

A SIMPLE PROOF OF KRAMKOV'S RESULT ON UNIFORM SUPERMARTINGALE DECOMPOSITIONS

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Abstract: we give a simple proof of Kramkov's uniform optional decomposition in the case where the class of density processes satisfies a suitable closure property. In this case the decomposition is previsible.

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§1 Introduction

In [4], Kramkov showed that for a suitable class of probability measures, \mathcal{P} , on a filtered measure space $(\Omega, \mathcal{F}, \mathcal{F}_t; t \geq 0)$, if S is a non-negative supermartingale under all $\mathbb{Q} \in \mathcal{P}$, then there is a uniform optional decomposition of S into the difference between a \mathcal{P} -uniform local martingale and an increasing optional process. In this note we give (in Theorem 2.2) a simple proof of this result in the case where the martingale logarithms of the density processes of the p.m.s in \mathcal{P} (taken with respect to a suitable reference p.m.) are closed under scalar multiplication (and hence continuous).

The applications in [4] refer to the financial set-up, where \mathcal{P} is the collection of Equivalent Martingale Measures for a collection of discounted securities \mathcal{X} , and S is the payoff to a superhedging problem for an American option, so that

$$S_t = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{P}} \operatorname{ess\,sup}_{\text{stopping times } \tau \geq t} \mathbb{E}[X_\tau | \mathcal{F}_t],$$

where X is the claims process for the option.

Other examples are a multi-period coherent risk-measure where the risk measure ρ_t is given by

$$\rho_t(X) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}[X | \mathcal{F}_t]$$

(see [4]) and the Girsanov approach to a control set-up, where S is given by the same formula, but \mathcal{P} corresponds to a collection of costless controls on X (see, for example, [1]).

§2 Uniform supermartingale decomposition

We assume that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, satisfying the usual conditions, and a collection, \mathcal{P} , of probability measures on (Ω, \mathcal{F}) such that $\mathbb{Q} \sim \mathbb{P}$, for all $\mathbb{Q} \in \mathcal{P}$.

We note that, since $\mathbb{Q} \sim \mathbb{P}$, $\Lambda_t^{\mathbb{Q}} \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t}$ is a positive \mathbb{P} -martingale, with $\Lambda_0^{\mathbb{Q}} = 1$.

Lemma 2.1. We may write $\Lambda_t^{\mathbb{Q}} = \mathcal{E}(\lambda^{\mathbb{Q}})_t$, where \mathcal{E} is the Doleans-Dade exponential and $\lambda_t^{\mathbb{Q}} = \int_0^t \frac{d\Lambda_s^{\mathbb{Q}}}{\Lambda_{s-}^{\mathbb{Q}}}$, so that $\lambda^{\mathbb{Q}}$ is a \mathbb{P} -local martingale with jumps strictly bounded below by -1 .

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Proof From Theorem II.8.3 of [4], if neither $\Lambda_t^{\mathbb{Q}}$ nor $\Lambda_{t-}^{\mathbb{Q}}$ vanishes then $\lambda^{\mathbb{Q}}$ exists. The fact that $\Lambda^{\mathbb{Q}}$ does not vanish follows from the stronger statement that, since $\mathbb{Q} \sim \mathbb{P}$, $\Lambda^{\mathbb{Q}}$ is strictly positive. This also implies, once we have established its existence, the condition on the jumps of $\lambda^{\mathbb{Q}}$.

The fact that $\Lambda_{t-}^{\mathbb{Q}}$ does not vanish follows via the following argument. First note that $\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t} = (\Lambda_t^{\mathbb{Q}})^{-1}$, so $(\Lambda_t^{\mathbb{Q}})^{-1}$ is a \mathbb{Q} -martingale. Now \mathcal{M}^{loc} , the class of local martingales, is equal to H_1^{loc} , the localisation of $H_1 = \{\text{martingales } M : \mathbb{E}[\sup_{0 \leq t < \infty} |M_t|] < \infty\}$ [see [2]]. So, it follows that there is a localising sequence (T_n) of stopping times increasing (\mathbb{Q} and hence) \mathbb{P} -a.s. to ∞ such that $\mathbb{E}[\sup_{t \leq T_n} (\Lambda_t^{\mathbb{Q}})^{-1}] < \infty$. It follows from the integrability that $\sup_{t \leq T_n} (\Lambda_t^{\mathbb{Q}})^{-1}$ is \mathbb{P} -a.s. finite and hence $\inf_{t \leq T_n} \Lambda_t^{\mathbb{Q}}$ is \mathbb{P} -a.s. strictly positive. Thus $\Lambda_{t-}^{\mathbb{Q}}$ is \mathbb{P} -almost surely positive on the stochastic interval $[[0, T_n]]$. Now letting $n \uparrow \infty$ we see that the second requirement is satisfied. \square

We denote by \mathcal{L} the collection $\{\lambda^{\mathbb{Q}}; \mathbb{Q} \in \mathcal{P}\}$ and by \mathcal{L}^{loc} the usual localisation of \mathcal{L} .

Theorem 2.2 *Suppose that*

- i) $\mathbb{P} \in \mathcal{P}$; and
- ii) \mathcal{L}^{loc} is closed under scalar multiplication;

then any \mathcal{P} -uniform non-negative supermartingale, S , possesses a class-uniform Doob-Meyer predictable decomposition, i.e. we may write S uniquely as

$$S = M - A,$$

where M is a \mathcal{P} -uniform local martingale and A is a locally integrable predictable increasing process with $A_0 = 0$.

Remark: Notice that condition (ii) implies that every element of \mathcal{L}^{loc} is continuous. This follows since: any element of \mathcal{L}^{loc} has jumps bounded below by -1 ; then if $\delta\lambda \in \mathcal{L}^{loc}$ for all $\delta \in \mathbb{R}$, by taking appropriately large positive and negative values of δ , we see that the jumps of λ must be of size zero.

Proof of Theorem 2.2: take $\mathbb{Q} \in \mathcal{P}$, with $\Lambda^{\mathbb{Q}} = \mathcal{E}(\lambda^{\mathbb{Q}})$. Now S is a non-negative \mathbb{P} -supermartingale. It follows that S is locally of class D and so we can take the (unique) Doob-Meyer decomposition of S with respect to \mathbb{P} : $S = M - A$, where M is a \mathbb{P} -local martingale and A is a predictable increasing process.

Now S is a non-negative \mathbb{Q} -supermartingale if and only if $S\Lambda^{\mathbb{Q}}$ is a non-negative \mathbb{P} -supermartingale. Thus, since it follows from the remark that $\Lambda^{\mathbb{Q}}$ is continuous,

$$\begin{aligned} S\Lambda^{\mathbb{Q}} &= S_0 + \int_0^\cdot S_{t-} d\Lambda_t^{\mathbb{Q}} + \int_0^\cdot \Lambda_t^{\mathbb{Q}} dS_t + \langle S, \Lambda^{\mathbb{Q}} \rangle. \\ &= S_0 + \int_0^\cdot S_{t-} d\Lambda_t^{\mathbb{Q}} + \int_0^\cdot \Lambda_t^{\mathbb{Q}} dM_t + \int_0^\cdot \Lambda_t^{\mathbb{Q}} (d \langle \lambda^{\mathbb{Q}}, M \rangle_t - dA_t) \end{aligned} \quad (2.2)$$

is a (non-negative) \mathbb{P} -supermartingale. Now since the first two terms in the last line of (2.2) are local martingales, whilst the last is a predictable process of finite variation, it follows that the last term must be decreasing.

Now we claim that we must then have

$$\langle \lambda^{\mathbb{Q}}, M \rangle^+ \ll A, \text{ with } \frac{d\langle \lambda^{\mathbb{Q}}, M \rangle^+}{dA} \leq 1, \quad (2.2)$$

where $\langle \lambda^{\mathbb{Q}}, M \rangle^+$ and $\langle \lambda^{\mathbb{Q}}, M \rangle^-$ are, respectively, the increasing processes corresponding to the positive and negative components in the Hahn decomposition of the signed measure induced by $\langle \lambda^{\mathbb{Q}}, M \rangle$.

This follows from the more general statement: if μ , m^+ and m^- are three σ -finite measures on a measurable space (Ω, \mathcal{F}) and

(i) m^+ and m^- are mutually singular

and

(ii) $\nu \stackrel{\text{def}}{=} \mu - m^+ + m^-$ is also a measure, then

$$m^+ \ll \mu, \text{ with } \frac{dm^+}{d\mu} \leq 1.$$

To see this, suppose that m^+ is supported by B and $m^-(B) = 0$ then take $A \in \mathcal{F}$ with $m^+(A) > 0$. It follows that $m^+(A) = m^+(A \cap B)$ and $m^-(A \cap B) = 0$ so

$$\nu(A \cap B) = \mu(A \cap B) - m^+(A \cap B) \geq 0.$$

Thus $\mu(A) \geq \mu(A \cap B) > 0$ and

$$m^+(A) = m^+(A \cap B) \leq \mu(A \cap B) \leq \mu(A).$$

Now \mathcal{L}^{loc} is closed under scalar multiplication so that, again localising if necessary, we may assume that $\delta\lambda \in \mathcal{L}$ and so, defining \mathbb{Q}^δ by $\Lambda^{\mathbb{Q}^\delta} \stackrel{\text{def}}{=} \mathcal{E}(\delta\lambda^{\mathbb{Q}})$, we see that (2.2) holds with $\lambda^{\mathbb{Q}}$ replaced by $\delta\lambda^{\mathbb{Q}}$ for any $\delta \in \mathbb{R}$. Letting $\delta \rightarrow \infty$ we see that $\frac{d\langle \lambda^{\mathbb{Q}}, M \rangle^+}{dA} = 0$, whilst letting $\delta \rightarrow -\infty$ we see that $\frac{d\langle \lambda^{\mathbb{Q}}, M \rangle^-}{dA} = 0$. It follows immediately that

$$\langle \lambda^{\mathbb{Q}}, M \rangle \equiv 0.$$

To complete the proof we need simply observe that

$$M\Lambda^{\mathbb{Q}} = M_0 + \int M_{t-}d\Lambda_t^{\mathbb{Q}} + \int \Lambda_t^{\mathbb{Q}}dM_t + \int \Lambda_t^{\mathbb{Q}}d\langle \lambda^{\mathbb{Q}}, M \rangle_t,$$

and hence M is a \mathbb{Q} -local martingale and since \mathbb{Q} is arbitrary, the result follows \square

Remark: We note that if \mathcal{P} consists of the EMMs for a vector-valued martingale M and the underlying filtration supports only continuous martingales (for example if it is the filtration of a multi-dimensional Wiener process), then the conditions of Theorem 2.2 are satisfied. This follows since, under these conditions, if λ is a \mathbb{P} -local martingale then

$$\lambda \in \mathcal{L}^{loc} \Leftrightarrow \langle \lambda, M \rangle = 0,$$

and the same then holds for any multiple of λ .

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