An Operational Measure of Riskiness

Sergiu Hart

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Joint work with

Dean P. Foster

The Wharton School

University of Pennsylvania

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Papers

- Dean Foster and Sergiu Hart
  "An Operational Measure of Riskiness" (2009)
  Journal of Political Economy
  `www.ma.huji.ac.il/hart/abs/risk.html`

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  "A Reserve-Based Axiomatization of the Measure of Riskiness" (2008)
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Papers (continued)

- Sergiu Hart
  "A Simple Riskiness Order Leading to the Aumann–Serrano Index of Riskiness" (2008)
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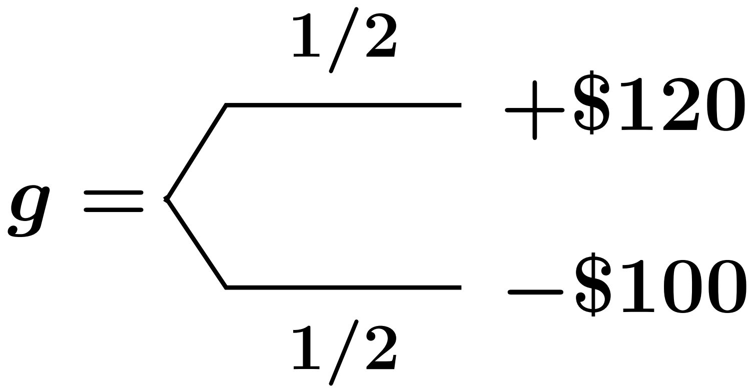
- Sergiu Hart
  "Comparing Risks by Acceptance and Rejection" (2009)
  www.ma.huji.ac.il/hart/abs/risk-u.html

I: Introduction

A gamble



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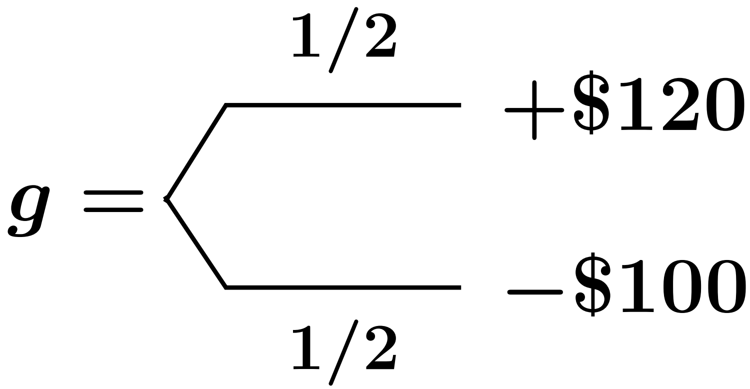

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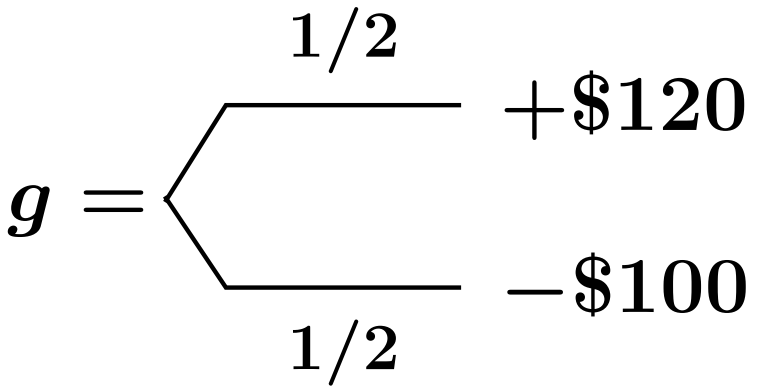

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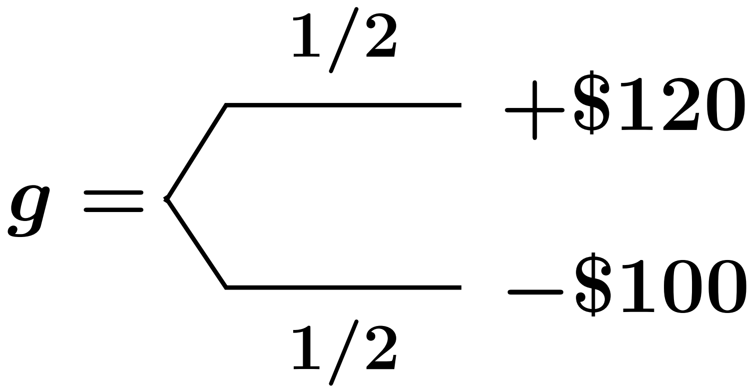

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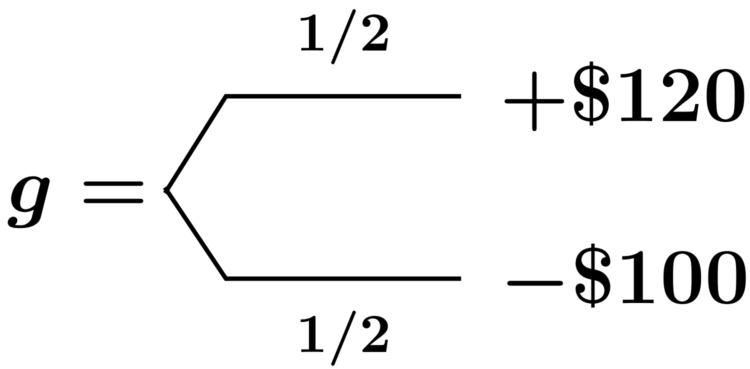

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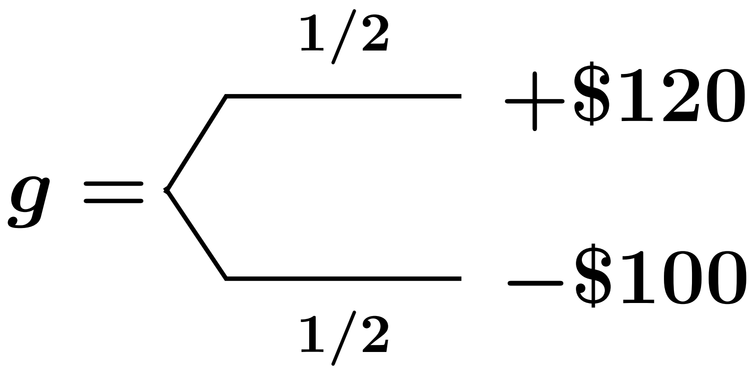

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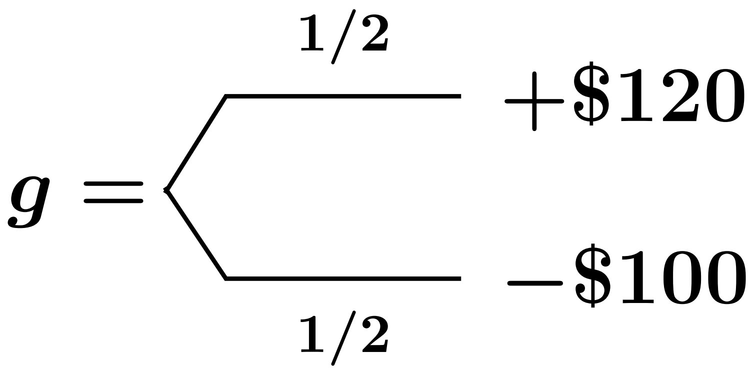

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Is there a "cutoff point" ?

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the* CRITICAL WEALTH *level
below which accepting g ***becomes* RISKY**

II: The Bankruptcy Model

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- Positive expectation: $\mathbf{E}[g] > 0$
- Some negative values: $\mathbf{P}[g < 0] > 0$
  (loss is possible)
- [technical] Finitely many values:
  g takes the values $x_1, x_2, \ldots, x_m$
  with probabilities $p_1, p_2, \ldots, p_m$

Gambles and Wealth

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- [technical] G is finitely generated: there is a finite collection of gambles such that every g_t is a multiple of one of them

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  - s **rejects** the gamble g at wealth W when $W < Q(g)$
  - s **accepts** the gamble g at wealth W when $W \geq Q(g)$

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$$\lim_{t \to \infty} W_t = 0$$

No-Bankruptcy

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No-Bankruptcy

A strategy **GUARANTEES** **NO-BANKRUPTCY**:

$\{ \lim_{t\to\infty} W_t = 0 \}$ has probability 0

for every $G = (g_1, g_2, ..., g_t, ...)$

and every $W_1 > 0$

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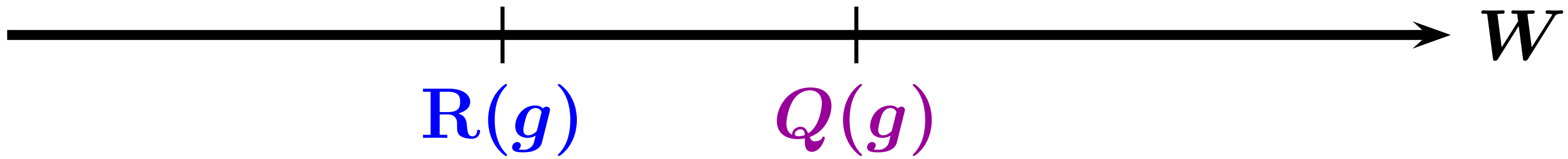

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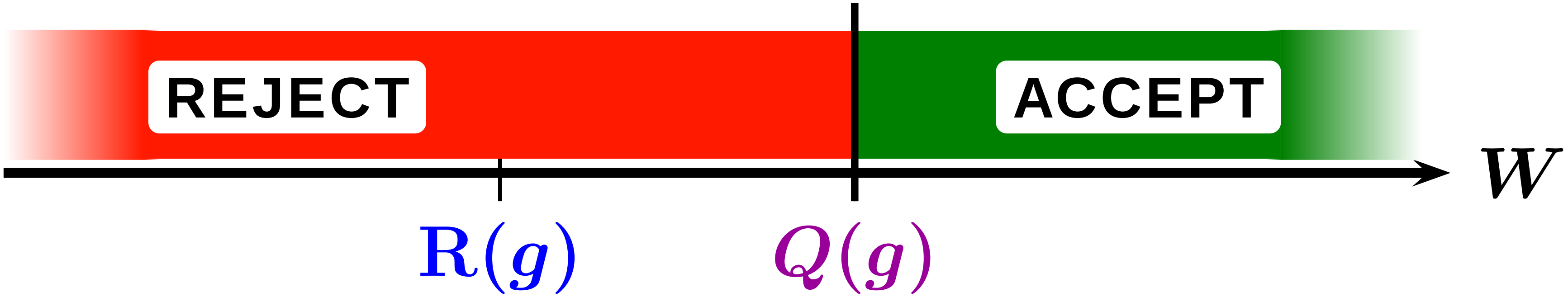

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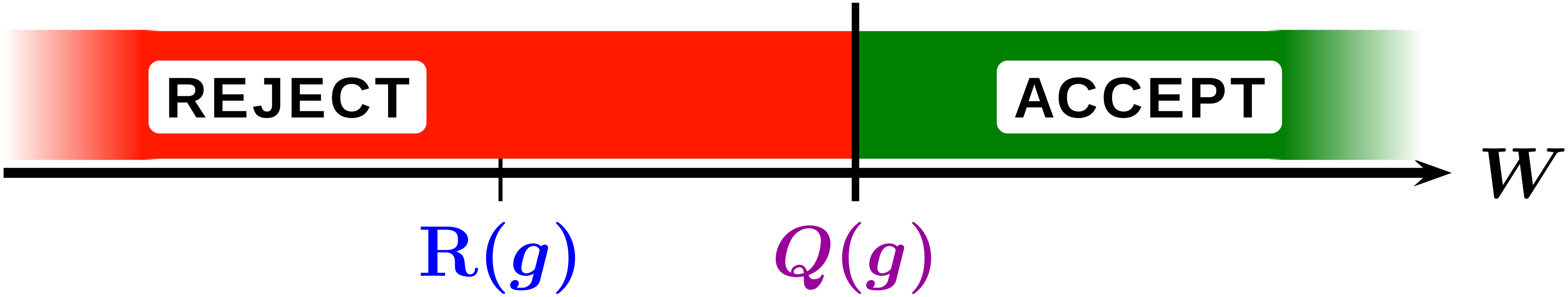

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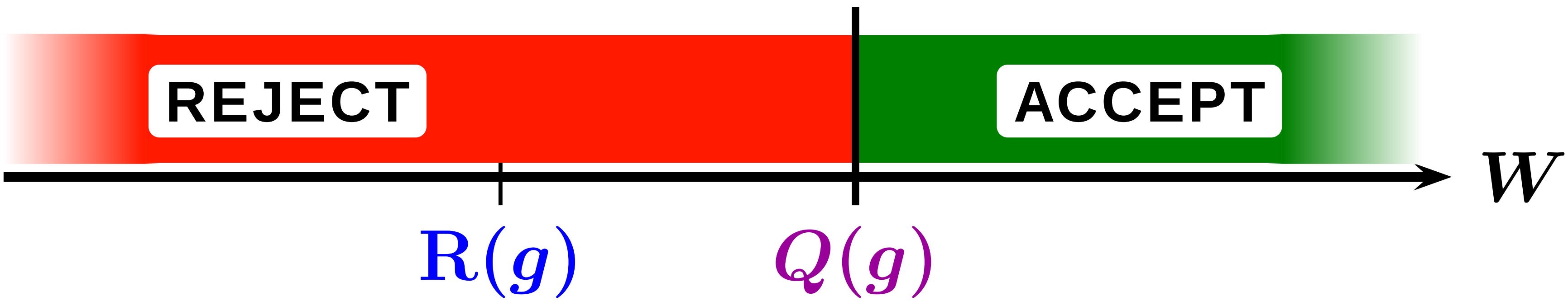

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- $Q(g) = \infty$ for all g: Always reject
- $Q(g) = \mathrm{R}(g)$ for all g: Reject $\Leftrightarrow W < \mathrm{R}(g)$
- Anything in between

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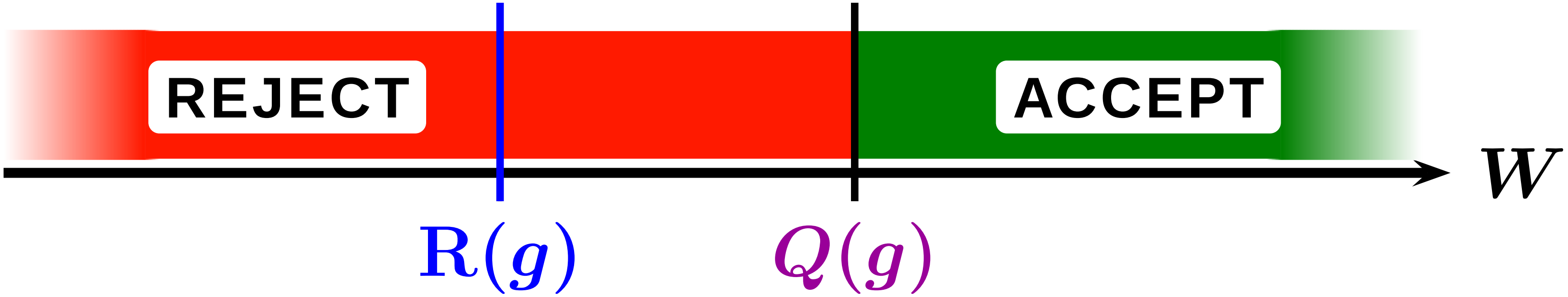

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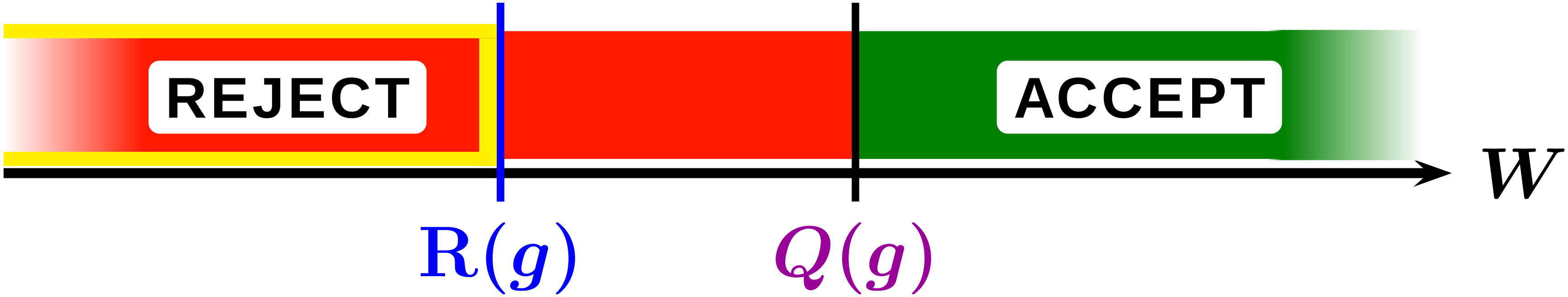

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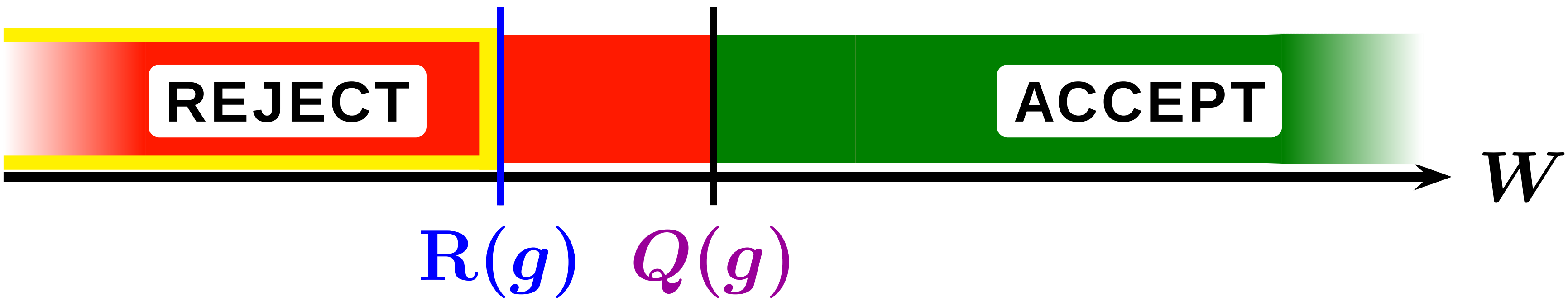

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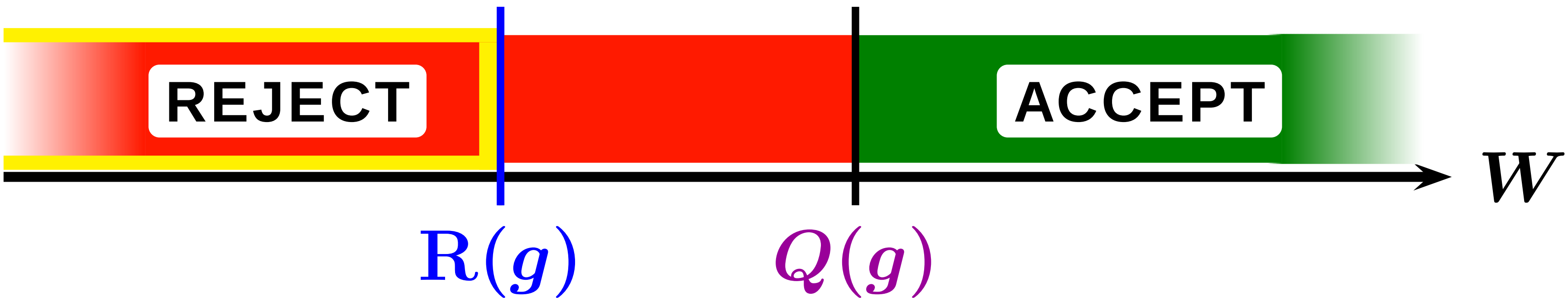

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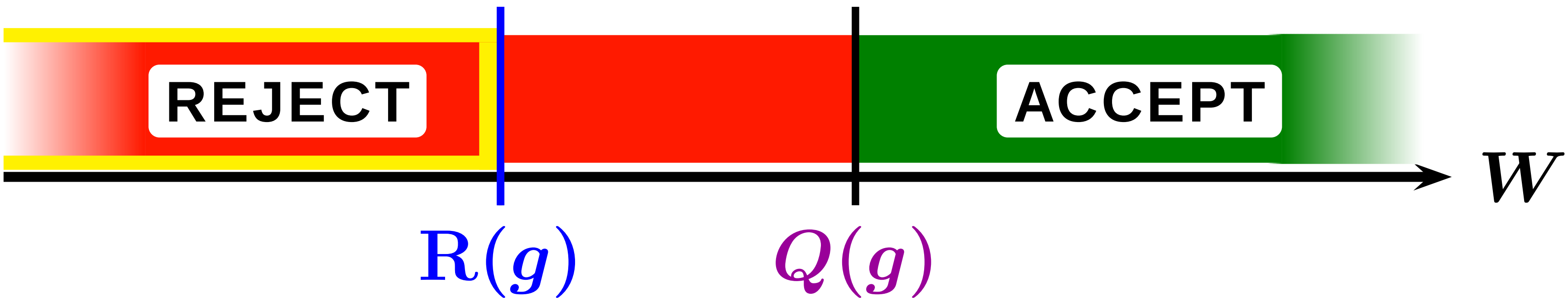

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if and only if

s ***rejects*** g ***when*** $W < \mathrm{R}(g)$

Main Result

$\mathrm{R}(g)$ = the **RISKINESS** of g

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RISKINESS $\sim$ ***"reserve"***

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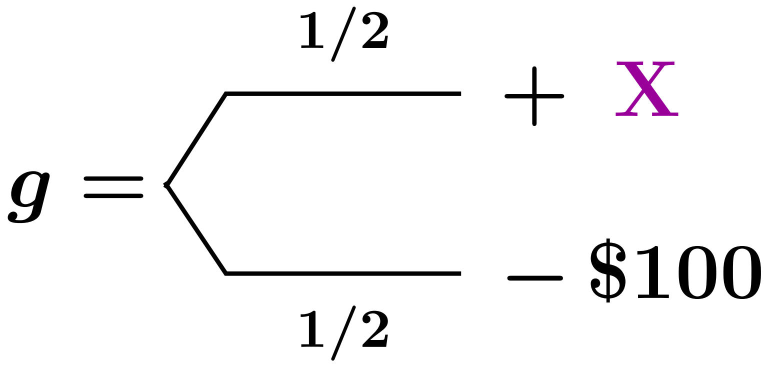
$$\mathbf{E}\left[\log\left(1 + \frac{1}{R}g\right)\right] = 0$$

The Riskiness of Some Gambles

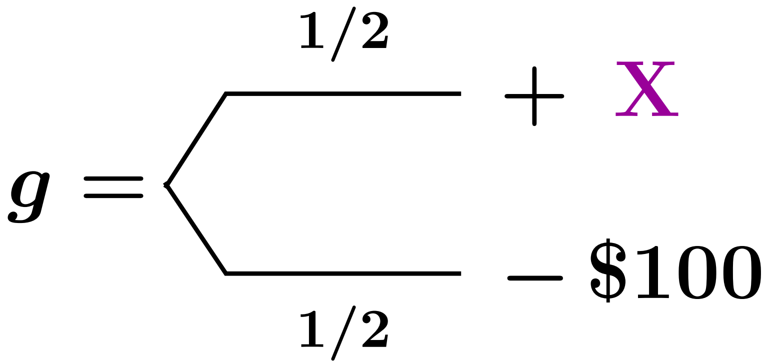


The Riskiness of Some Gambles

$$g = \begin{cases} \frac{1}{2} & +\ \mathrm{X} \\ \frac{1}{2} & -\ \$100 \end{cases}$$

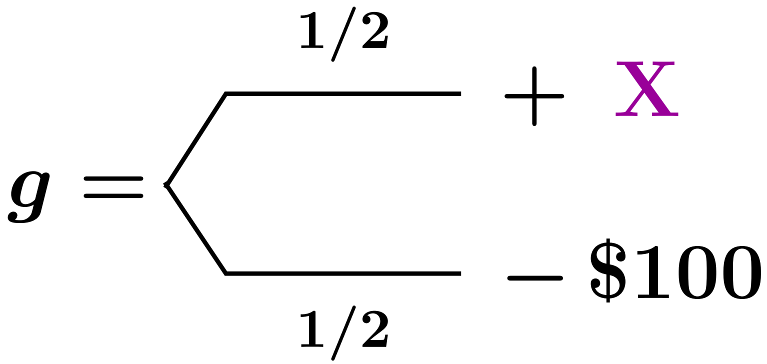
X	$\mathrm{E}[g]$	$\mathrm{R}(g)$
\$120	\$10	\$600

The Riskiness of Some Gambles



X	E [g]	R(g)
$200	$50	$200
$120	$10	$600

The Riskiness of Some Gambles



X	E [*g*]	R(*g*)
$300	$100	$150
$200	$50	$200
$120	$10	$600

The Riskiness of Some Gambles

$$g = \begin{cases} +\,X & \text{with prob. } 1/2 \\ -\,\$100 & \text{with prob. } 1/2 \end{cases}$$

X	E $[g]$	R(g)
\$300	\$100	\$150
\$200	\$50	\$200
\$120	\$10	\$600
\$105	\$2.5	\$2100

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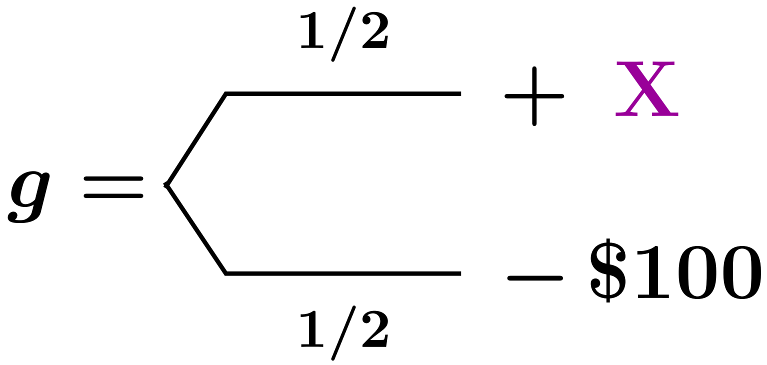

X	E [g]	R(g)
$300	$100	$150
$200	$50	$200
$120	$10	$600
$105	$2.5	$2100
$102	$1	$5100

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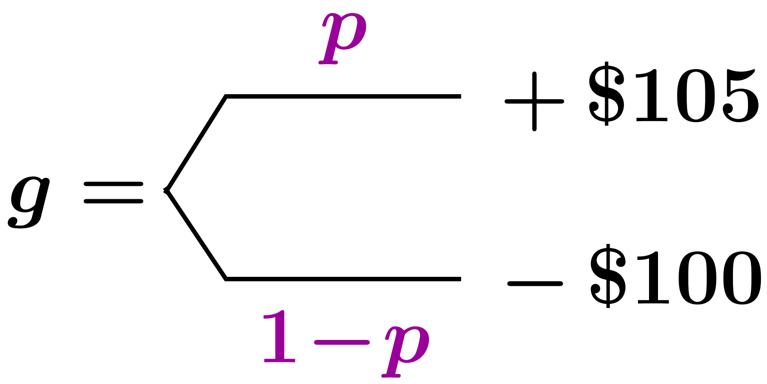

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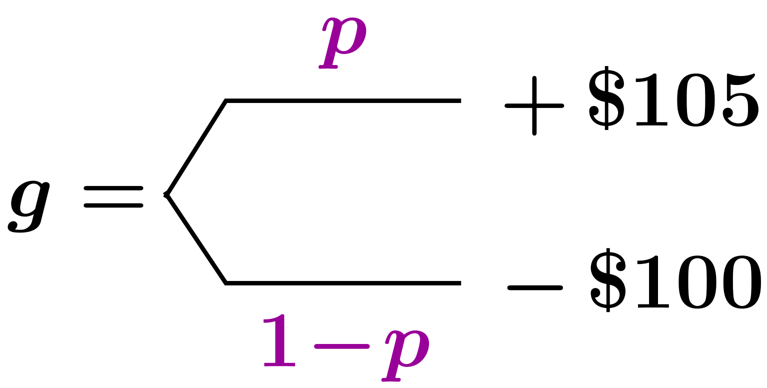

p	$\mathrm{E}[g]$	$\mathrm{R}(g)$
0.5	\$2.5	\$2100
0.6	\$23	\$235.23
0.8	\$64	\$106.93
0.9	\$84.5	\$100.16

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(... more to follow ...)

III: The Shares Model

The Shares Model

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  - if $Q(g) > R(g)$ *for every gamble* g
  - only if $Q(g) \geq R(g)$ *for every gamble* g

Example

$$g = \begin{cases} \xrightarrow{1/2} & +\$120 \\ \xrightarrow{1/2} & -\$100 \end{cases}$$

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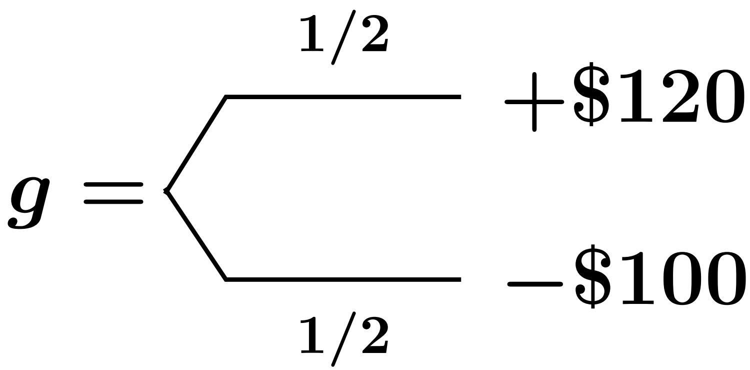
- Consider an i.i.d. sequence $(g_t)_t$ with $g_t \sim g$

Example: $Q(g) = \$200$



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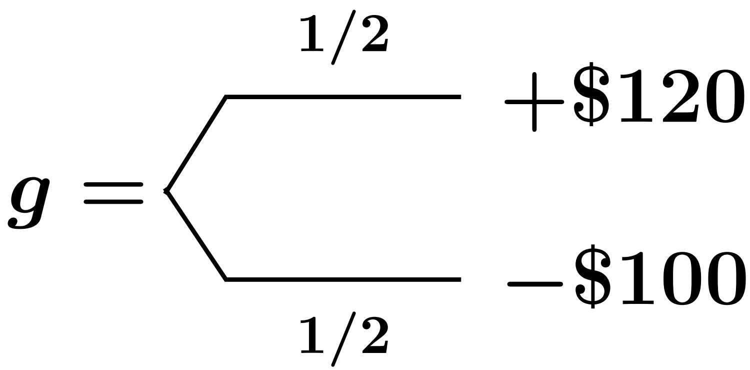

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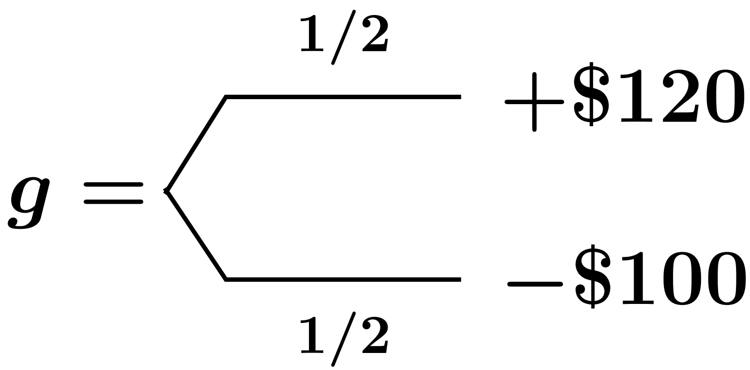

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$$\Rightarrow \quad W_{t+1} = W_t + \left(\frac{W_t}{200}\right) g_t = W_t \left(1 + \frac{g_t}{200}\right)$$

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$$\frac{g}{200} = \begin{cases} +\dfrac{120}{200} & 1/2 \\[2ex] -\dfrac{100}{200} & 1/2 \end{cases}$$

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Example: $Q(g) = \$200$

$$\frac{g}{200} = \begin{cases} \xrightarrow{1/2} \; +\dfrac{120}{200} = +\,60\% \\[2ex] \xrightarrow[1/2]{} \; -\dfrac{100}{200} = -\,50\% \end{cases}$$

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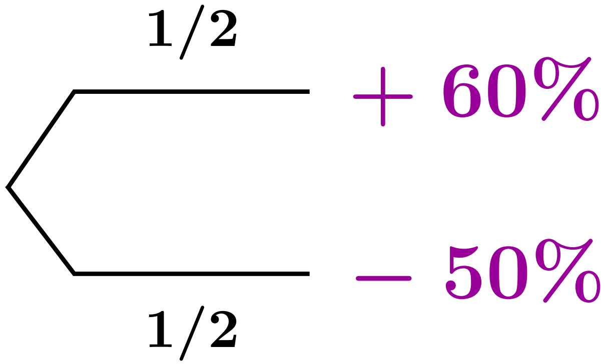

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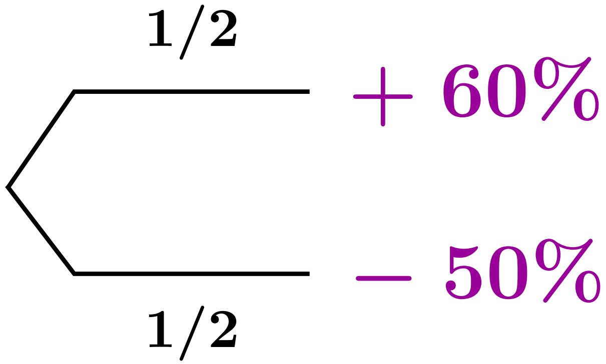

Proposition. $W_t \to 0$ (a.s.)

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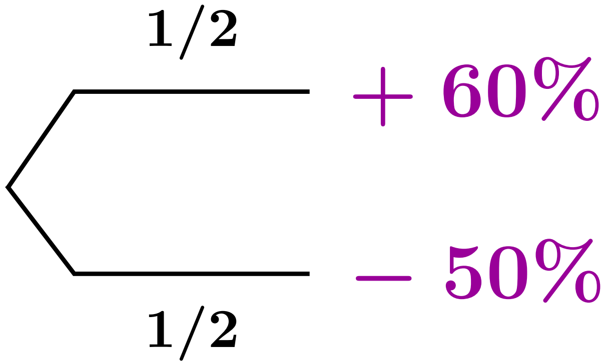

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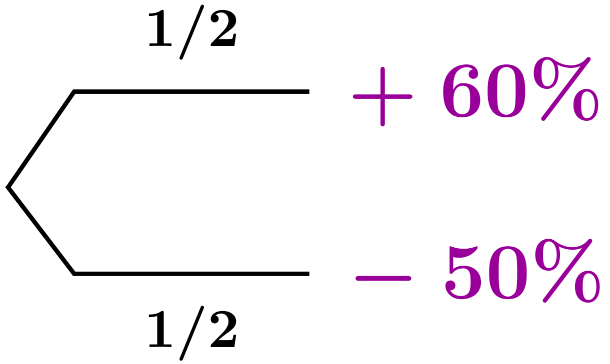

$$W_{t+1} = W_t \times 1.6$$

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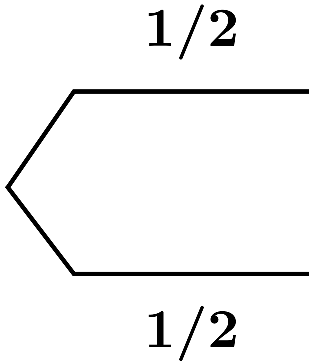

$+\,60\%$ $\qquad W_{t+1} = W_t \times 1.6$

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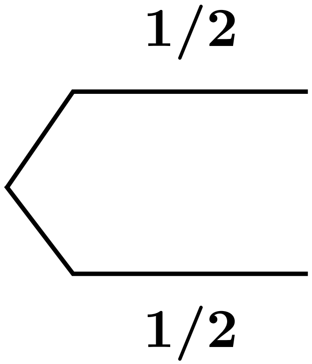

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Proof. The Law of Large Numbers $\Rightarrow$

- about half the days wealth is multiplied by 1.6
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Example: $Q(g) = \$200$

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$\Rightarrow$ A factor of $\approx \sqrt{1.6 \cdot 0.5}$ per period

Example: $Q(g) = \$200$

$$\begin{array}{lll} \frac{1}{2} & +\,60\% & W_{t+1} = W_t \times 1.6 \\ \frac{1}{2} & -\,50\% & W_{t+1} = W_t \times 0.5 \end{array}$$

Proposition. $W_t \to 0$ (a.s.)

Proof. The Law of Large Numbers $\Rightarrow$

- about half the days wealth is multiplied by $\mathbf{1.6}$
- about half the days wealth is multiplied by $\mathbf{0.5}$

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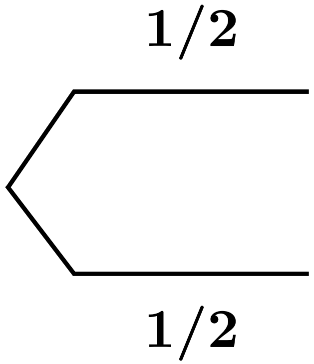

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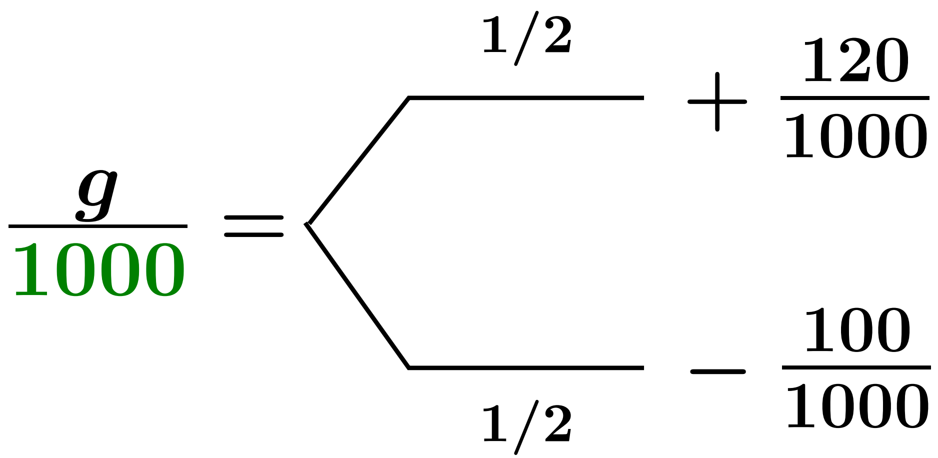

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$$\frac{g}{1000} = \begin{cases} \xrightarrow{1/2} \; +\dfrac{120}{1000} = +\,12\% \\[2ex] \xrightarrow[1/2]{} \; -\dfrac{100}{1000} = -\,10\% \end{cases}$$

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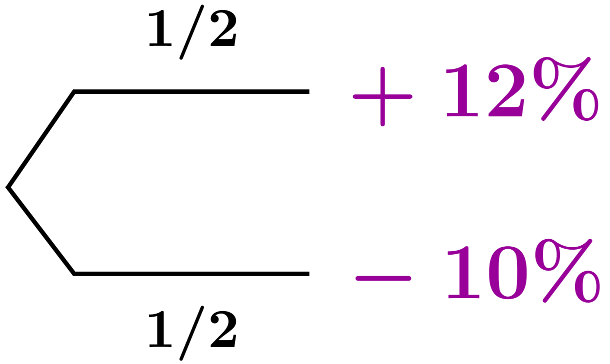

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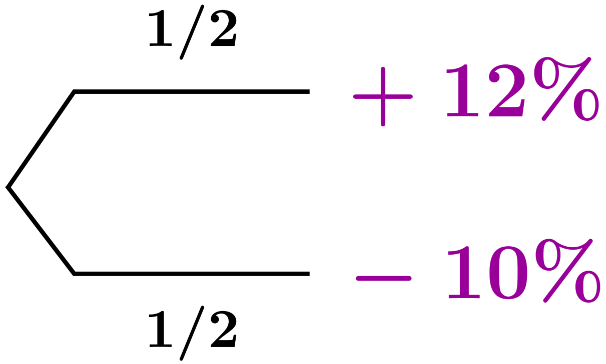

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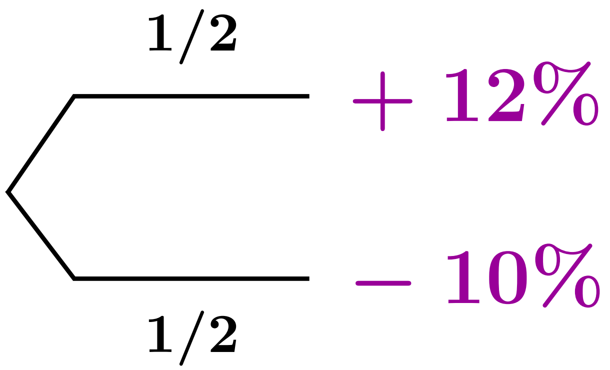

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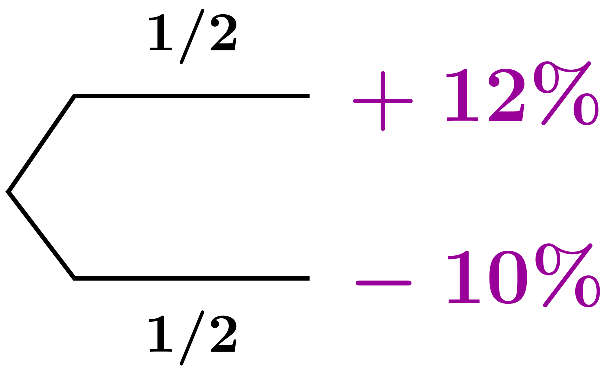

$$W_{t+1} = W_t \times 1.12$$

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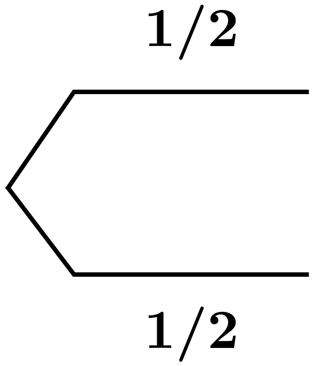

$+\,12\%$ $\qquad W_{t+1} = W_t \times 1.12$

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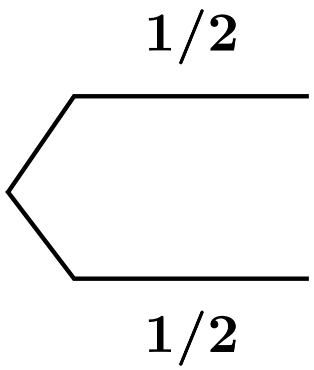
Proof.

Example: $Q(g) = \$1000$



$+\ 12\%$ $\qquad W_{t+1} = W_t \times 1.12$

$-\ 10\%$ $\qquad W_{t+1} = W_t \times 0.90$

Proposition. $W_t \to \infty$ (a.s.)

Proof. The Law of Large Numbers $\Rightarrow$

- $\approx$ half the days wealth is multiplied by 1.12
- $\approx$ half the days wealth is multiplied by 0.90

Example: $Q(g) = \$1000$

$1/2$	$+\,12\%$	$W_{t+1} = W_t \times 1.12$
$1/2$	$-\,10\%$	$W_{t+1} = W_t \times 0.90$

Proposition. $W_t \to \infty$ (a.s.)

Proof. The Law of Large Numbers $\Rightarrow$

- $\approx$ half the days wealth is multiplied by 1.12
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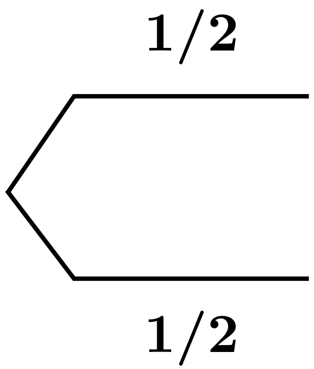

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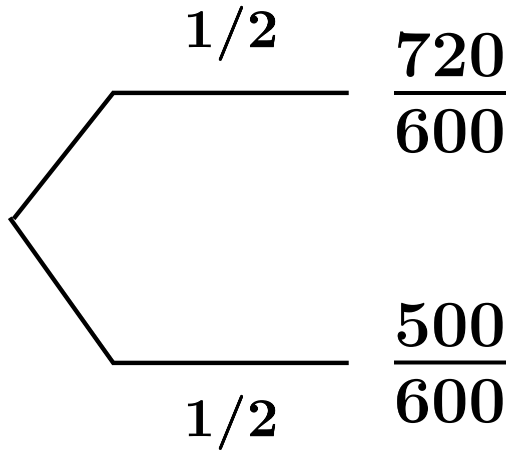

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t	$Q(g)$	$\mathrm{P}[W_{t+1} \geq W_1]$	MED
100	\$200	2.7%	0.0014%
100	\$1000	64%	148%
1000	\$200	10^{-7}%	10^{-46}%
1000	\$1000	87%	5373%

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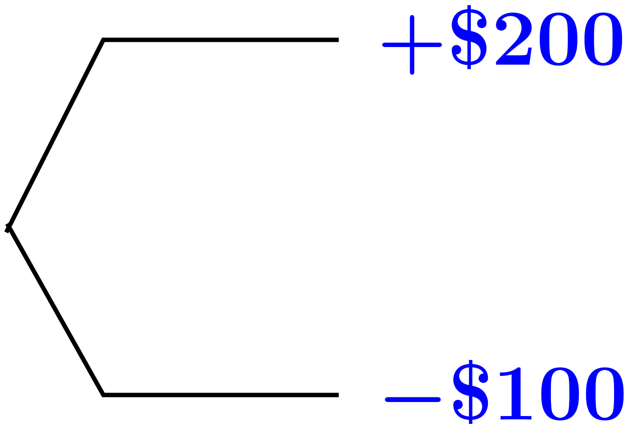

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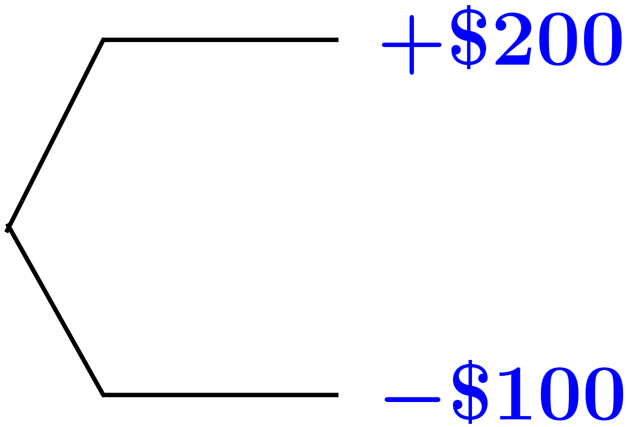

Notation: Q

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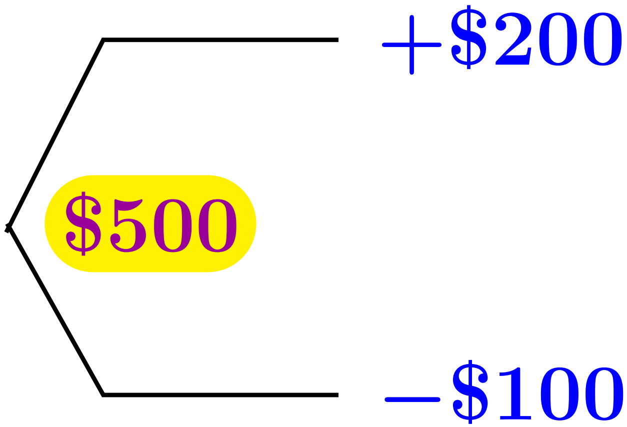

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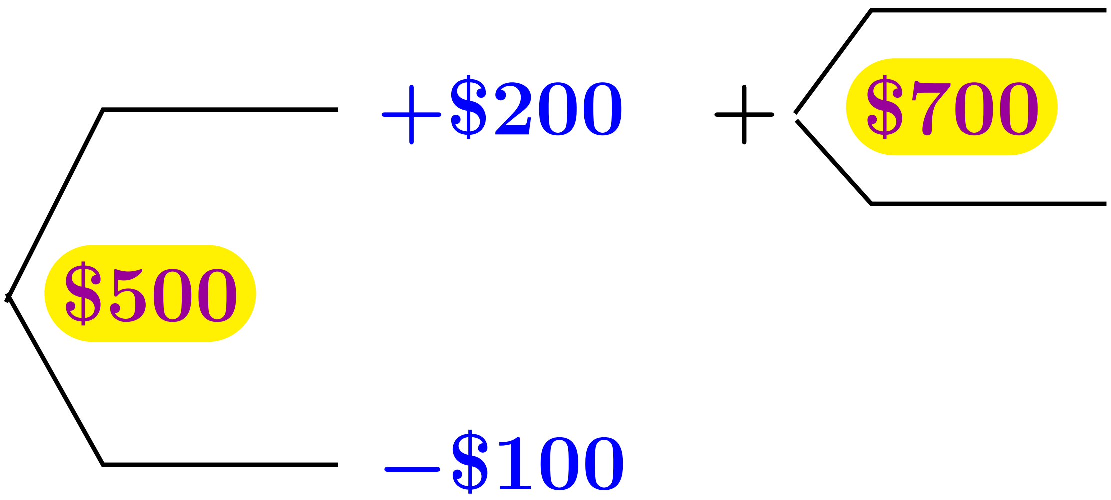

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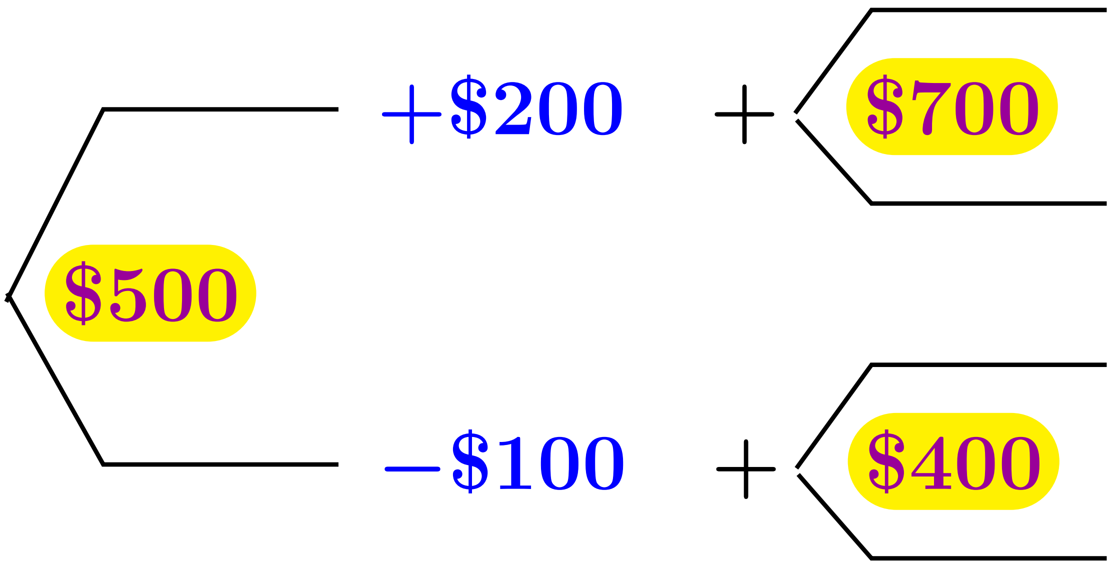

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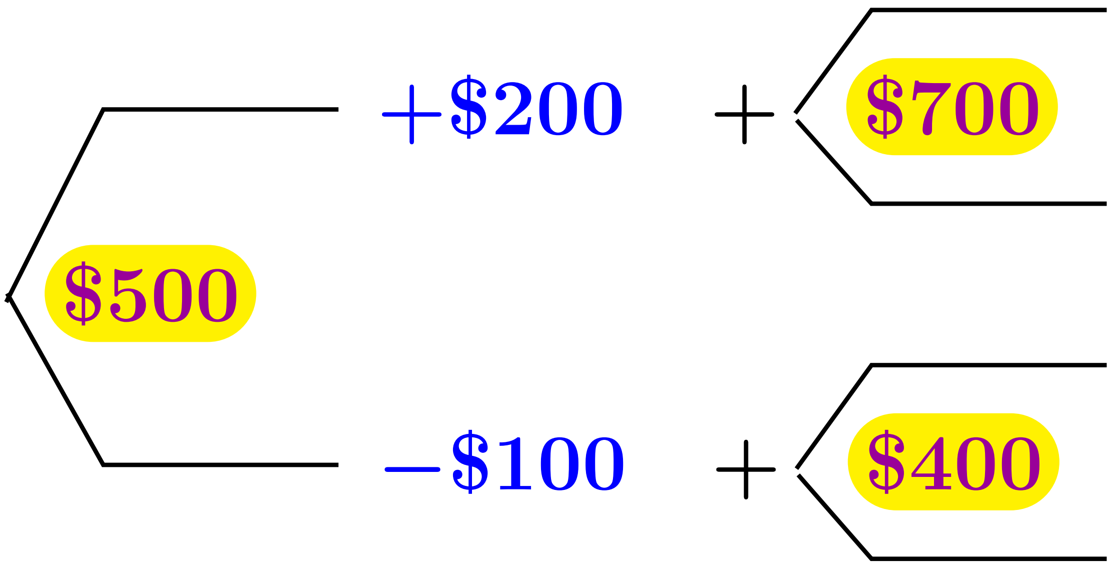

$\Rightarrow \$500$

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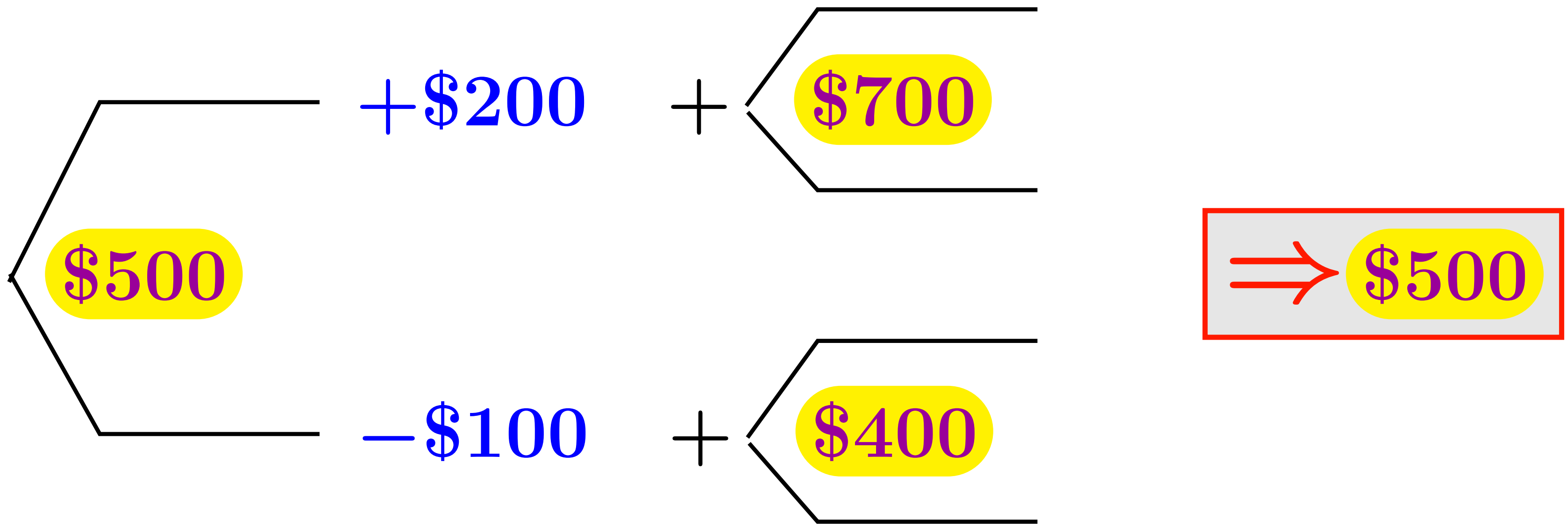

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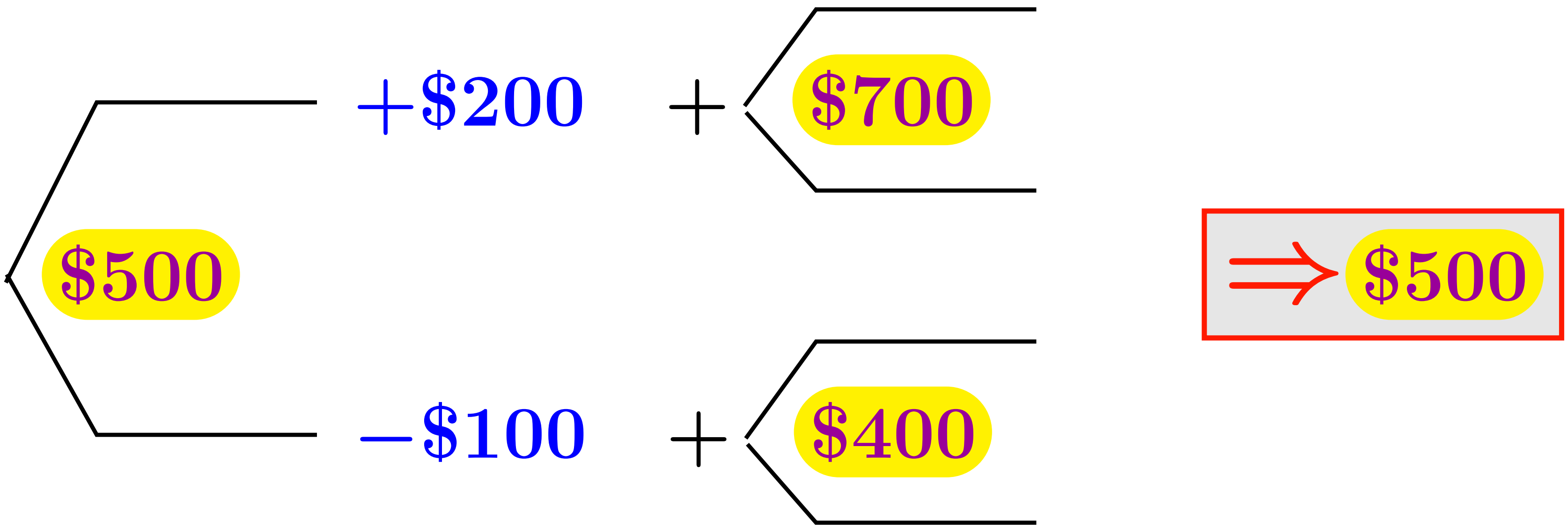

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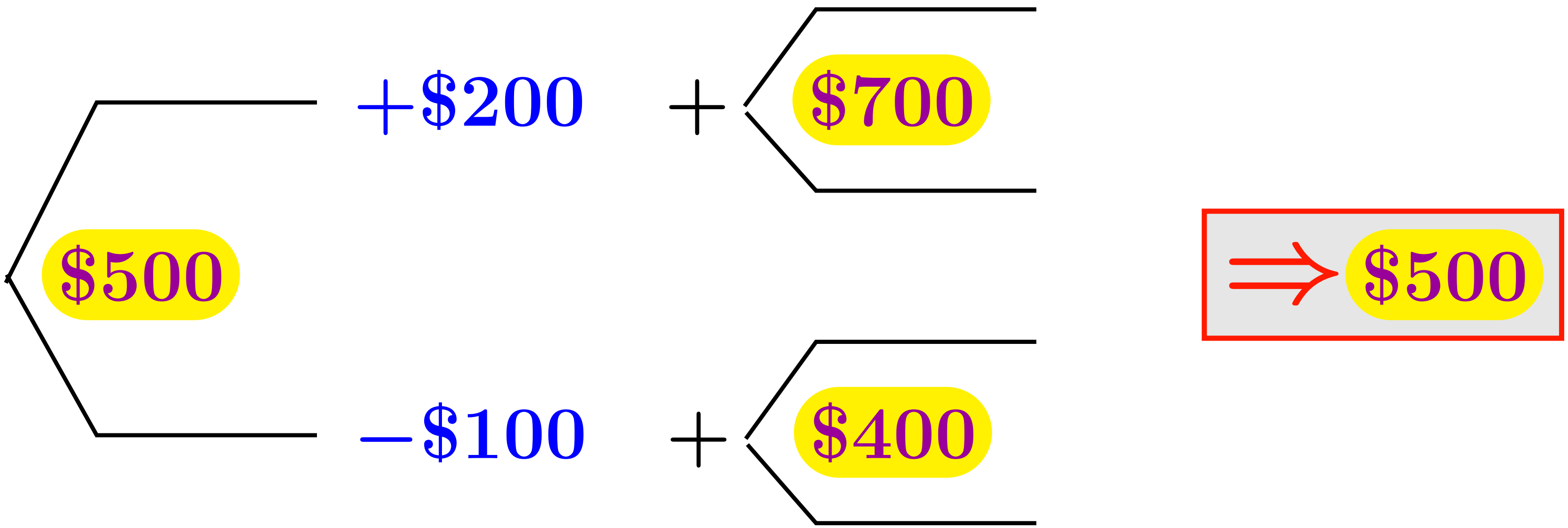

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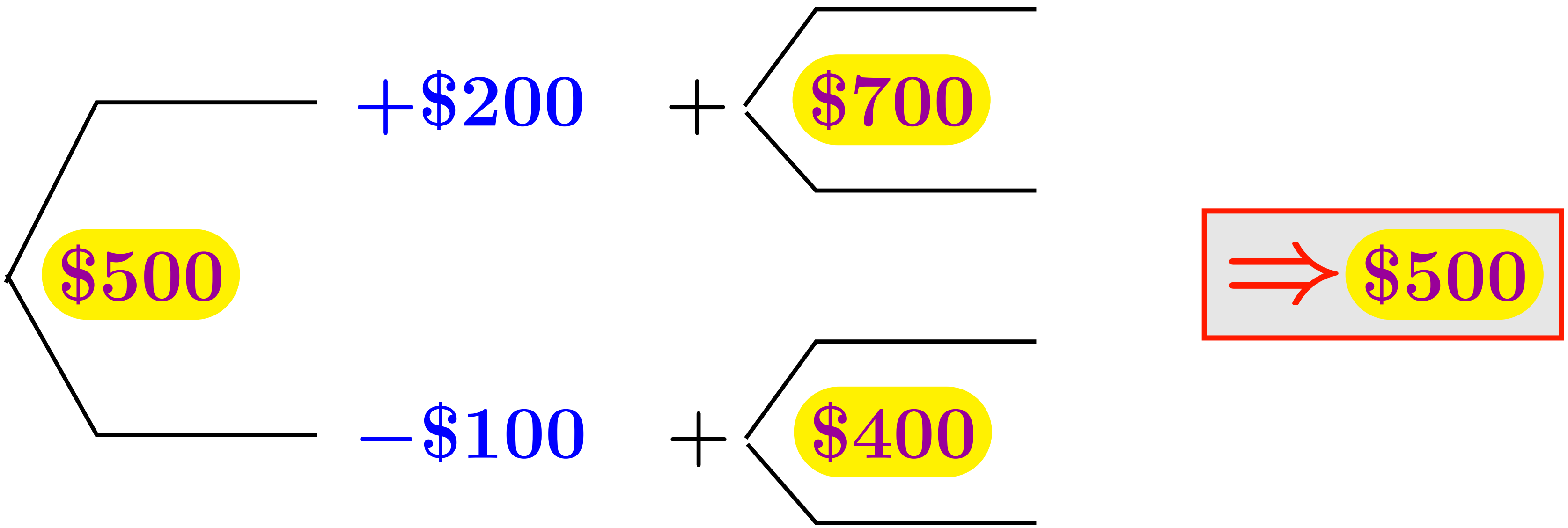

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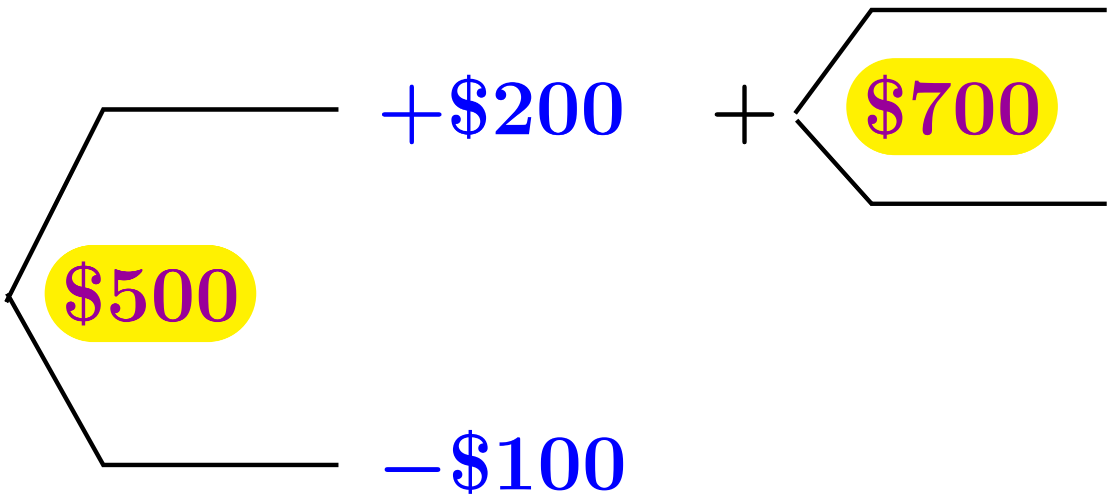

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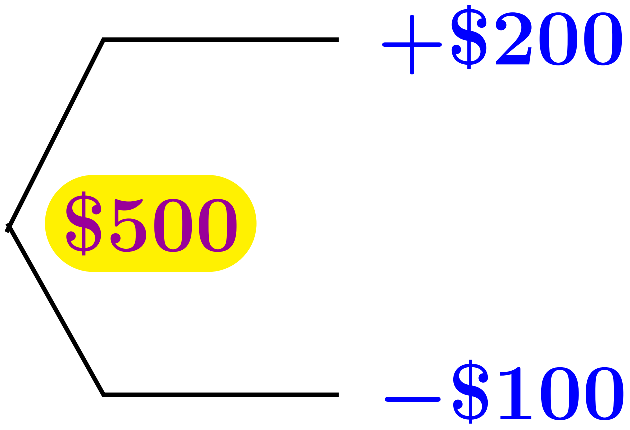

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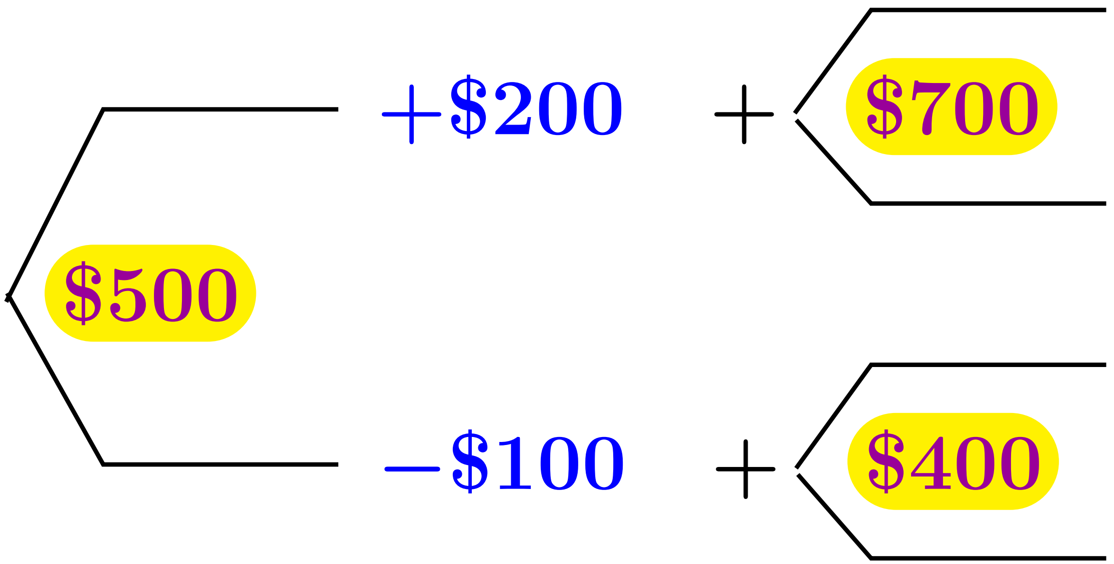

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V: Connections

Aumann & Serrano (2008)

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Index of Riskiness Q

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- **HOMOGENEITY**: $Q(\lambda g) = \lambda Q(g)$ for $\lambda > 0$

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There is $c > 0$ such that
$Q(g) = c\, R^{AS}(g)$ for every gamble g

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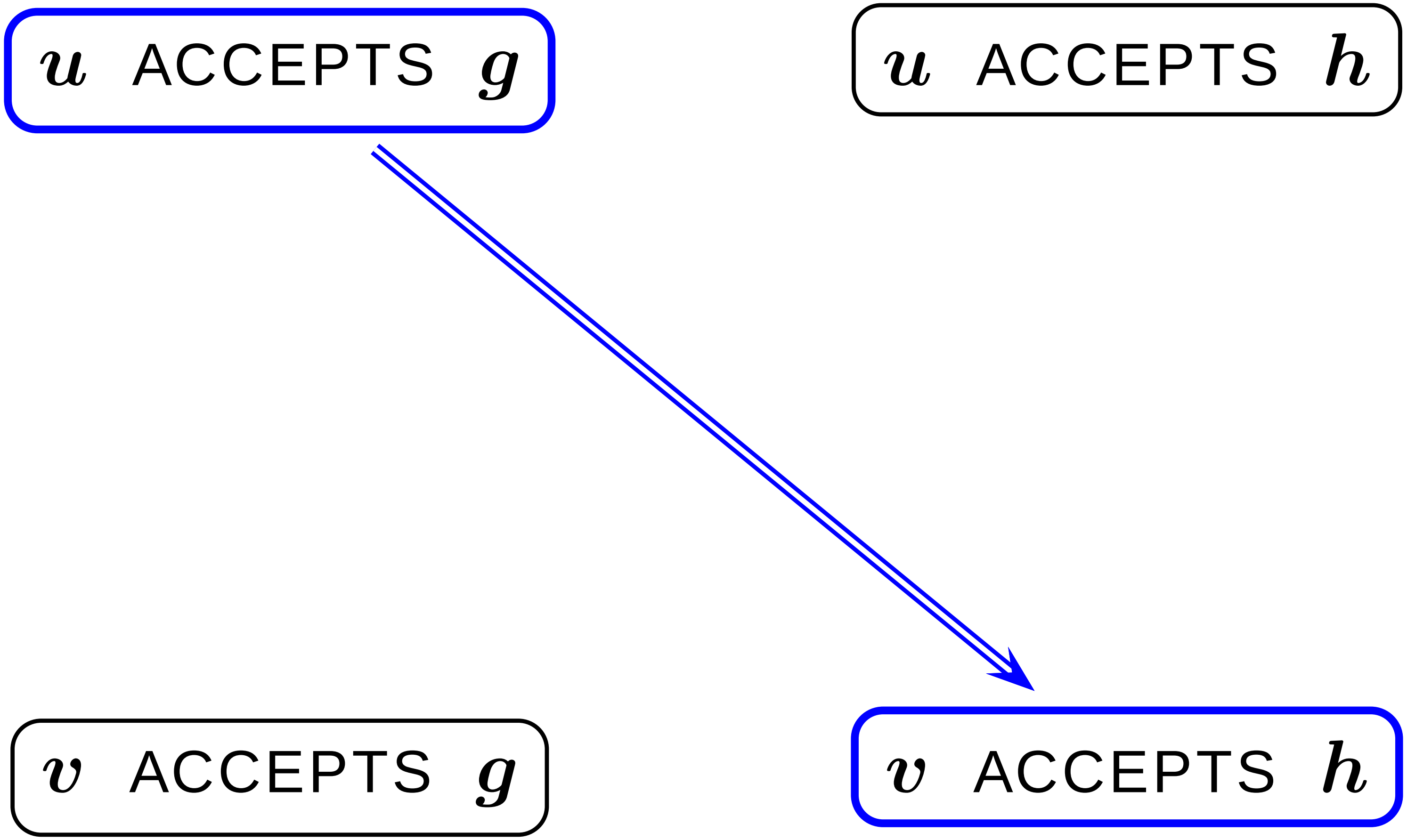
$$\boxed{u \text{ ACCEPTS } g} \Longrightarrow \boxed{v \text{ ACCEPTS } h}$$

Duality



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\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{u \text{ ACCEPTS } g} \arrow[d, "u \rhd v"'] \arrow[dr, Rightarrow] & \boxed{u \text{ ACCEPTS } h} \arrow[d, "u \rhd v"] \\
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  ($\Leftrightarrow$ coefficient of absolute risk-aversion is nonincreasing in wealth)

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- Together with homogeneity: R^{AS} is *unique* up to multiplication by a positive constant

Comparing R and R^{AS}

$R^{AS}(g)$ is the unique solution $R > 0$ of

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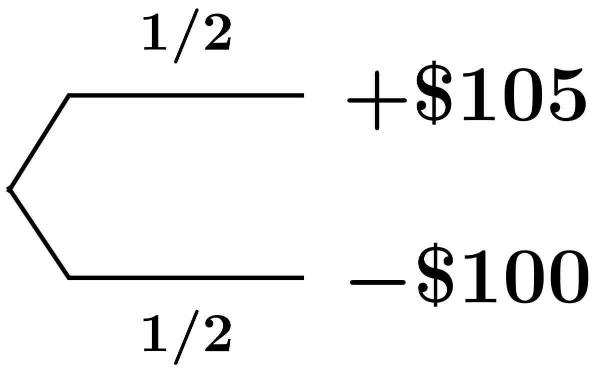

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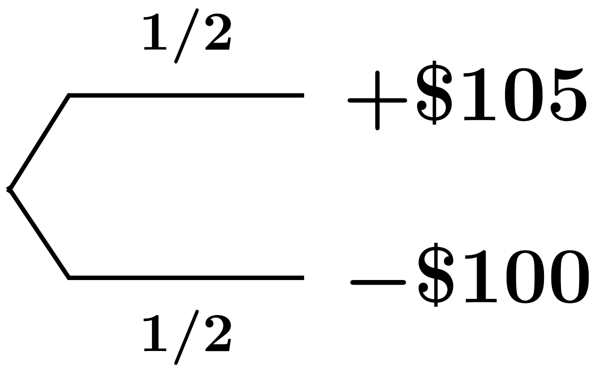

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$$g = \begin{cases} \xrightarrow{1/2} & +\$105 \\ \xrightarrow[1/2]{} & -\$100 \end{cases}$$

$\mathrm{R}(g) = \$2100 \qquad R^{AS}(g) = \$2100.42\ldots$

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Nevertheless: similar in many respects !!

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OUR RESULT: **reject** g

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- what is “wealth”?

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Rejecting $\boldsymbol{g}$ when $\boldsymbol{W} < \underline{\boldsymbol{W}} + \mathbf{R}(\boldsymbol{g})$

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(Proof: replace $\mathbf{0}$ with $\underline{W}$)

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The End

"We're recommending a risky strategy for you;
so we'd appreciate if you paid before you leave."