Short-length routes in low-cost networks

(joint work with David Aldous)

Wilfrid Kendall
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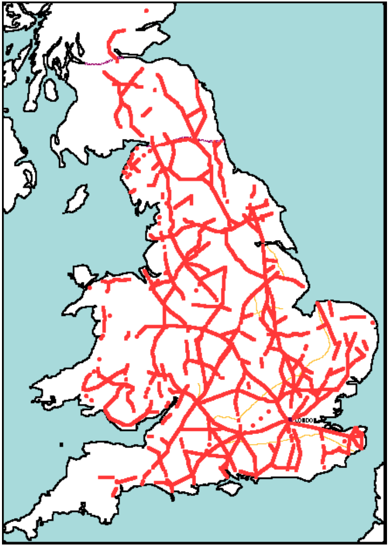
Colloquium talk

THE UNIVERSITY OF WARWICK

An ancient optimization problem



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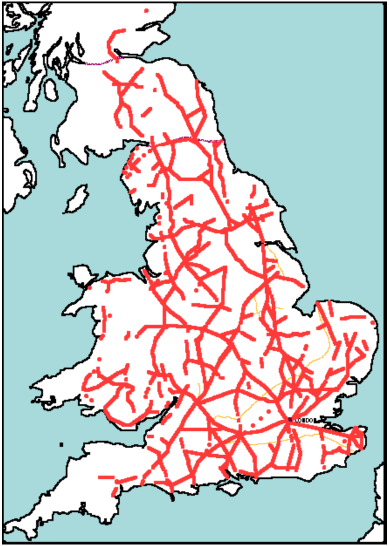

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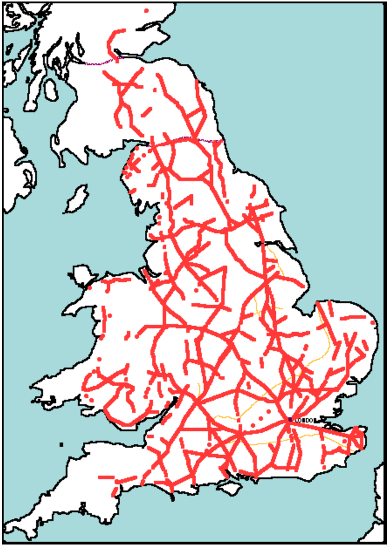

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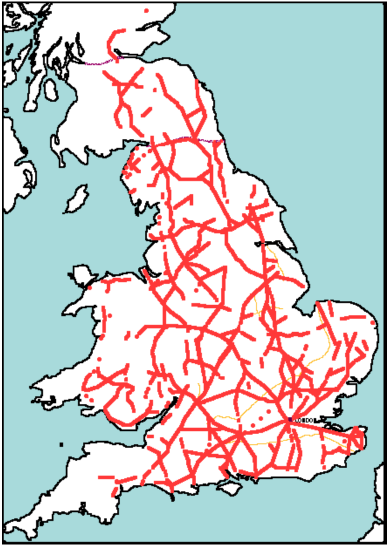

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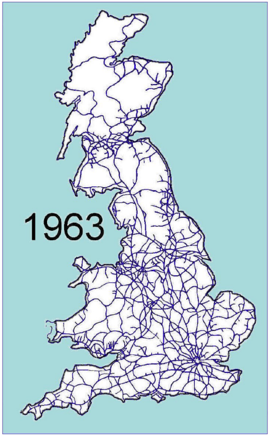
British Railway network before Beeching



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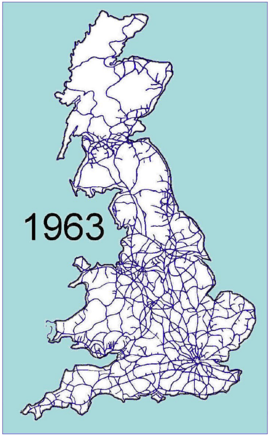

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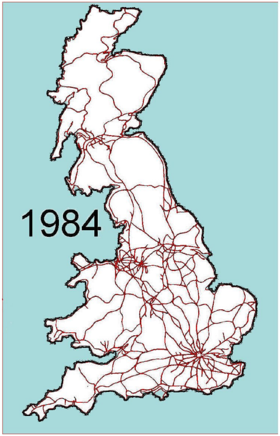

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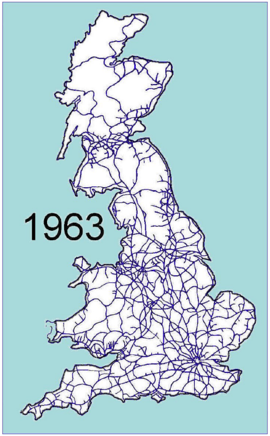

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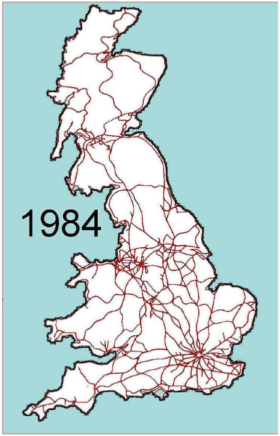

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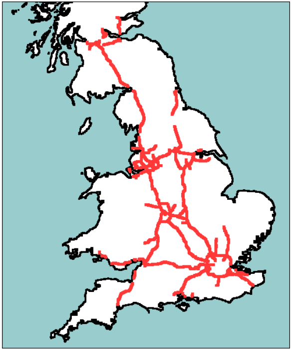

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Aldous and Kendall (2008) provide answers for the

First Question

Consider a configuration $x^{(N)}$ of N cities in $[0, \sqrt{N}]^2$ as above, and a well-chosen connecting network $G = G(x^{(N)})$. How does large-N trade-off between $\operatorname{len}(G)$ and $\operatorname{average}(G)$ behave?

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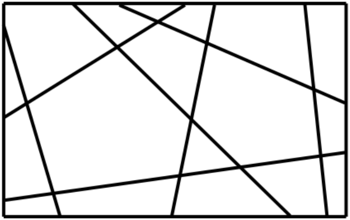
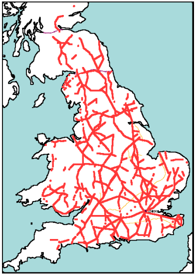
Question about flows

Consider a network which exhibits good trade-offs. What can be said about flows in this network?

First question (I)





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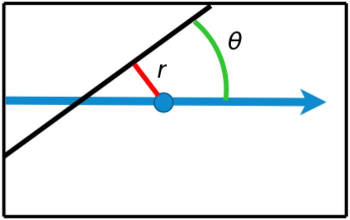

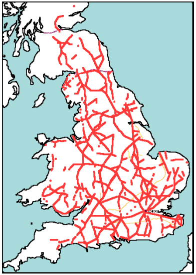

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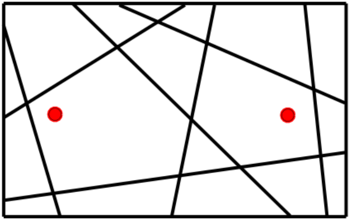

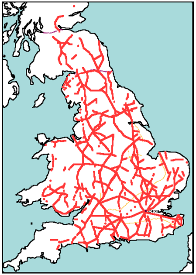

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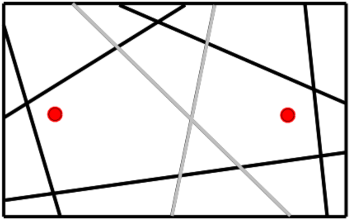

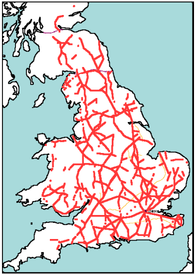

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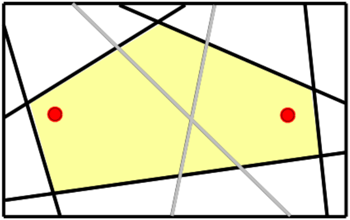

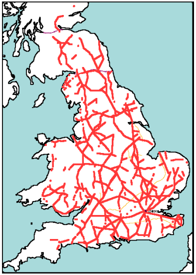

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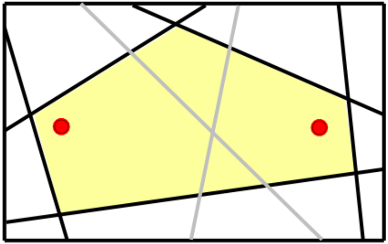
  focus on cell $\mathcal{C}_{x,y}$ containing the two cities.

First question (II)



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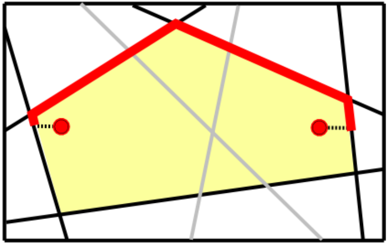

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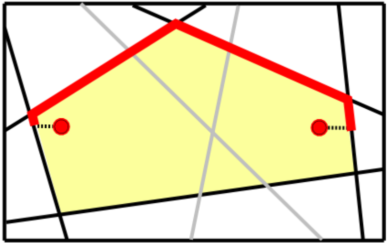

- Upper-bound "network distance" between two cities by mean semi-perimeter of cell, $\frac{1}{2}\mathbb{E}\left[\text{len}\,\partial C_{x,y}\right]$.
- Aldous and Kendall (2008) answer First Question using this, and use other methods from stochastic geometry to show that the resolution is nearly optimal.

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- A famous conjecture (late 1940's) by D. G. Kendall is that large cells in the line process tessellation are nearly circular. This is now known to be true (Miles, Kovalenko).
- the project builds on a wide range of work: from 300-year-old French encyclopaedist to recent calculations on self-similar random processes.

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- (H. Steinhaus) compute length of *regularizable* curve by counting mean number of hits by unit-intensity invariant Poisson line process.

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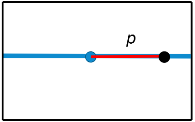

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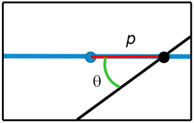

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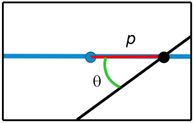
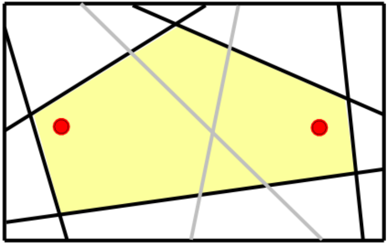


The key construction



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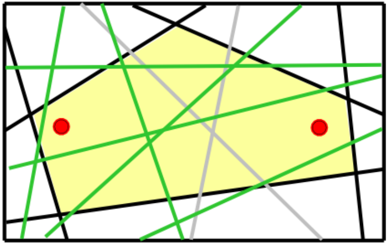

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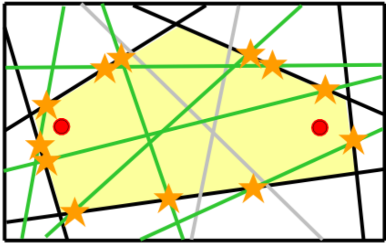

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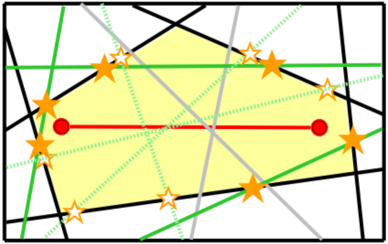

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- Compute mean length of $\partial C_{x,y}$ by use of independent unit-intensity invariant Poisson line process Π_2, and determine the mean number of hits.
- It is convenient to form Π_2^* by deleting from Π_2 those lines separating x from y. (Mean number of hits: $2|x-y| = 2n$.)

Mean perimeter length as a double integral

Theorem

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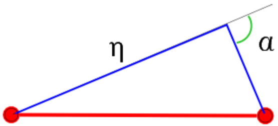

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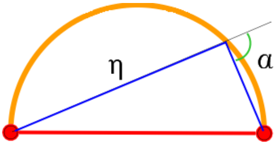

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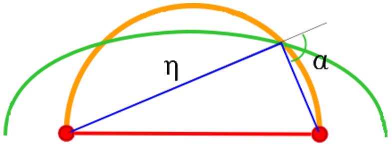

- Fixed α: locus of z is circle.
- Fixed η: locus of z is ellipse.

Asymptotics

Theorem

Careful asymptotics for $n \to \infty$ show that

$$\mathbb{E}\left[\tfrac{1}{2}\operatorname{len}\partial C_{x,y}\right] = n + \tfrac{1}{4}\iint_{\mathbb{R}^2}(\alpha - \sin\alpha)\exp\left(-\tfrac{1}{2}(\eta - n)\right)\mathrm{d}\,z \approx n + \frac{4}{3}\left(\log n + \gamma + \frac{5}{3}\right)$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

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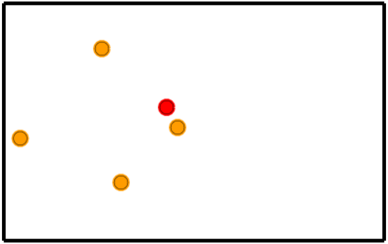
Thus a unit-intensity invariant Poisson line process is within $O(\log n)$ of providing connections which are as efficient as Euclidean connections.

Illustration of the final construction



Use a hierarchy

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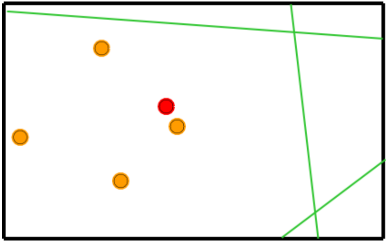

Use a hierarchy of:

1. a (sparse) Poisson line process;

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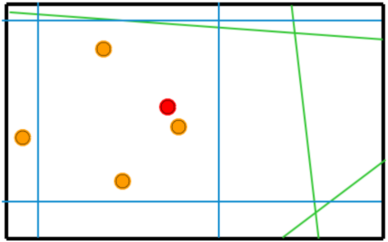

Use a hierarchy of:

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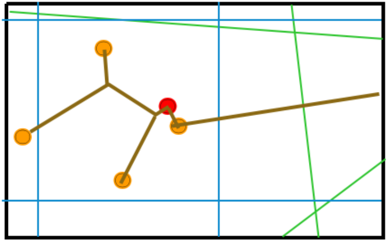

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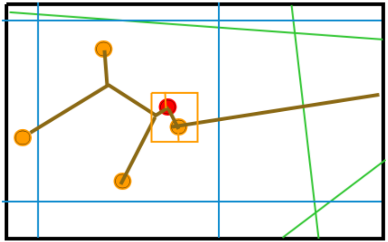

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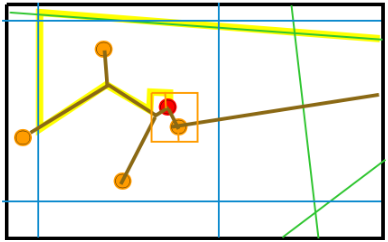

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Answering the first question

Theorem

For any configuration $x^{(N)}$ in square side $\sqrt{N}$ and for any sequence $w_N \to \infty$ there are connecting networks G_N such that:

$$\begin{aligned} \text{len}(G_N) &= \text{len}(\text{ST}(x^{(N)})) + o(N) \\ \text{average}(G_N) &= \frac{1}{N(N-1)} \sum_{i \neq j} \sum \|x_i - x_j\| + o(w_N \log N) \end{aligned}$$

The sequence $\{w_N\}$ can tend to infinity arbitrarily slowly.

A complementary result

Theorem

Given a configuration of N cities in $[0, \sqrt{N}]^2$ which is $L_N = o(\sqrt{\log N})$-equidistributed: random choice X_N of city can be coupled to uniformly random point Y_N so that

$$\mathbb{E}\left[\min\left\{1, \frac{|X_N - Y_N|}{L_N}\right\}\right] \quad \longrightarrow \quad 0\,;$$

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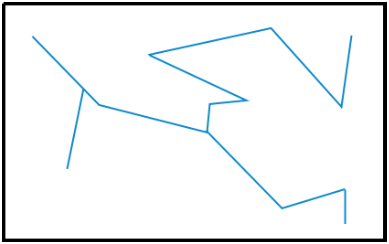
Sketch of proof



Use tension between two facts:

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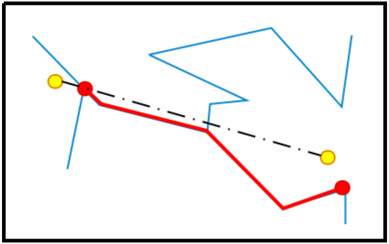

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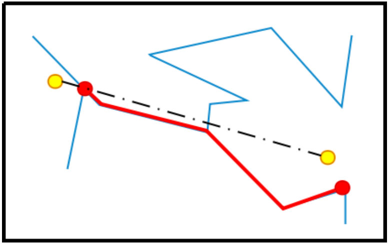

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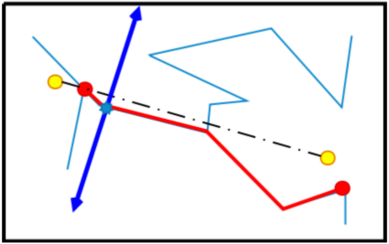

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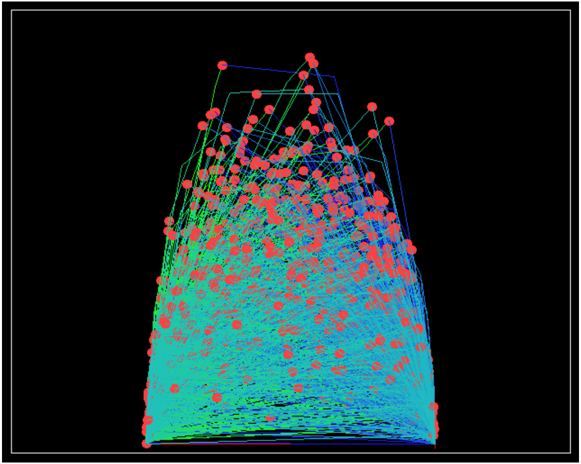
so a random perpendicular to the Euclidean path is almost a uniformly random line.

Simulations (example)



1000 simulations
at $n = 1000000$:
average 21.22,
s.e. 0.23,
asymptotic 21.413.

Vertical exaggeration:
$\sqrt{n}$

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QUESTIONS?

Bibliography

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