded continuously and the manager faces idiosyncratic risk. We find the manager becomes risk seeking and accepts a fair gamble despite his concave utility function. The reason for this is that his implied utility function is not concave in wealth.

The paper is structured as follows. Section I treats the asset sale problem in an incomplete market where there is no access to a traded financial asset. Section II extends this framework to allow trading in a market asset which has zero Sharpe ratio and zero correlation with the real asset. We discuss

a one-period version of our continuous time model in Section III. Interpretation, comparisons and extensions are treated in Section IV and Section V concludes. Proofs are contained in Appendix A whilst the main results of Sections I and II are extended to CRRA in Appendix B.

I The Asset Sale Problem with No Market Asset

Consider a manager wishing to choose the optimal time to sell an indivisible real asset over an infinite horizon. The real asset cannot be traded continuously and the manager is exposed to idiosyncratic risk by having the right to sell this asset. In this section, we consider the optimal behavior of the manager assuming he does not have access to any other investment opportunities. Without any hedging possibilities (via other assets such as a market asset), all risks that the manager faces from the right to sell are idiosyncratic and unhedgeable.

In order to keep the formulation as simple as possible we assume that interest rates are zero, or equivalently that all quantities are expressed in discounted terms and utility is measured with respect to discounted wealth. this does not affect our results.

Assume the manager is risk-averse. The problem facing the manager is to choose the sale time τ to maximize the expected utility of total wealth, or in other words to find

$$V^N(x, y) = \sup_{\tau} \mathbb{E}[U(x + Y_{\tau}) | Y_0 = y], \quad (1)$$

where U is logarithmic utility³, $U(x) = \ln x$. The superscript N on the value function refers to the fact that we are working in the No-market setting. Note that we do not introduce a subjective discount factor into the utility function. This is because we have an interesting problem without an extra discounting parameter, and moreover we do not want to introduce artificial incentives for the manager to accelerate the sale.

³In the body of the paper we use logarithmic utility since the resulting formulae are more compact. Logarithmic utility is the special case with unit risk aversion of a constant relative risk aversion utility, and we treat the general case in Appendix B.

In (1), the quantity x is initial wealth, and Y_t is the price process of the indivisible asset. It is assumed that Y_t is known at time t , so that the problem facing the manager is to choose the optimal disinvestment (or sale) time τ . Although Y_t is known, there is no continuous trading available in this asset and it is not possible to construct a replicating portfolio. Hence the manager faces an incomplete market. An example would be the sale of a factory where it is possible to obtain an estimate of its sale value at a particular time but not possible to buy and sell the factory continuously. The price process Y_t is assumed to satisfy constant parameter lognormal dynamics

$$dY_t = Y_t(\sigma dW_t + \mu dt) \tag{2}$$

with volatility σ and drift μ . Since we assume zero interest rates, μ is also the risk premium on Y . This equation has solution $Y_t = y \exp\{\sigma W_t + (\mu - \sigma^2/2)t\}$, where $Y_0 = y$. Note that if $\mu > \sigma^2/2$ then Y_t grows to ∞ whereas, if $\mu < \sigma^2/2$, then Y_t tends to zero, almost surely. A key parameter is $\gamma = \frac{2\mu}{\sigma^2}$ which is twice the Sharpe ratio per unit of volatility, in which case the critical threshold is $\gamma = 1$.

We first briefly mention the analogous problem under the assumption that utility is linear, $U(x) = x$. In this case, the manager is risk-neutral and he simply maximizes $\mathbb{E}Y_\tau$ over possible selling times. In such a set-up, the problem is degenerate. The manager either waits indefinitely when Y has positive risk premium or sells immediately if the risk premium is negative. Since the manager is risk-neutral, he simply considers whether (discounted) Y is growing or not. This can also be restated in terms of the parameter γ . In the risk-neutral situation, the asset is sold immediately if $\gamma < 0$ and never sold if $\gamma > 0$.

Now we return to the situation where the manager is risk-averse with logarithmic utility, $U(x) = \ln(x)$. It will turn out that since the manager is risk-averse, he will require a greater value of γ to hold onto Y indefinitely. Recall that $\gamma = 1$ differentiates between cases where Y tends to infinity or zero. For $\gamma > 1$ the real asset grows without bound and the manager waits. For $0 < \gamma < 1$, the manager's risk aversion means it may be optimal to reduce uncertainty by selling even though the asset has positive risk premium. This will depend on how large Y is compared with wealth and the larger this proportion, the greater the incentive to disinvest.

The problem without a market is amenable to entirely classical optimal stopping techniques. However, we provide an outline of the solution to the problem, to facilitate comparison with the problem we study in Section II. The problem in this section is very close to that studied in Kadam et al (2004). There are however some differences which make it difficult to compare directly. Kadam et al (2004) assume $\gamma > 1$ (which in our setup leads to holding onto Y indefinitely) and counterbalances this with an extra subjective discounting factor to artificially encourage sale of the asset.⁴

Given the structure and scalings of the problem, it is plausible that the optimal stopping rule is of the form to sell the real asset the first time Y_t exceeds a critical value y^* . Further, since initial wealth x is constant, the stopping rule becomes to stop the first time that the ratio of the real asset price to wealth exceeds some level z . It will be convenient in later sections to work in terms of the ratio of wealth to real asset price, so that the stopping rule can be represented as the first time that x/Y_t falls below w , for $w = 1/z$. With all this in mind, we consider stopping times of the form

$$\tau_w = \inf_u \{x/Y_u \leq w\} = \inf_u \{Y_u \geq x/w\}.$$

Under this strategy, and assuming $y < x/w$, the expected utility of the manager for any trigger level w , is given by

$$F(w) = \mathbb{E}[\ln(x + Y_{\tau_w})] = \mathbb{P}(\tau_w = \infty) \ln x + \mathbb{P}(\tau_w < \infty) \ln(x + x/w) = \ln x + \ln(1 + 1/w) \mathbb{P}(\tau_w < \infty).$$

Now we maximize over choice of level w . For $\gamma \geq 1$ the real asset grows without bound and $\mathbb{P}(\tau_w < \infty) = 1$, and it is clear in this case that F is maximized over w by taking $w^* = 0$. That is, it is never optimal for the manager to sell the asset. For $\gamma < 1$ we have (see for example Durrett (1991))

$$\mathbb{P}(\tau_w < \infty) = \left(\frac{wy}{x}\right)^{1-\gamma}. \quad (3)$$

In that case the derivative of F becomes

$$F'(w) = \left(\frac{y}{x}\right)^{1-\gamma} w^{-\gamma} \left[(1-\gamma) \ln(1 + 1/w) - \frac{1}{1+w} \right]$$

⁴Kadam et al (2004) also assume an initial wealth of zero, and hence they restrict to relative risk aversion less than one to avoid utility of negative infinity. We do not need this restriction in the Appendix where we treat general power utilities.

and since, for $w > 0$, $\ln(1 + 1/w) > 1/(1 + w)$ it follows that for $\gamma \leq 0$ we have $F'(w) \geq 0$, and F is maximized by the choice $w^* = \infty$. In this case the manager sells immediately. The interesting case is when $0 < \gamma < 1$. In that case F is maximized by the unique solution w^* in $(0, \infty)$ to $F'(w) = 0$. Putting these observations together gives

Proposition 1 For $\gamma \leq 0$, $V^N(x, y) = \ln(x + y)$, and for $\gamma \geq 1$, $V^N(x, y) = \infty$.

In the non-degenerate case $0 < \gamma < 1$, in the exercise region $y \geq x/w^*$ we have $V^N(x, y) = \ln(x + y)$, and in the continuation region $y < x/w^*$,

$$V^N(x, y) = \ln x + \left(\frac{yw^*}{x}\right)^{1-\gamma} \ln(1 + 1/w^*). \quad (4)$$

The optimal exercise ratio $w^* = w^*(\gamma)$ is the unique solution in $(0, \infty)$ to

$$(1 - \gamma) \ln(1 + 1/w) - \frac{1}{1 + w} = 0 \quad (5)$$

Proof: A formal proof of the proposition follows on checking that V^N is a supermartingale under any stopping rule, and a true martingale under the stopping rule τ_{w^*} . Given the formula above this is an exercise in Itô calculus. This proof also verifies that the optimal stopping rule is of the form presupposed above, namely to sell the real asset the first time its price exceeds x/w^* . \square

It follows immediately from (5) that the optimal exercise ratio w^* is decreasing in γ . Equivalently, the optimal exercise ratio $z^* = 1/w^*$ for real asset value to wealth is increasing in γ . Thus, as the risk premium of the real asset increases, the real asset becomes worth more to the manager and he waits longer before selling. Indeed, given the explicit form for the value function, the certainty equivalent value p^N for the real asset Y is given as the solution to $\ln(x + p^N) = V^N(x, y)$. If $\gamma \leq 0$ then $p^N = y$ and if $\gamma \geq 1$ then $p^N = \infty$. Otherwise, for $\gamma \in (0, 1)$, the critical exercise ratio exists in $(0, \infty)$ and for $y \geq x/w^*$ we have $p^N = y$, and in the continuation region

$$p^N = x \left((1 + 1/w^*)^{(yw^*/x)^{1-\gamma}} - 1 \right), \quad y < x/w^*. \quad (6)$$

The certainty equivalent value in (6) is increasing in y , and (for $y < x/w^*$) $p^N < x/w^*$. A few lines

of algebra also shows that p^N given by (6) satisfies $p^N > y$. That is, the certainty equivalent value of the right to sell is greater than the current value, because the asset has a positive risk premium.

II The Asset Sale Problem Allowing Trading in the Market

In this section we revisit the optimal disinvestment problem in a more general setting in which the manager has alternative investment opportunities. One very interesting case is when this market asset is strongly correlated to the non-traded real asset. However, in this paper we are interested in the opposite situation in which the financial asset is uncorrelated with the real asset.

The problem which the manager now faces is to maximize the expected utility of total wealth

$$V^M(x, y) = \sup_{\tau} \sup_{X_{\tau}} \mathbb{E}[U(X_{\tau} + Y_{\tau}) | X_0 = x, Y_0 = y], \quad (7)$$

for logarithmic utility, where X_t is the manager's trading wealth⁵ (from investment in the financial asset) and, as before, Y_t is the price process of the indivisible, non-traded asset and τ is the sale time chosen by the manager.

If the financial asset were correlated to the real asset then there would be a hedging motive for trading in the financial asset. This trading would allow the manager to offset some of the risk associated with holding the asset Y . Such hedging motives arise in many portfolio choice problems in incomplete markets including Detemple and Sundaresan (1999) and Kahl et al (2003). Stochastic labor income problems, where the labor income is unhedgeable but access to financial markets provides a reduction in risk, provide another example, see Viceira (2001).

However, since we assume that the financial asset is uncorrelated with Y , the manager has no obvious incentive to hedge via the market. Furthermore, we suppose that the financial asset has a Sharpe ratio of zero. Temporarily ignoring the presence of Y , if the manager were to solve the classical Merton (1969, 1971) problem over any fixed horizon T , then his optimal portfolio would be to invest

⁵For example, if the financial asset price is denoted P and the manager holds θ shares and $(X - \theta P)$ in a riskless bond then his self-financing wealth dynamics are of the form $dX = \theta dP$. Recall we are working with discounted units to keep our notation as simple as possible, and that the traded asset forms a fair gamble, so that P is a martingale.

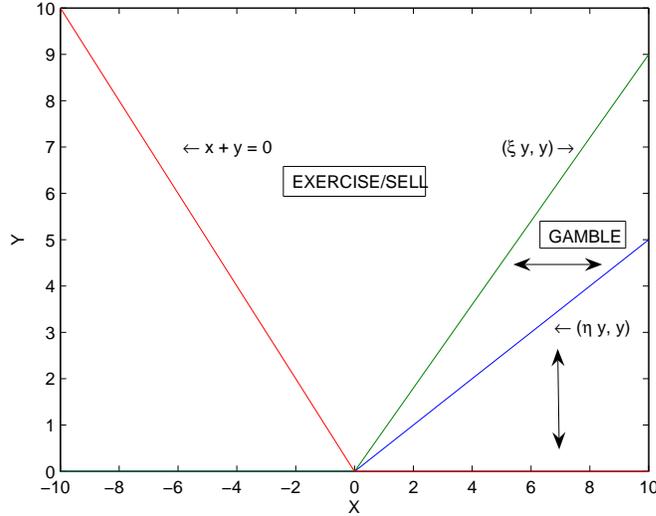


Figure 1: **Regions of (X, Y) space where the manager with access to a market asset will wait, gamble in the market, or exercise/sell the real asset.** *The manager does nothing if $x > \eta y$, gambles in the market if $\xi y \leq x \leq \eta y$ and sells the real asset when $x < \xi y$.*

nothing in the (risky) financial asset. Thus, in the case of zero correlation and zero Sharpe ratio it seems natural to conjecture that embedding the problem of Section I in this larger market would have no effect, and the value function and optimal exercise strategy would be unchanged. However this is *not* the case as we now seek to demonstrate.

Fix $\xi \leq \eta < \infty$ with $\xi > -1$ and $\eta > 0$, and define a combined investment/exercise strategy as follows, and as illustrated in Figure 1. If $\xi Y_t \geq X_t$ then exercise/sell immediately. If $\eta Y_t \leq X_t$ invest nothing in the financial asset, and wait until the first time that $\eta Y_t = X_t$, if ever. Finally, if $\xi Y_t < X_t < \eta Y_t$, then take a large position in the market asset. Since the financial asset has zero risk premium it is a martingale under \mathbb{P} , and this property is inherited by X . Investing in the market asset can therefore be thought of as a fair gamble. In the limit of larger and larger positions the trading wealth X_t leaves the interval $(\xi Y_t, \eta Y_t)$ immediately, with the probabilities of leaving either

end determined by the martingale property. If we leave at the left-hand endpoint then $\xi Y_t = X_t$ and we sell the asset Y . Conversely, if we leave at the right-hand endpoint then $\eta Y_t = X_t$ and we cease investment in X and wait for changes in relative wealth X/Y through fluctuations in the value of Y_t . In the limiting case $\eta = \xi = w$ we recover the strategies of the previous section without trading opportunities.

Implicit in our definitions is the fact that the manager is allowed to borrow⁶ against his holdings in the real asset. Hence his wealth constraint is $X_t + Y_t \geq 0$. This explains why we insist on $\xi > -1$. Further, this also means that we must allow for negative trading wealth (excluding the real asset). For this reason it is natural to work with the ratio of trading wealth to real asset value, since in that case the denominator is positive by modeling assumption.

Proposition 2 *Under the above strategy, specified by the thresholds (ξ, η) , the value function is given by*

$$V^M(x, y) = \begin{cases} \ln(x + y) & x \leq \xi y \\ G(x, y) & \xi y \leq x < \eta y \\ H(x, y) & \eta y < x \end{cases}$$

where

$$G(x, y) = \frac{\xi}{\eta - \xi} \left(\frac{x}{\xi y} - 1 \right) (\ln y + \ln \eta + \Theta) + \frac{\eta}{\eta - \xi} \left(1 - \frac{x}{\eta y} \right) (\ln y + \ln(1 + \xi))$$

and

$$H(x, y) = \ln x + \left(\frac{\eta y}{x} \right)^{1-\gamma} \Theta$$

with

$$\Theta \equiv \Theta(\eta, \xi) = \left[\frac{\eta - \xi + \eta (\ln(1 + \xi) - \ln \eta)}{\eta + (\eta - \xi)(1 - \gamma)} \right]. \quad (8)$$

Proof: The proofs of all the results in this section are contained in Appendix A.

It is clear from the expression for H that in order to maximize the value function it is necessary to choose η and ξ to maximize $\eta^{1-\gamma}\Theta$. It turns out that for $\gamma \in (0, 1)$ there are two distinct ranges of

⁶If the manager is constrained to keep his trading wealth X_t positive then the form of his optimal strategy may change, although the general character of our solutions does not change. We return to this issue in Section IV D.

values of γ for which the character of the maxima are different. That is, the three regions in Figure 1 change, depending on the value of γ . We display the resulting ranges for γ in Figure 2 which we discuss after presenting the lemma.

Lemma 3 *Let γ_- be the unique solution in $(0, 1)$ of $\Gamma_-(\gamma) = 0$ where*

$$\Gamma_-(\gamma) = (1 - \gamma)(2 - \gamma) \ln \left(\frac{2 - \gamma}{1 - \gamma} \right) - 1.$$

Then $\gamma_- = 0.3492$. Consider now the problem of finding the maximum of $\eta^{1-\gamma}\Theta$ over $-1 < \xi \leq \eta < \infty$. For $0 < \gamma \leq \gamma_-$ the maximum is attained at $\eta = w^$, $\xi = w^*$ where $w^* = w^*(\gamma)$ is the solution to (5).*

For $\gamma_- < \gamma \leq 1$ the maximum is attained at $\eta = \eta^$ where*

$$\eta^* = \eta^*(\gamma) = \frac{1 - \gamma}{2 - \gamma} \left(\frac{1}{1 - \gamma} - \ln \left(\frac{2 - \gamma}{1 - \gamma} \right) \right)^{-1} \quad (9)$$

and $\xi = \xi^$ where*

$$\xi^* = \xi^*(\gamma) = \frac{\eta^*(2 - \gamma)}{1 - \gamma} - 1 = \frac{(1 - \gamma) \ln((2 - \gamma)/(1 - \gamma)) - \gamma}{1 - (1 - \gamma) \ln((2 - \gamma)/(1 - \gamma))}. \quad (10)$$

Figure 2 (and the lemma) show that the regions of waiting, investing (or gambling) in the market and selling the real asset differ depending on in which range for γ the parameters lie. In the situation where $\gamma < \gamma_-$, there will be no gambling region and the manager will behave as if there were no investment opportunities. On Figure 2 this is depicted via the exercise ratio w^* .⁷ However, as long as $1 > \gamma > \gamma_-$, the manager will invest in the market, despite it having zero Sharpe ratio and zero correlation with the real asset. In the case $\gamma_- < \gamma < 1$, there are three distinct regions in the wealth/real asset value coordinates where the manager either waits, gambles in the market or sells the real asset. Figure 2 plots the critical exercise level ξ^* given in (10) and the critical level where the manager gambles, η^* given in (9).

⁷The critical exercise ratio w^* , which is optimal for $0 < \gamma < \gamma_-$, is continued for $\gamma > \gamma_-$ on the graph for comparison. This is the exercise ratio which would be optimal over the whole range $0 < \gamma < 1$ in the model with no market asset of Section I.

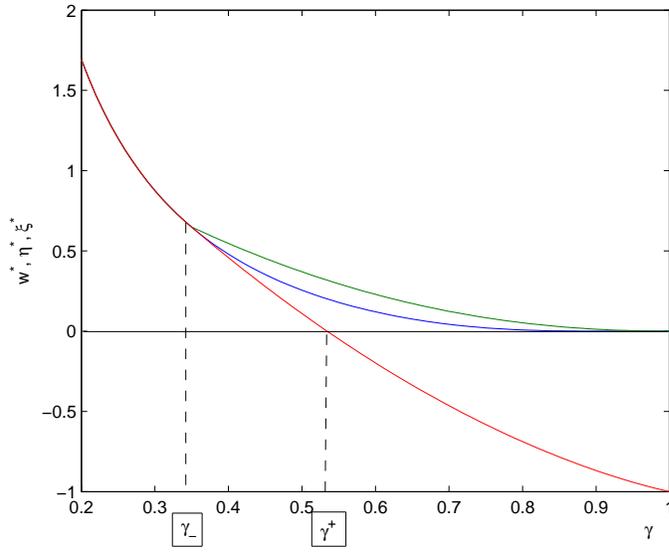


Figure 2: **The different regions when $0 < \gamma < 1$.** Plot of w^*, η^*, ξ^* for $0 < \gamma < 1$. Over the range $0 < \gamma < \gamma_-$, $\xi^* = \eta^* = w^*$. For $\gamma_- < \gamma < 1$, η^*, ξ^* are given in (9) and (10) and w^* by (5). Over the range $\gamma > \gamma_-$, w^*, η^* and ξ^* can be identified by the fact that $\xi^* < w^* < \eta^*$. The level $\gamma = \gamma^+$ is where ξ^* crosses zero and the manager starts borrowing against the real asset.

Figure 2 shows that as γ increases from γ_- to one, the exercise boundary governed by ξ^* falls, crosses zero and tends to -1. The interpretation is that the exercise boundary on Figure 1 moves from the positive (x,y) quadrant to the boundary $x = -y$ at $\xi^* = -1$. If $\xi^* < 0$, the exercise boundary on Figure 1 is in the top-left quadrant and the manager is borrowing against the real asset. The value of γ at which ξ^* equals zero has been denoted γ^+ on Figure 2. This is the point at which the optimal behavior of the manager may first involve negative trading wealth and borrowing.⁸ Also as γ increases to one, the boundary where the manager invests or gambles in the market governed by η^* becomes the line $x = 0$. That is, for γ close to but below one, the manager will take a gamble in the market if the real asset becomes extremely valuable, and only sell the real asset if his total wealth $X + Y$ becomes very small. Recall, when $\gamma > 1$ the manager waits indefinitely to take advantage of the growth of the real asset value.

Now we can state the solution to the optimal stopping problem (7). For $\gamma \leq 0$ the optimal stopping rule is to exercise immediately, and for $\gamma \geq 1$, there is no optimal stopping rule (in the sense any candidate stopping rule which is finite can be improved upon by waiting longer). These results are exactly as in the no-market case of Section I, and so attention switches to the case $0 < \gamma < 1$. The content of the next proposition is that for γ in this range the optimal stopping rule is of the form described before Proposition 2, where the values of ξ and η are chosen to maximize a certain quantity.

Proposition 4 (i) For $\gamma \leq 0$, $V^M(x, y) = \ln(x + y)$.

(ii) For $0 < \gamma \leq \gamma_-$ the value function is given by $V^M(x, y) = \ln(x + y)$ in the exercise region $y \geq x/w^*$, and in the continuation region $y < x/w^*$

$$V^M(x, y) = \ln x + \left(\frac{w^* y}{x}\right)^{1-\gamma} \ln(1 + 1/w^*)$$

where w^* solves (5).

(iii) For $\gamma_- < \gamma < 1$ the value function is given by, in the exercise region $x \leq y\xi^*(\gamma)$,

$$V^M(x, y) = \ln(x + y)$$

⁸We will return to the case where borrowing is not allowed in Section IV D and characterize γ^+ .

for $y\xi^*(\gamma) < x < y\eta^*(\gamma)$,

$$V^M(x, y) = \frac{\xi^*}{\eta^* - \xi^*} \left(\frac{x}{\xi^* y} - 1 \right) (\ln y + \ln \eta^* + \Theta^*) + \frac{\eta^*}{\eta^* - \xi^*} \left(1 - \frac{x}{\eta^* y} \right) (\ln y + \ln(1 + \xi^*))$$

and for $x \geq y\eta^*(\gamma)$,

$$V^M(x, y) = \ln x + \left(\frac{\eta^* y}{x} \right)^{1-\gamma} \Theta^*$$

where η^* and ξ^* are given by (9) and (10) and $\Theta^* = \Theta(\eta^*, \xi^*)$.

(iv) For $\gamma \geq 1$ the value function is given by $V^M(x, y) = \infty$.

The key result in the above proposition is that for $\gamma_- < \gamma < 1$, the manager improves his expected utility by investing in the market asset, despite it being a fair gamble.⁹ We state this result below.

Theorem 5 For $\gamma \leq \gamma_-$ and $\gamma \geq 1$ we have that $V^N(x, y) = V^M(x, y)$ and the solution with the market is identical to the solution of the optimal stopping problem with no market given in Section I. For $\gamma_- < \gamma < 1$ we have that $V^N(x, y) < V^M(x, y)$ and the solution to the optimal stopping problem (7) is different to the solution of (1).

Given the value function V^M it is possible to define the manager's certainty equivalent value p^M for the real asset via $p^M = e^{V^M(x, y)} - x$ and we return to this in Section IV B. We delay the interpretation and intuitive explanation of our results until after we have the benefit of insights from a one-period model described in the following section.

III A One-period Binomial Model

In this section we show how the same phenomena we described in the continuous model can arise in a single period, binomial model. It is the case that in a one-period model, a manager who is deciding when to sell a real asset and has access to a fair gamble can outperform a manager with no access to such a gamble.

⁹The precise forms of the value function for the problem stated in (7) are less interesting than how they relate to the value function for the corresponding problem (1) with no market.

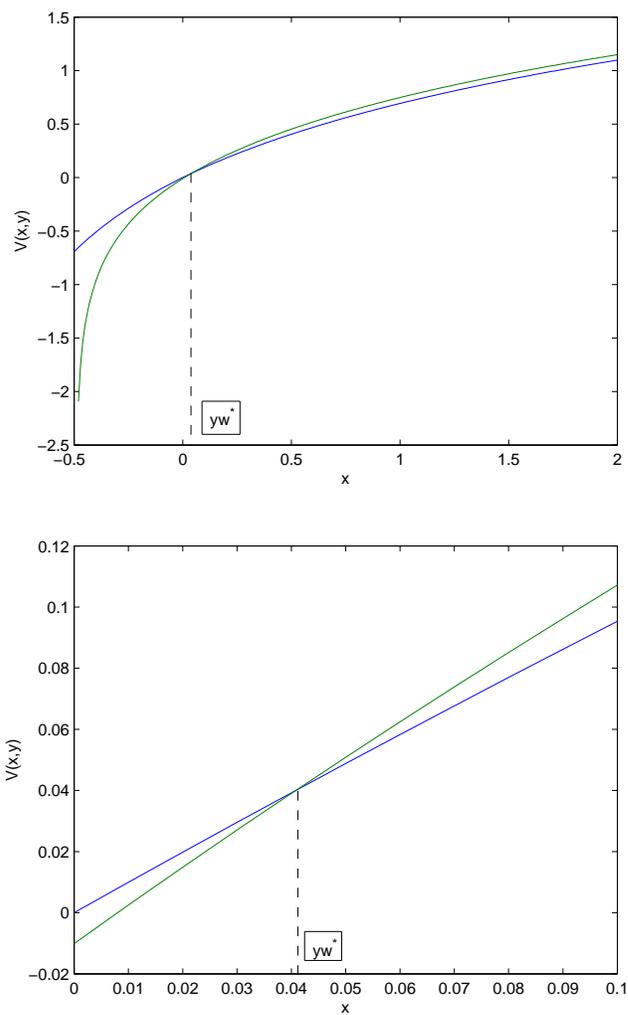


Figure 3: **Value functions in the one-period setting.** *The two graphs plot the value functions in the no-market case against initial wealth x , with $y = 1$. In the top panel, $\ln(x+y)$ and (11) are plotted, with their intersection at $yw^* = 0.041$. Parameter values are $u = 2.0, d = 0.49$. The lower panel zooms in to focus on the point where the manager is indifferent between selling and waiting.*

Consider a manager with initial wealth x and with ownership of a real asset with current value y . After one time-period the value of the real asset will change to either yu or yd where $u > d$. We assume that these two events have equal probability, and this yields a significant simplification of the subsequent formulae, although the results can be extended for any probability p . We suppose that $u + d > 2$, so that in expectation the value of the real asset is rising, but $ud < 1$ so that the expected logarithm of the value is falling.¹⁰ Recall that interest rates have been set to zero.

The problem facing the manager in the no-market situation is either to sell the real asset immediately at time zero, or to wait until time 1. We suppose the manager has logarithmic utility. In the first case the utility is $\ln(x + y)$ whereas in the second case the expected utility is

$$\frac{1}{2} \ln(x + yu) + \frac{1}{2} \ln(x + yd). \quad (11)$$

It is straightforward to show that it is optimal for the manager to wait until time 1 to sell the real asset provided $y < x/w^*$ where w^* is the solution to

$$(1 + w^*)^2 = (w^* + u)(w^* + d), \quad (12)$$

or equivalently, $w^* = (1 - ud)/(u + d - 2)$. Note that the parameters u and d have been chosen to ensure that (12) has a unique positive solution.

Now suppose that at time 0, but before¹¹ the manager decides whether to sell the real asset, the manager is offered the chance to make a small bet of size ϵ which results in net gains of $\pm\epsilon$ with probability one-half in each case. He may postpone the decision about selling the real asset until after the outcome of this bet is known. The outcome of the bet is independent of the behavior of the

¹⁰This is analogous to the case $0 < \gamma < 1$ in the continuous time model of Section II.

¹¹We are trying to model the situation where there is a single gambling opportunity and a single movement in the real asset value. In discrete time there are three ways to model this situation: either the gambling opportunity may occur before the asset value movement, or the asset value may move before the gamble is available, or the two random events may happen simultaneously. In a multi-period model, and in the continuous time limit these distinctions are not important. However, in the single period model, we only find the feature of a risk-seeking manager if we assume the gambling opportunity arises first. This is the motivation for our choice of ordering for the stochastic opportunities in the single period model.

real asset. If the manager did not own the real asset then the concavity of his utility function would mean that he would choose not to accept the bet; further, since the outcome is uncorrelated with the behavior of the real asset, the bet cannot be used to hedge the fluctuations in the price of the real asset. However, we will show that there are circumstances under which it is optimal for the manager to accept the gamble.

Suppose the initial value of the real asset is $y = x/w^*$, and that the manager chooses to accept the bet. If the bet is successful then the value of the real asset $y = x/w^*$ is below the exercise ratio threshold $(x + \epsilon)/w^*$, and he will choose to keep the real asset until time one; otherwise, if the bet is unsuccessful, he will sell the real asset at time 0. His expected utility is

$$\frac{1}{4} \{ \ln(x + \epsilon + xu/w^*) + \ln(x + \epsilon + xd/w^*) \} + \frac{1}{2} \ln(x - \epsilon + x/w^*).$$

Writing $\epsilon = x\tilde{\epsilon}/w^*$ and using (12) this becomes

$$\ln x + \ln(1 + 1/w^*) + \frac{1}{4} \left\{ \ln \left(1 + \frac{\tilde{\epsilon}}{w^* + u} \right) + \ln \left(1 + \frac{\tilde{\epsilon}}{w^* + d} \right) + 2 \ln \left(1 - \frac{\tilde{\epsilon}}{w^* + 1} \right) \right\}.$$

This utility exceeds $\ln x + \ln(1 + 1/w^*)$ if the final bracket is positive, and expanding this term in $\tilde{\epsilon}$ we see that this happens (for some positive ϵ) provided

$$0 < \frac{1}{w^* + u} + \frac{1}{w^* + d} - \frac{2}{w^* + 1} = \frac{(u + d - 2)}{(w^* + 1)^2}.$$

However, as noted above, we chose $u + d > 2$ to ensure that the asset-sale problem in the no-market setting had a non-degenerate solution. Hence, the value function for a manager with the additional investment opportunity exceeds that of a manager with no such opportunities for a side-bet.

We can now examine this result more closely via Figure 3. The graph displays the expected utility of the manager without the opportunity to take a fair gamble. The top panel plots the utility if Y is sold at time 0 and the expected utility if the manager waits, respectively $\ln(x + y)$ and the value in (11), with their intersection at the threshold $x = yw^*$ where w^* solves (12). For wealth below this threshold, $\ln(x + y)$ is higher and selling the asset at time 0 is optimal. To the right of the wealth threshold, waiting is optimal. The lower panel is a close-up of the threshold area. We see the value

functions to the left and right of the point of indifference (between selling and waiting) have different slopes. This shows why the manager can benefit from a fair gamble at this indifference point. Locally, the manager is risk-seeking.

This one-period example can be extended in many ways. For example, one could discuss examples in which the up and down probabilities of the real asset are unequal, in which case the same general conclusions hold but the analysis is more complicated. Further one could determine the optimal (fair) gamble that the manager would wish to enter at time zero. In contrast our aim is to show simply that there is some fair gamble which the manager would wish to enter at time zero. It is also possible to extend the model to a multi-period setting. However, the interpretation of all these extensions is more complicated than in the continuous time setting where there is a single key parameter which determines the form of the optimal strategy for the manager, and the optimal strategy is specified by at most two free boundary levels.

IV Discussion

We saw in the one-period model that at the point of indifference between selling the real asset and waiting one period a fair gamble improves the manager's expected utility. This section first provides some interpretation in the continuous time model. Following this, we examine the certainty equivalent values of the real asset in the two cases (with and without the market asset), and we provide implications concerning the optimal sale time in the two models. We also consider the impact of introducing a no-borrowing constraint on the trading wealth, and finally describe how the results should be modified under CRRA preferences.

A. Interpretation

One of the main contributions of this paper is to give a relatively simple example of a problem where the naive intuition that uncorrelated assets can be omitted from the model is deficient. The optimal stopping problems described in Sections I and II are designed such that there is no incentive to hold the market asset, in the sense that it cannot be used to hedge the fluctuations in value of

the real asset, and since the financial asset is a martingale, a risk-averse manager would not normally include it in his portfolio. Nonetheless, we showed the presence of the market asset changes his optimal strategy.

Theorem 5 shows that for sufficiently large values of γ the equivalence between the no-market and market problems breaks down. The reason why for these parameter values the manager can take advantage of the market is as follows. The real asset has a relatively large risk premium, and hence the manager would like to hold it for as long as possible. When this asset forms a large part of his total wealth, maintaining a position in the real asset becomes too risky.¹² However, there is one way in which the manager can potentially reduce the proportion of wealth that he has invested in the real asset (under some scenarios), and that is by trading on the market. If he trades successfully then the proportion of his wealth invested in the real asset drops, it is optimal to continue to hold the real asset and he benefits from the expected future growth. If he trades unsuccessfully, then it will be optimal to sell the real asset and, given that his trading wealth has dropped, he is worse off than before. The overall benefit from trading on the market depends on the balance between the benefit from growth in both trading wealth together with potential future growth in the price of the real asset, and the loss of utility from a loss of trading wealth. When γ is sufficiently large the first effect dominates. This is precisely what we saw in the one-period model.

The key point is that although for fixed wealth x the value function of the manager who cannot trade in the market is concave in the price level y of the the real asset, it is not concave in x . This was illustrated in Figure 3 for the one period model, and is also true in the continuous time model, see (4). As a result the manager can increase his utility with an investment in a fair gamble.

B. The Certainty Equivalent Value of the Right to Sell the Real Asset

The fact that for certain parameters the value function changes when we introduce (or omit) the market asset, means that the certainty equivalent value of the right to sell the real asset changes when we move from one setting to the other. In Figure 4 we present the certainty equivalent value for the

¹²If the real asset could be sold in small parts then he might choose to reduce his holdings by a partial sale of the real asset, but in our model the real asset is indivisible.

risk-averse manager with logarithmic utility under the two scenarios. As described earlier these values are given by the solutions to $\ln(x + p^N) = V^N(x, y)$ and $\ln(x + p^M) = V^M(x, y)$.

Recall that in the case $0 < \gamma < \gamma_-$, the manager does not invest in the market even when it is available, see Proposition 4. The top left panel shows this situation for unit initial wealth. Here, $p^N = p^M$ and in both cases, the manager sells when y reaches the level $1/w^*$. If γ is larger than this, the market or gamble has an impact.¹³

In the top right panel we take $\gamma = 0.5$. Now there is a difference between the value without the market and that with the market, the with-market value is the higher one. With this value of γ , if the manager does not have access to a market, then he sells the real asset if Y/x reaches the level $1/w^* = 3.92$. In contrast, when he has access to the market, the manager invests in the market if the real asset value is above x/η^* , where $1/\eta^* = 2.7$, and sells the real asset if Y reaches x/ξ^* , where $1/\xi^* = 9.14$. This last point is off the scale of the graph, but it is clear that as y increases the with-market value is getting close to no-market value. Access to a market or fair gamble increases his expected utility and as such, the value of the right to sell is higher also. The difference in values appears small, but if we consider the time-value of the right to sell¹⁴ then the proportional increase due to the presence of the market exceeds 50% at $y = x/\eta^*$. Once the ratio of the value of the real asset to wealth is above $1/w^* = 3.92$, the time-value of the right to sell without the market is zero, whilst the right to sell in the presence of trading still has value.

The lower panel sets $\gamma = 0.7$. There is now a striking difference between the value of the right to sell with and without the fair gamble. For these parameter values, the manager with unit initial wealth but without access to the market, sells the real asset if Y is above $1/w^* = 23.46$. Now consider the manager with unit initial wealth, and with access to a market gamble. He invests in the market if the real asset value y is above $1/\eta^* = 8.1$, and continues to invest in the market until his trading

¹³We also note in passing that the certainty equivalent value of the right to sell is not a convex function of y . This is a consequence of the incomplete market and the risk-aversion of the agent.

¹⁴We define the time value of the right to sell to be the difference between the certainty equivalent value p^N or p^M and the intrinsic value y .

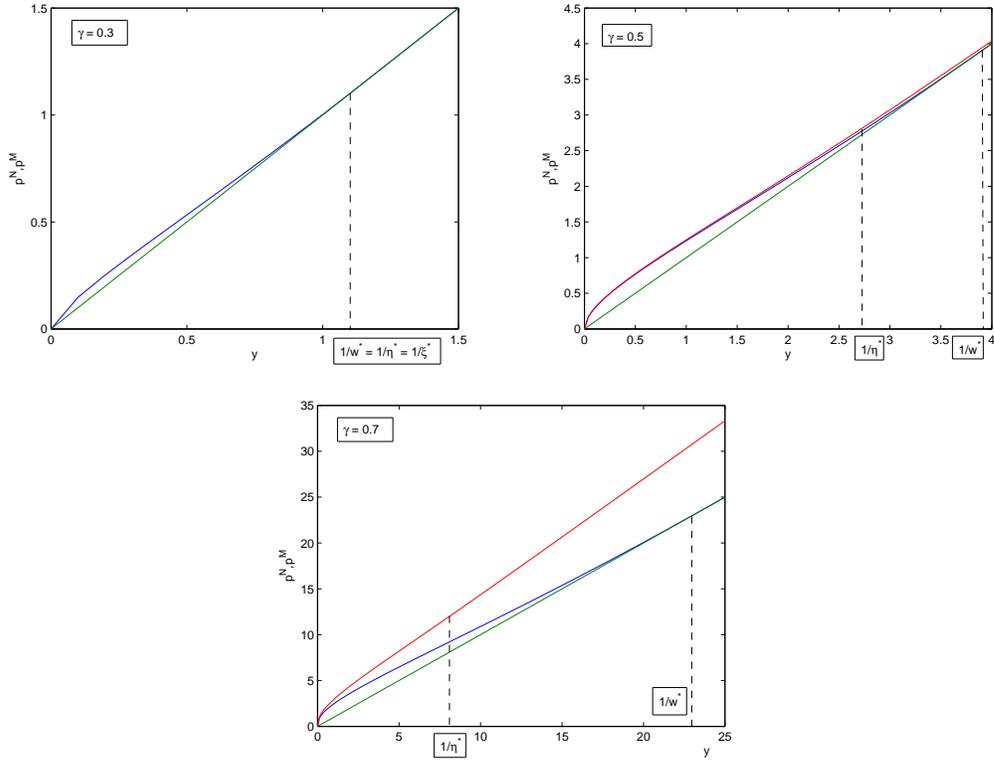


Figure 4: **The manager's certainty equivalent value in the no-market versus market case.** Each panel uses a different value of γ and $x = 1$ for all graphs. The top left graph takes $\gamma = 0.3$, so $1/w^* = 1/\eta^* = 1/\xi^* = 1.14$ and $p^N = p^M$. The top right graph uses $\gamma = 0.5$ giving $1/w^* = 3.92$, $1/\eta^* = 2.7$ and $1/\xi^* = 9.14$. The lower graph is for $\gamma = 0.7$ and $1/w^* = 23.46$ and $1/\eta^* = 8.1$. In this case, $\xi^* = -0.45$.

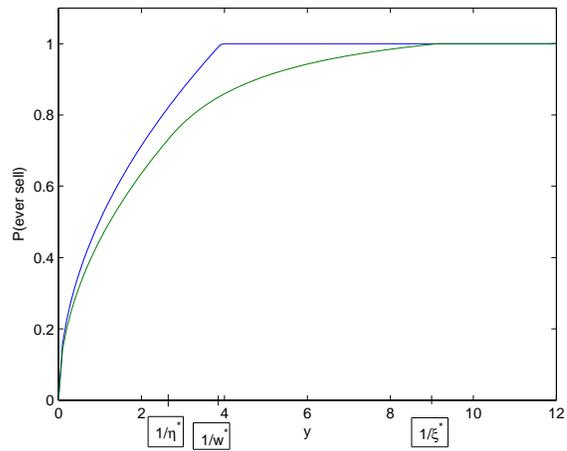


Figure 5: **Probability the manager ever sells the real asset.** *Initial wealth $x = 1$ and $\gamma = 0.5$. The higher line is the probability where there is no access to a market. The lower line is the probability when the manager can trade in a market asset.*

wealth drops below ξ^*y , where $\xi^* = -0.45$ (at which point he sells the real asset), or until his trading wealth exceeds $8.1y$ at which point the process is repeated. Since the lower plot in Figure 4 is for fixed x ($x = 1$), but the wealth of a manager with access to the market is not constant, the option value when the real asset is sold is not represented in the figure. In particular the time-value of the right to sell does not converge to zero as y increases.

C. The probability of exercise

We now investigate the probability that the option to sell the real asset is ever exercised. As we have seen, one of the effects of the presence of the market is that it allows the the manager to trade in such a way that the real asset is only sold when it forms a higher proportion of his total wealth compared to the no-market case. However, this comes at the “cost” of reducing the probability that the real asset is ever sold.

Without the market, for $x > yw^*$ the probability of ever reaching the optimal exercise ratio is (see (3))

$$\left(\frac{y}{x}\right)^{1-\gamma} (w^*)^{1-\gamma};$$

whereas, for $x \leq yw^*$ the probability is one. This is displayed as the upper line in Figure 5, for fixed $\gamma = 0.5$ and initial wealth of one. For $\gamma \leq \gamma_-$ this is also the probability of exercise in the presence of the market since the optimal stopping rule is unchanged.

Consider now the case with the market. Suppose $\gamma_- < \gamma < 1$, then $\xi^* < w^* < \eta^*$. For $x \geq \eta^*y$ the probability the asset is ever sold is

$$\left(\frac{y}{x}\right)^{1-\gamma} \frac{(\eta^*)^{2-\gamma}}{1-\gamma};$$

for $\xi^*y \leq x \leq \eta^*y$ the probability the asset is ever sold is

$$\eta^* \left(\frac{2-\gamma}{1-\gamma}\right) - \frac{x}{y};$$

whereas for $x \leq \xi^*y$ the option is exercised immediately and the probability of exercise is one. The lower line in Figure 5 displays the probability of sale when $\gamma = 0.5$, in the market case.

From Figure 5 we see that for $\gamma = 0.5$ the probability of selling the real asset is higher if there is no market. Direct comparisons of these probabilities for all values of γ ($\gamma_- < \gamma < 1$), results in the following proposition.

Proposition 6 *For fixed positive initial wealth, for $\gamma_- < \gamma < 1$ and for any initial value y for the real asset, the probability that the real asset is sold is lower in the model with the market than in the no-market situation.*

Proof: The proof of this proposition, and of some of the preceding statements are contained in Appendix A.

Roughly stated, when the manager has access to the market, it is possible for him to trade in such a way that at least part of the time his wealth is increased and then he is able to hold on to the real asset, and benefit from the positive returns, for a longer period. However, because he now sets a higher exercise threshold, the probability of exercise is reduced. However, care should be taken in interpreting these results as saying that the manager holds on to the real asset for longer in the market situation. When the real asset price to wealth ratio first hits $1/\eta^*$ the manager takes a large position in the market. If these investments are unsuccessful then he sells the real asset. Furthermore, in the large position limit, this can happen instantaneously. Hence there are scenarios in which the manager with access to the market exercises sooner than the manager without such access.

D. No-borrowing constraints.

In the paper we have assumed that the manager is free to borrow against his holdings in the real asset. The constraint on his wealth has been that he is forced to keep positive the sum of trading wealth and the intrinsic value of his position in the real asset. In effect, the constraint $X_t \geq -Y_t$ means that we allow him to borrow against the real asset. It is also possible to consider the optimal sale timing problem under the restriction that the manager must keep his trading wealth X_t positive and that his position is liquidated if ever his trading wealth reaches 0.

If $\xi^* \geq 0$ then the optimal strategy of the manager never allows his wealth to go negative and the imposition of a no-borrowing constraint has no effect. By the definition of ξ^* as a function of γ , the

critical value of γ where the no-borrowing constraint first starts to bite is the unique solution γ^+ in $(0, 1)$ of $\Gamma^+(\gamma) = 0$ where

$$\Gamma^+(\gamma) = (1 - \gamma) \ln \left(\frac{2 - \gamma}{1 - \gamma} \right) - \gamma.$$

We find $\gamma^+ = 0.5341$.

In the no-borrowing case, Proposition 2 remains valid, except that now when optimizing, we restrict attention to $0 \leq \xi \leq \eta$. For $\gamma \leq \gamma^+$ and for $\gamma \geq 1$ the solution is unchanged. For $\gamma^+ < \gamma < 1$ we find that the no-borrowing solution is as follows:

Proposition 7 *For $\gamma^+ < \gamma < 1$, the maximum of $\eta^{1-\gamma} \Theta(\eta, \xi)$ over $0 \leq \xi \leq \eta$ is attained at $\xi^* = 0$ and $\eta = \eta^*$ where $\eta^* = \eta^*(\gamma) = e^{-\gamma/(1-\gamma)}$. Under the no-borrowing constraint the value function becomes*

for $xe^{\gamma/(1-\gamma)} < y$

$$V^M(x, y) = \frac{e^{\gamma/(1-\gamma)}x}{y} \left(\ln y + \frac{1 - \gamma}{2 - \gamma} \right) + \left(1 - \frac{e^{\gamma/(1-\gamma)}x}{y} \right) \ln y,$$

and for $y \leq xe^{\gamma/(1-\gamma)}$

$$V^M(x, y) = \ln x + \left(\frac{y}{x} \right)^{1-\gamma} \frac{e^{-\gamma}}{(2 - \gamma)(1 - \gamma)}.$$

Proposition 7 tells us that the manager (when $\gamma^+ < \gamma < 1$) only sells or exercises when he becomes insolvent. Insolvency now refers to zero trading wealth, X . This contrasts with the earlier case of borrowing where (see Figure 2) the manager exercised before insolvency, where insolvency occurred when total wealth $X + Y$ reached zero. Since the manager who is able to borrow against the real asset wishes to do so when $\gamma > \gamma^+$, when borrowing constraints are imposed, he optimally chooses to exercise only when he reaches the constraint.

Given the value function it is straightforward to calculate the certainty equivalent value for the real option. However the results are not substantially different from the unconstrained case.

E. Other CRRA utilities.

The analysis of this paper has been completed under the assumption of logarithmic utility. However, it is interesting to consider the extent to which the conclusions are robust to a change in utility

function. Although logarithmic utility is very convenient for calculations and leads to closed form solutions for many of the quantities of interest, there is every reason to expect that for a large class of utility functions the introduction of an extra trading opportunity (albeit an opportunity in which it appears a manager would choose not to invest) would have an impact upon the value function.

To illustrate this claim, in Appendix B we present results for constant relative risk aversion with relative risk aversion parameter R . It turns out that the analysis of Sections I and II carries through to this more general case with only minor modifications. The key quantity of interest is the function $\gamma_-(R)$ which denotes the critical value at which the omission of the market asset begins to have an effect on the manager's behavior. Of secondary interest is the critical parameter value $\gamma^+(R)$ at which the optimal strategy of the manager first involves borrowing against the implicit wealth held in the real asset.

For logarithmic utility, for values of $\gamma > 1$ the manager never chooses to sell the real asset. In the constant relative risk aversion model, the relevant upper threshold for γ above which the manager waits indefinitely becomes $\min\{R, 1\}$.

Figure 6 displays $\gamma_-(R)$, $\gamma^+(R)$ and $\min\{R, 1\}$ for various values of R . For $R = 1$, the values correspond to those given previously in Lemma 3 and Proposition 7. Each of the functions $\gamma_-(R)$, $\gamma^+(R)$ and $\min\{R, 1\}$ are increasing in R so that the critical thresholds increase as risk aversion increases. We are most interested in the interval $(\gamma_-(R), \min\{R, 1\})$ which is the range of parameter values for which the omission of an underlying market makes a difference to the manager's behavior. This interval is largest for $R = 1$, corresponding to logarithmic utility.

We can now consider some limiting special cases. As $R \rightarrow 0$, both $\gamma_-(R)$ and $\gamma^+(R)$ tend to zero. Additionally, $\min\{R, 1\} \rightarrow 0$ so there is only one critical value of γ . The limiting case of $R \rightarrow 0$ is the risk-neutral case. In that situation a manager retains a real asset with positive expected returns indefinitely, and sells an asset with negative expected returns immediately. The presence of the market makes no difference. If we took the McDonald and Siegel (1986) set-up and specialized to our situation of asset sale (corresponding to an option with zero strike) it would give this degenerate solution. That

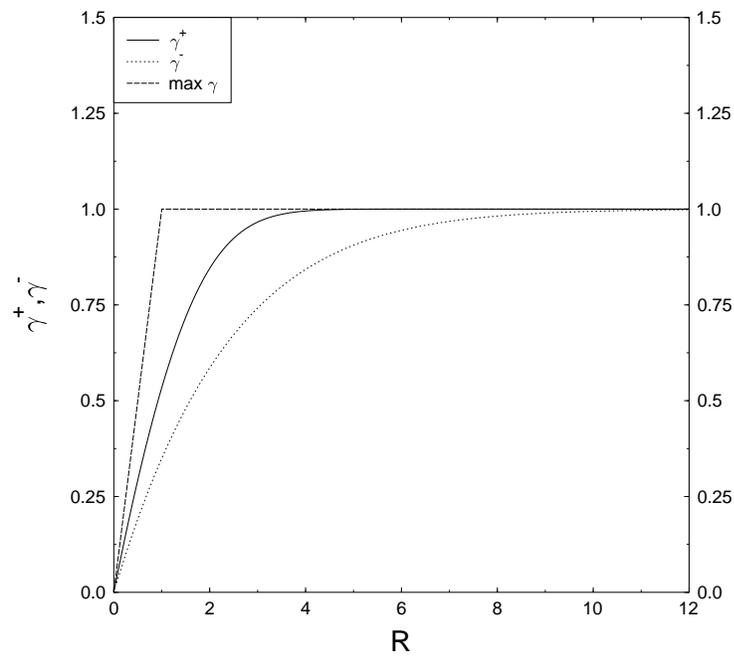


Figure 6: **Plot of the critical values of γ for a range of relative risk aversion levels R .** The lower curve is γ_- and the upper curve is γ^+ . The line labeled $\max \gamma$ is $\min\{R, 1\}$ and is the upper limit on γ above which it is never optimal for the manager to sell the real asset.

is, the special case where the manager is risk-neutral corresponds to that of McDonald and Siegel (1986).

As $R \rightarrow \infty$, both $\gamma_-(R), \gamma^+(R)$ tend to one, and the presence of a fair gamble or market does not affect the manager's actions. That is, the solution is exactly that of a manager with no market asset available for trading. This limiting case corresponds to exponential utility which has constant absolute risk aversion. Another way of seeing that the fair gamble has no effect under exponential utility is to return to Figure 3. We showed that the gambling of the manager takes place because his value function without the gamble is not concave everywhere in wealth. Under exponential utility, wealth scales out of the problem and appears in exponential form. Thus in this case, the value function is immediately concave in wealth and the gamble cannot improve the situation. Henderson (2004b) considered a manager with exponential utility and the option to invest. Her model took non-zero correlation between the real asset and the market, as her focus was to compare this situation to that of a complete market where the correlation is one. However, correlation could be set to zero in her model and consistent with what we have found here, her manager would not be affected by the fair gamble.

V Conclusion

We have shown in this paper that it can be optimal for a risk-averse manager to accept a fair gamble. In particular this happens if the manager is facing idiosyncratic risk in an incomplete market arising from the right to sell an indivisible asset. Without the right to sell the real asset, the manager's risk aversion would make him unwilling to invest in the fair gamble. Curiously, we find that the presence of the real asset causes him to become locally risk-seeking.

We presented our results both in a continuous time framework and in a simple one-period model. The latter is useful for intuition. In the region of parameter values where the manager can benefit from the gamble, the manager wants to hold the real asset for as long as possible, but sells when it becomes too large a proportion of his wealth. At this point, his risk-aversion to idiosyncratic risk

outweighs the benefit of holding onto the asset for future growth. However, the opportunity to invest in the fair gamble gives the manager a more varied strategy choice. If the gamble pays off, this allows him to retain the real asset for longer, as the real asset now makes up a smaller proportion of his wealth. This is the intuition behind the surprising finding that the manager benefits from being able to enter a fair gamble which is uncorrelated with his risk exposure.

The range of parameters (driving the real asset) for which the manager accepts the gamble is greatest when utility is logarithmic. In limiting cases, the manager does not benefit from having access to the gamble. As relative risk aversion increases, the limiting case of exponential utility is recovered and the manager never takes the gamble. As risk-aversion decreases to zero, we recover the risk-neutral model of McDonald and Siegel (1986). Again, the presence of the market does not change the optimal behavior of the manager.

It is also important to note that our finding that the manager accepts a fair gamble would not apply in a model where the real asset were divisible. There the manager could sell the real asset in small pieces over time. An important finding is that access to the gamble reduces the probability that the real asset is ever sold. Equivalently, a manager without the opportunity to invest in a fair gamble would have a higher probability of ever selling his real asset.

Our paper demonstrates that unexpected behavior may occur in a relatively straightforward model, and therefore care must be taken when modeling. Omission of assets which appear to have no impact on optimal behavior can lead to incorrect conclusions. This has implications for portfolio choice problems in incomplete markets, particularly those with an optimal timing component, such as those arising in the areas of real options and executive stock options.

Appendix

A Proofs for Section II

Proof of Proposition 2:

For $x \leq \xi y$ we exercise immediately and we have $V^M(x, y) = U(x + y) = \ln(x + y)$. For $\xi y < x < \eta y$, the strategy is to invest in the market until wealth reaches the extremes of this range. In the limit the investment positions are chosen so that this happens before the value of Y changes. Thus, using the martingale property for the investments in x ,

$$G(x, y) = \frac{x - \xi y}{\eta y - \xi y} G(\eta y, y) + \frac{\eta y - x}{\eta y - \xi y} G(\xi y, y),$$

and, by value matching on the boundaries $\xi y = x$, $\eta y = x$,

$$G(x, y) = \frac{\xi}{\eta - \xi} \left(\frac{x}{\xi y} - 1 \right) H(\eta y, y) + \frac{\eta}{\eta - \xi} \left(1 - \frac{x}{\eta y} \right) \ln(y + \xi y).$$

To complete the proof it is sufficient to show that

$$H(x, y) = \ln x + \left(\frac{\eta y}{x} \right)^{1-\gamma} \Theta$$

where Θ is as given in (8).

We have that for $y < x/\eta$

$$H(x, y) = \ln x \mathbb{P}(Y_t \text{ never reaches } x/\eta | Y_0 = y) + H(x, x/\eta) \mathbb{P}(Y_t \text{ hits } x/\eta | Y_0 = y)$$

and, at least in the case $\gamma \leq 1$, by (3) this becomes

$$H(x, y) = \ln x + [H(x, x/\eta) - \ln x] \left(\frac{\eta y}{x} \right)^{1-\gamma}.$$

It remains to prove that $H(x, x/\eta) - \ln x = \Theta$.

By the homogeneity of the problem we must have $V^M(\lambda x, \lambda y) = \ln \lambda + V^M(x, y)$. Applying this to H we deduce

$$x \frac{\partial H}{\partial x} + y \frac{\partial H}{\partial y} = 1. \tag{A1}$$

The derivative with respect to y is easy to calculate:

$$y \frac{\partial H}{\partial y} = (1 - \gamma)[H(x, x/\eta) - \ln x] \left(\frac{\eta y}{x}\right)^{1-\gamma}. \quad (\text{A2})$$

Further, by smooth pasting at $x = \eta y$

$$x \frac{\partial H}{\partial x} \Big|_{(x, x/\eta)} = x \frac{\partial G}{\partial x} \Big|_{(x, x/\eta)} = \frac{\eta}{\eta - \xi} [H(x, x/\eta) - \ln x + \ln \eta - \ln(1 + \xi)]$$

Collecting these expressions together, and evaluating (A1) and (A2) at $(x, x/\eta)$ we obtain

$$\frac{\eta}{\eta - \xi} [H(x, x/\eta) - \ln x + \ln \eta - \ln(1 + \xi)] + (1 - \gamma)[H(x, x/\eta) - \ln x] = 1,$$

and finally

$$H(x, x/\eta) - \ln x = \frac{(\eta - \xi) + \eta[\ln(1 + \xi) - \ln \eta]}{\eta + (1 - \gamma)(\eta - \xi)}.$$

□

Proof of Lemma 3:

The function Γ_- is monotonic functions on $(0, 1)$ with a positive value near zero, and a negative value at 1. Hence it has a unique root in $(0, 1)$.

For fixed $\gamma \in (0, 1)$ we want to maximise $\eta^{1-\gamma}\Theta(\eta, \xi)$. It is convenient to reparameterize the independent variables as η and $\delta = (\eta - \xi)/\eta$, so that $0 < \delta < 1 + 1/\eta$. Then, if $\Phi(\eta, \delta) = \eta^{1-\gamma}\Theta(\eta, \xi)$, we have

$$\Phi(\eta, \delta) = \eta^{1-\gamma} \left[\frac{\delta + \ln(1 - \delta + 1/\eta)}{1 + \delta(1 - \gamma)} \right]$$

For fixed δ it is clear that the maximum of Φ over η is attained at an interior point. However, this need not be the case for fixed η , and the maximum value of Φ may occur at $\delta = 0$. Thus we have to investigate the possibility of maxima of Φ which occur on the boundaries. We have

$$\frac{\partial \Phi}{\partial \eta} = \frac{\eta^{-\gamma}}{1 + \delta(1 - \gamma)} \left[(1 - \gamma) [\delta + \ln(1 - \delta + 1/\eta)] - \frac{1}{1 + \eta(1 - \delta)} \right] \quad (\text{A3})$$

and

$$\frac{\partial \Phi}{\partial \delta} = \frac{\eta^{1-\gamma}}{(1 + \delta(1 - \gamma))^2} \left[(1 + \delta(1 - \gamma)) \frac{1 - \eta\delta}{1 + \eta(1 - \delta)} - (1 - \gamma)(\delta + \ln(1 - \delta + 1/\eta)) \right] \quad (\text{A4})$$

Setting both expressions equal to zero and eliminating the logarithmic term we find that at a turning point

$$\delta\{\delta\eta(1-\gamma) + (\gamma + \eta - 1)\} = 0$$

Hence, for fixed η , there are at most two turning points given by

$$\delta_1 = 0 \quad \text{and} \quad \delta_2 = \frac{1 - \gamma - \eta}{\eta(1 - \gamma)}.$$

Note that $0 < \delta_2$ only if $\eta < (1 - \gamma)$.

Consider the turning point corresponding to $\delta_1 = 0$. For $\delta = 0$,

$$\frac{\partial\Phi}{\partial\eta} = \eta^{-\gamma} \left[(1 - \gamma) \ln(1 + 1/\eta) - \frac{1}{(1 + \eta)} \right].$$

Hence, recall (5), $(\eta = w^*(\gamma), \delta = 0)$ is a turning point of Φ . In order to determine whether this point is a (local) maximum it is necessary to consider the Hessian matrix of second derivatives. This is given by

$$\Phi''|_{(w^*, 0)} = \frac{(w^*)^{-\gamma}}{(1 + w^*)^2} \begin{bmatrix} -\frac{1}{w^*}[(1 - \gamma)(1 + w^*) - w^*] & (1 - \gamma)(1 + w^*) - w^* \\ (1 - \gamma)(1 + w^*) - w^* & -(w^*)^3 \end{bmatrix}.$$

This matrix is negative definite, and hence the turning point is a local maximum, provided $0 < (1 - \gamma)(1 + w^*) - w^* < (w^*)^2$. On substituting for γ using (5), we find that the first inequality is always satisfied, and the second is satisfied provided $w^*(1 + w^*) \ln(1 + 1/w^*) > 1$. When there is the equality $(1 - \gamma)(1 + w^*) - w^* = (w^*)^2$ we have $(1 - \gamma) = w^*$ and using the definition of Γ_- the necessity condition for a local maximum translates to $\gamma < \gamma_-$. Otherwise, for $\gamma > \gamma_-$, $(w^*(\gamma), 0)$ is a saddle point.

Finally consider the value $\delta = (1 - \gamma - \eta)/(\eta(1 - \gamma))$. Substituting this expression into $\partial\Phi/\partial\eta = 0$ we find

$$\eta = \frac{1 - \gamma}{2 - \gamma} \left[\frac{1}{1 - \gamma} - \ln \left(\frac{2 - \gamma}{1 - \gamma} \right) \right]^{-1}$$

and then

$$\delta = \frac{1}{(1 - \gamma)^2} \left[1 - (1 - \gamma)(2 - \gamma) \ln \left(\frac{2 - \gamma}{1 - \gamma} \right) \right] \quad (\text{A5})$$

Again, for $\delta > 0$ we see that we need $\gamma > \gamma_-$. The value of δ given in (A5) translates to (10). \square

Proof of Proposition 4:

Proposition 4 gives the value function for a general form of candidate strategy. In order to prove that the specific form of this strategy given by the optimizing values of η and ξ from Lemma 3 is optimal we need to show that the associated value function is maximal. It is sufficient to show that $V^M(x, y) \geq \ln(x + y)$ with equality in the stopping region, that $V^M(X_t, Y_t)$ is a supermartingale, and that $V^M(X_t, Y_t)$ is a martingale in the continuation region. These last properties are a straightforward exercise in stochastic calculus, using the fact that by definition the trading wealth X_t must be a martingale. Crucial in the proof is the fact that for the optimal η^*, ξ^* we have smooth fit at $x = \xi^*y$.

Note that in case (iv) we know from Section I that when $\gamma \geq 1$ there is a strategy for which wealth is constant and for which the value function is infinite. Since this strategy remains feasible in the case with a market, the value function must be infinite in this case also. \square

Proof of Proposition 6:

Suppose we are in the situation with the market and that $\gamma_- < \gamma < 1$. The expressions for general (x, y) for the probability that the asset is ever sold follow easily if it can be shown that, for $x = \eta^*y$, the probability q that the asset is ever sold is $\eta^*/(1 - \gamma)$. An expression for q can be deduced by considering the effect of trading until wealth reaches $\eta^* \pm \epsilon$ for small ϵ . This expression simplifies greatly using the fact that $\eta^* + (\eta^* - \xi^*)(1 - \gamma) = (1 - \gamma)$.

It remains to show that $q^N(x, y) \geq q^M(x, y)$ where q^N and q^M are the probability of exercise in the no-market and market cases respectively. By scaling, both functions can be rewritten as $q(w)$ where $w = x/y$, and then $q^N(w) = \min\{1, (w^*/w)^{1-\gamma}\}$ and $q^M(w) = 1$ for $w \leq \xi^*$, $q^M(w) = \eta^*(2 - \gamma)/(1 - \gamma) - w$ for $\xi^* < w \leq \eta^*$ and $q^M(w) = (\eta^*/w)^{1-\gamma}(\eta^*/(1 - \gamma))$ for $w > \eta^*$. It is easy to see from the graph of these functions that a sufficient condition for $q^M(w) \leq q^N(w)$ is $q^M(\eta^*) \leq q^N(\eta^*)$ and $|(q^M)'|(\eta^*) \leq |(q^N)'|(\eta^*)$, where $'$ denotes derivative. Both these conditions

reduce to $(1 - \gamma)(w^*)^{1-\gamma}/(\eta^*)^{2-\gamma} \leq 1$ which can be checked, for example by plotting the various quantities as functions of γ . \square

Proof of Proposition 7:

The key step is to show that $\eta^{1-\gamma}\Theta(\eta, \xi)$ is maximised over the given range at $\xi = 0, \eta = e^{-\gamma/(1-\gamma)}$.

The constraint $\xi \geq 0$ becomes $\delta \leq 1$. At $\delta = 1$ we have from (A3) that

$$\frac{\partial \Phi}{\partial \eta} = \frac{\eta^{-\gamma}}{(2 - \gamma)} [(1 - \gamma)(1 - \ln \eta) - 1]$$

so that it follows the maximum value when $\delta = 1$ is attained at $\eta = e^{-\gamma/(1-\gamma)}$. It is easy to check that this is a global maximum by checking the derivative $\partial \Phi / \partial \delta$ in (A4). \square

B Constant Relative Risk Aversion

In this section we extend the results for logarithmic utility to power law utilities of the form

$$U(x) = U_R(x) = \frac{x^{1-R} - 1}{1 - R}, \quad R \in (0, \infty), R \neq 1.$$

The slightly non-standard form of the utility function is chosen so that in the limit $R \rightarrow 1$ we recover logarithmic utility: recall that $\lim_{R \rightarrow 1} (u^{1-R} - 1)/(1 - R) = \ln u$. As a result the logarithmic case of Sections I and II can be recovered immediately from the results of this section, in the limit $R \rightarrow 1$. In all cases the ideas behind the proofs are identical to those in the logarithmic case so we omit them.

In the no-market case the expected utility of the agent using the stopping rule τ_w and power utility is

$$\begin{aligned} F(w) = \mathbb{E}[U(x + Y_{\tau_w})] &= \frac{(x^{1-R} - 1)}{1 - R} \mathbb{P}(\tau_w = \infty) + \frac{x^{1-R}(1 + 1/w)^{1-R} - 1}{1 - R} \mathbb{P}(\tau_w < \infty) \\ &= \frac{x^{1-R}[1 + \{(1 + 1/w)^{1-R} - 1\} \mathbb{P}(\tau_w < \infty)] - 1}{1 - R}. \end{aligned}$$

For $\gamma \geq 1$, $\mathbb{P}(\tau_w < \infty) = 1$, and it is clear in this case that F is maximized over w by taking $w = 0$.

Otherwise, using the formula (3)

$$F'(w) = \left(\frac{y}{x}\right)^{1-\gamma} x^{1-R} w^{-\gamma} \left[(1-\gamma) \frac{(1+1/w)^{1-R} - 1}{1-R} - \frac{(1+1/w)^{-R}}{w} \right]$$

Since, for $w > 0$

$$\frac{(1+1/w)^{1-R} - 1}{1-R} > \frac{(1+1/w)^{-R}}{w} > (1+1/w)^{1-R} - 1$$

it follows that for $\gamma \leq 0$ there is no solution to $F'(w) = 0$ and F is maximized by the choice $w = \infty$, and for $\gamma \geq R$, there is again no solution to $F'(w) = 0$ and F is maximized by the choice $w = 0$. The interesting case is when $0 < \gamma < R \wedge 1$. In that case F is maximized by the unique solution w^* in $(0, \infty)$ to $F'(w) = 0$. The following proposition describes the optimal behavior and value function for a manager with power utility.

Proposition 8 *For all $R \neq 1$ and $\gamma \leq 0$, $V^N(x, y) = ((x+y)^{1-R} - 1)/(1-R)$, for $R < 1$ and $\gamma \geq R$, $V^N(x, y) = \infty$ and for $R > 1$ and $\gamma \geq 1$, $V^N(x, y) = 1/(R-1)$.*

In the non-degenerate cases $0 < \gamma < R \wedge 1$, $V^N(x, y) = ((x+y)^{1-R} - 1)/(1-R)$ in the exercise region $y \geq x/w^$, and in the continuation region $y < x/w^*$*

$$V^N(x, y) = \frac{x^{1-R} - 1}{1-R} + \left(\frac{yw^*}{x}\right)^{1-\gamma} x^{1-R} \frac{[(1+1/w^*)^{1-R} - 1]}{1-R}. \quad (\text{A6})$$

The optimal exercise ratio w^ solves*

$$(1-\gamma) \frac{(1+1/w)^{1-R} - 1}{1-R} - \frac{(1+w)^{-R}}{w} = 0 \quad (\text{A7})$$

Note that the formulæ (4) and (5) follow immediately on taking the limit $R = 1$ in (A6) and (A7).

One feature of the problem with relative risk aversion R is the fact that the optimal stopping problem has a degenerate solution for $\gamma \geq R \wedge 1$. When $\gamma > 1$ the real asset drifts to plus infinity, and so it is clearly never optimal to exercise at any finite threshold, the real asset is simply too good an investment. However in the case $R < \gamma \leq 1$, even though the value of the real asset will converge to zero almost surely, the expected value of $(Y^{1-R} - 1)/(1-R)$ tends to plus infinity, and the real asset is worth an infinite amount.

Now consider the optimization problem embedded in a market in the sense of Section II.

Proposition 9 Under the strategy described before Proposition 2, and specified by η, ξ , the value function is given by

$$V^M(x, y) = \begin{cases} U(x + y) & x \leq \xi y \\ G(x, y) & \xi y \leq x < \eta y \\ H(x, y) & \eta y < x \end{cases}$$

where

$$G(x, y) = \left(\frac{x}{\xi y} - 1 \right) \frac{\xi}{\eta - \xi} H(\eta y, y) + \left(1 - \frac{x}{\eta y} \right) \frac{\eta}{\eta - \xi} \left[\frac{y^{1-R}(1 + \xi)^{1-R} - 1}{1 - R} \right]$$

and

$$H(x, y) = \frac{x^{1-R} - 1}{1 - R} + y^{1-\gamma} x^{\gamma-R} \eta^{1-\gamma} \Theta$$

with

$$\Theta \equiv \Theta(\eta, \xi) = \left[\frac{\eta - \xi + \eta \left(\frac{\eta^{R-1}(1+\xi)^{1-R} - 1}{(1-R)} \right)}{\eta + (\eta - \xi)(R - \gamma)} \right].$$

In order to find the optimal strategy from stopping rules of this class we need to find the maximum of $\eta^{1-\gamma} \Theta$.

Lemma 10 Let $\gamma_-(R)$ be the unique solution in $(0, R \wedge 1)$ of

$$(R - \gamma)^R (R + 1 - \gamma) = (2R - \gamma)^R (1 - \gamma)$$

and let $\gamma^+(R)$ be the unique solution in $(0, R \wedge 1)$ to

$$(R - \gamma)^R (R + 1 - \gamma)^{1-R} = R^R (1 - \gamma).$$

Then $\gamma_-(R) < \gamma^+(R)$.

For $0 < \gamma \leq \gamma_-(R)$ the maximum of $\eta^{1-\gamma} \Theta$ is attained at $\xi^*(\gamma, R) = \eta^*(\gamma, R) = w^*$ where w^* is the solution to (A7).

For $\gamma_-(R) \leq \gamma \leq 1$ the maximum of $\eta^{1-\gamma} \Theta(\eta, \xi)$ over $\xi > -1$, $\eta > 0$ and $\eta \geq \xi$ is attained at (η^*, ξ^*) which are given by

$$\eta^* = \frac{(R - \gamma)(1 - R)}{R} \left[(R + 1 - \gamma) - (R - \gamma) \left(\frac{1 + R - \gamma}{1 - \gamma} \right)^{1/R} \right]^{-1}$$

and

$$\xi^* = \frac{\eta^*(R+1-\gamma)}{(R-\gamma)} - \frac{1}{R}.$$

If the no-borrowing constraint is introduced, then for $\gamma \leq \gamma^+(R)$ the maximising η and ξ are unchanged, otherwise for $\gamma^+(R) \leq \gamma \leq R \wedge 1$ the maximum of $\eta^{1-\gamma}\Theta$ is attained at $\xi = 0$ and

$$\eta = \eta^*(\gamma, R) = \left(\frac{R-\gamma}{R(1-\gamma)} \right)^{1/(1-R)}.$$

This lemma allows us to state the analogue of Theorem 5.

Theorem 11 For $\gamma \leq \gamma_-(R)$ the solution with the market is identical to the solution of the optimal stopping problem with no market given in Section I.

For $\gamma_-(R) < \gamma < R \wedge 1$ the solution to the optimal stopping problem (7) with the market asset is different to the solution of (1) without the market asset.

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