About Nash, Stackelberg, Inverse Stackelberg and Conjectural Variations

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In game theory: decision maker=player=agent=person=firm=actor=....

In Stackelberg games: leader, follower.

Outline

- Nash equilibrium
- Stackelberg games
- Inverse Stackelberg games
- More than one leader and/or follower
- Continuous time considerations
- Consistent conjectural variations
- Existence questions
- References

Nash equilibrium

Restriction to two players. Criteria are $J_i(u,v),\ i=1,2.$

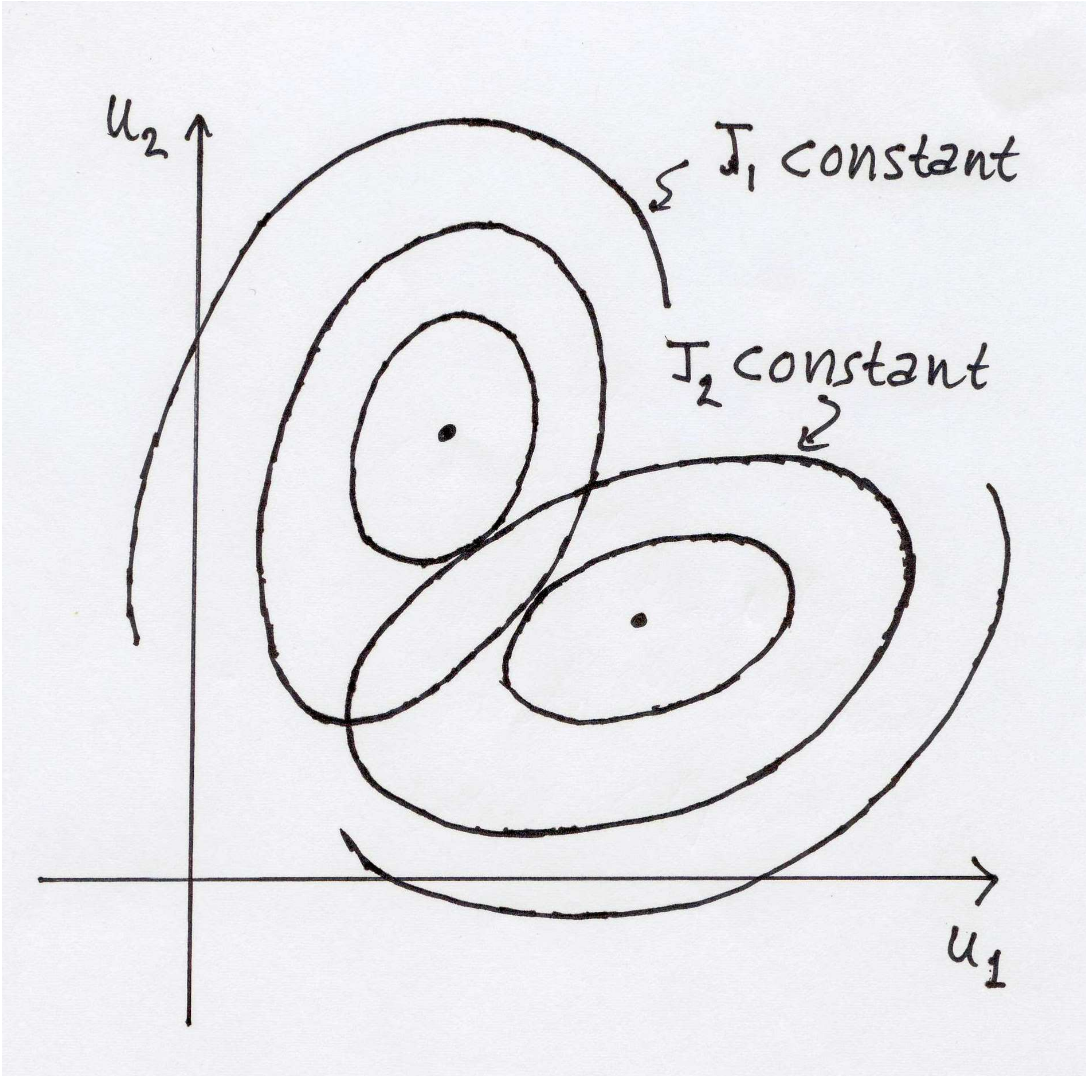
Definition Nash equilibrium

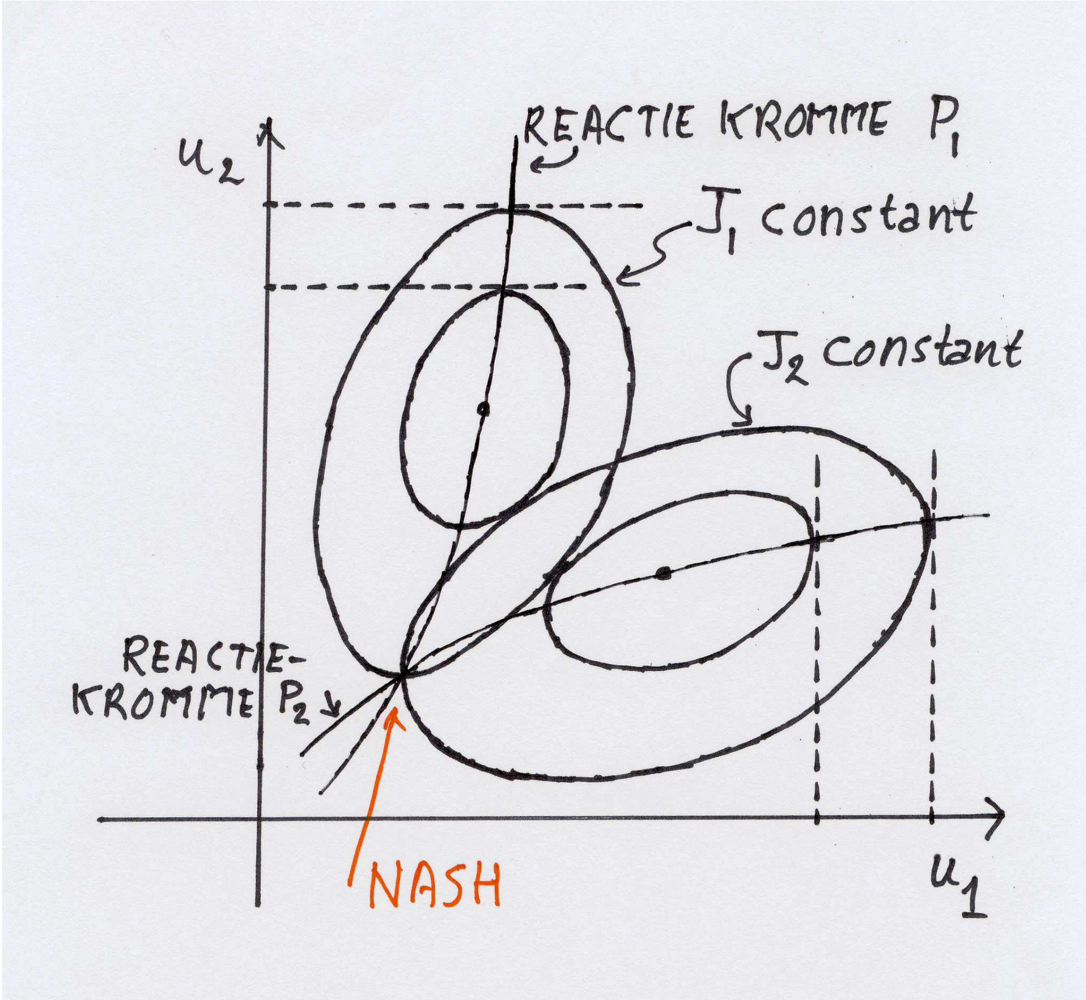
$$J_1(u^*,v^*) \leqslant J_1(u,v^*),\ J_2(u^*,v^*) \leqslant J_2(u^*,v),\ \forall u,v.$$

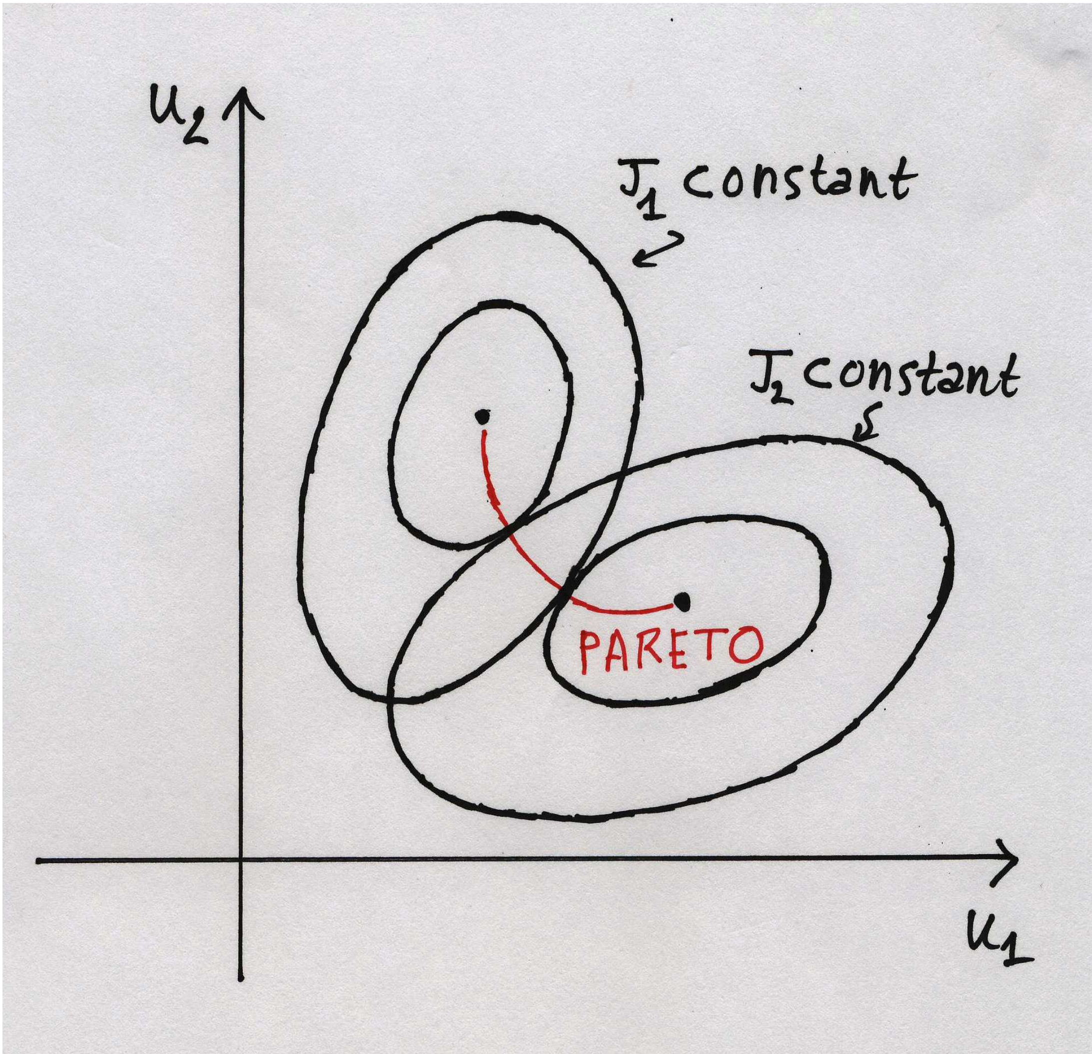
Definition Pareto solution. The pair (u^*,v^*) is called a Pareto solution if there is no other solution pair which is better for both players. In other words, a pair u,v with

$$J_i(u^*,v^*) > J_i(u,v),\ i=1,2$$

does not exist.


u_2
J_1 constant
J_2 constant
u_1


REACTIE KROMME P_1
u_2
J_1 constant
J_2 constant
REACTIE-
KROMME P_2
NASH
u_1


u_2
J_1 constant
J_2 constant
PARETO
u_1

Stackelberg games

Two players: (L)eader and (F)ollower, with cost functions (to be minimized)

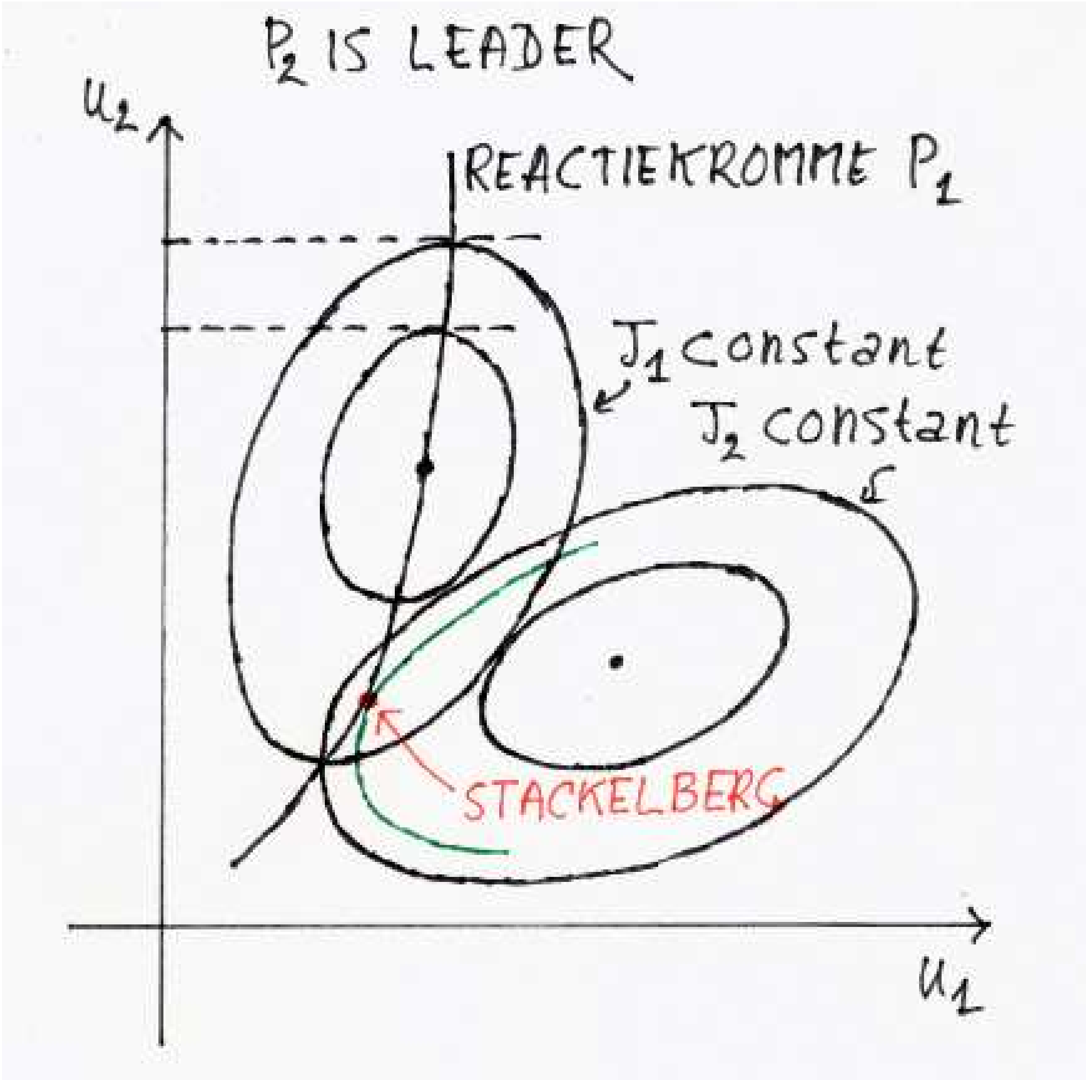
$$J_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}}),\ J_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}})$$

respectively. Leader announces u_{L}. Follower chooses u_{F} according to

$$\min_{u_{\mathrm{F}}} J_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = J_{\mathrm{F}}(u_{\mathrm{L}}, l_{\mathrm{F}}(u_{\mathrm{L}})).$$

Function $l_{\mathrm{F}}(\cdot)$ is called the reaction curve. Leader chooses u_{F} according to

$$\min_{u_{\mathrm{L}}} J_{\mathrm{L}}(u_{\mathrm{L}}, l_{\mathrm{F}}(u_{\mathrm{L}}));\ \frac{\partial J_{\mathrm{L}}}{\partial u_{\mathrm{L}}} + \frac{\partial J_{\mathrm{L}}}{\partial u_{\mathrm{F}}}\frac{d l_{\mathrm{F}}}{d u_{\mathrm{L}}} = 0.$$


P2 IS LEADER
u2
REACTIEKROMME P1
J1 constant
J2 constant
STACKELBERG
u1

Inverse Stackelberg games

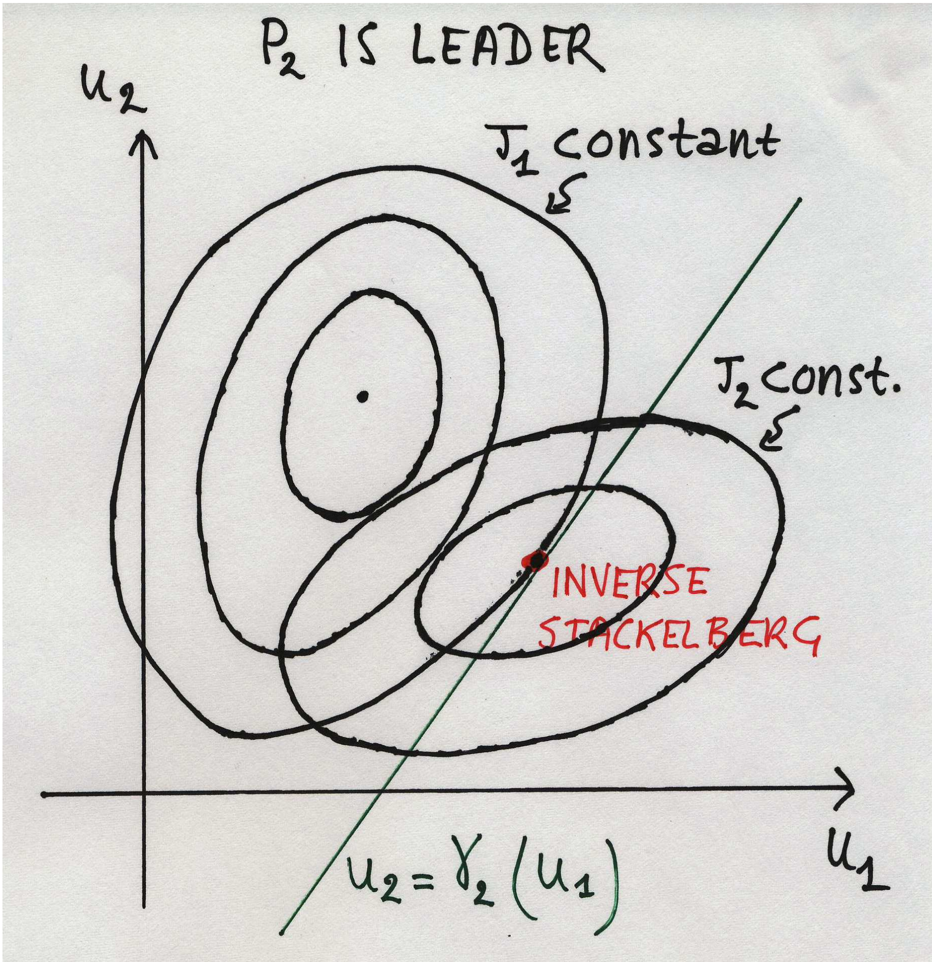
Leader announces the function $\gamma_{\mathrm{L}}(\cdot)$, which maps the u_{F}-space into the U_{L}-space. Follower will choose

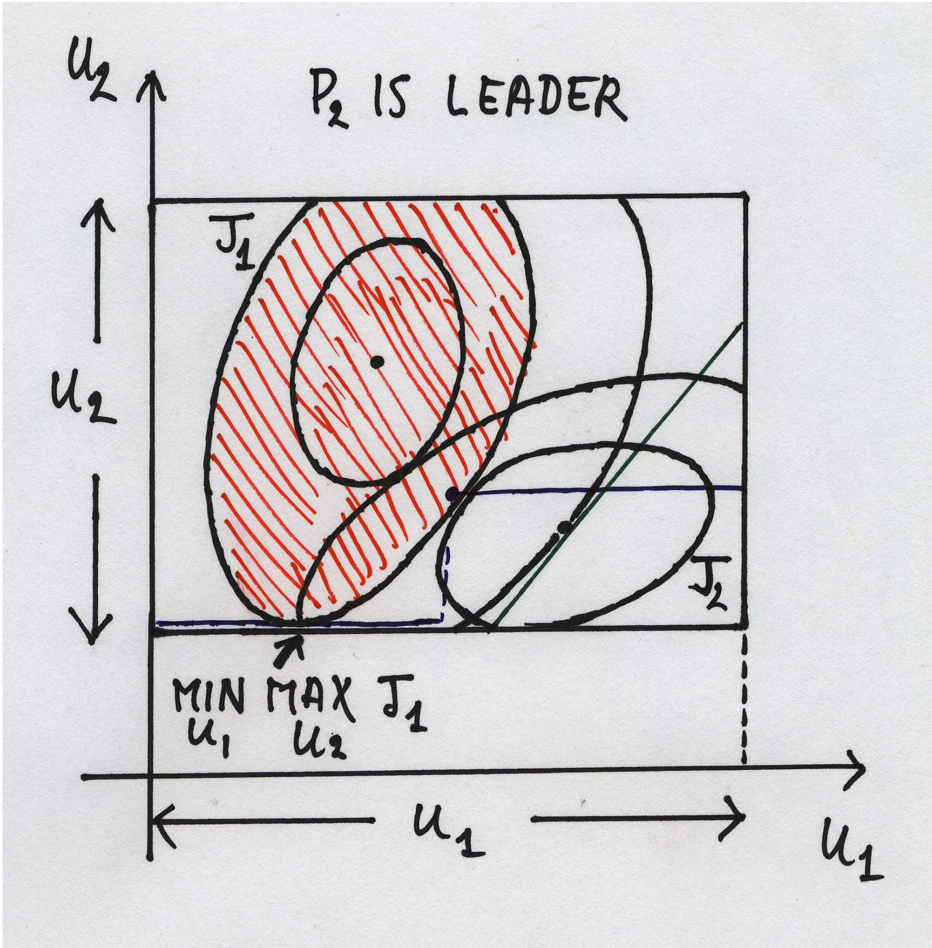
$$u_{\mathrm{F}}^* = \arg\min_{u_{\mathrm{F}}} J_F(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}}).$$

Subsequently, $\gamma_{\mathrm{L}}^*(\cdot)$ should satisfy

$$\gamma_{\mathrm{L}}^*(\cdot) = \arg\min_{\gamma_{\mathrm{L}}(\cdot)} J_{\mathrm{L}}(\gamma_{\mathrm{L}}(u_{\mathrm{F}}(\gamma_{\mathrm{L}}(\cdot))), u_{\mathrm{F}}(\gamma_{\mathrm{L}}(\cdot))).$$

Thus one enters the realm of composed functions; not very appetizing!


P2 IS LEADER
u2
J1 constant
J2 const.
INVERSE
STACKELBERG
u1
u2 = γ2 (u1)


u_2
P_2 IS LEADER
J_1
U_2
J_2
MIN MAX J_1
u_1 u_2
U_1
u_1

Examples:

- Leader is the government, γ_{L} represents the tax rules, and the Follower is the citizen who decides how much to earn (u_{F}).
- Leader is the bank, γ_{L} represents transaction costs, and the Follower is an investor who can buy/sell stocks (to an amount u_{F}).
- Leader is producer of electricity, γ_{L} represents the price if u_{F} MW is bought by the consumer, or by a consortium of consumers (who is the Follower).
- traffic-dependent toll on freeways (Ministry of Transportation is Leader, drivers are the Followers).

The static problem

Consider an investor and his bank with

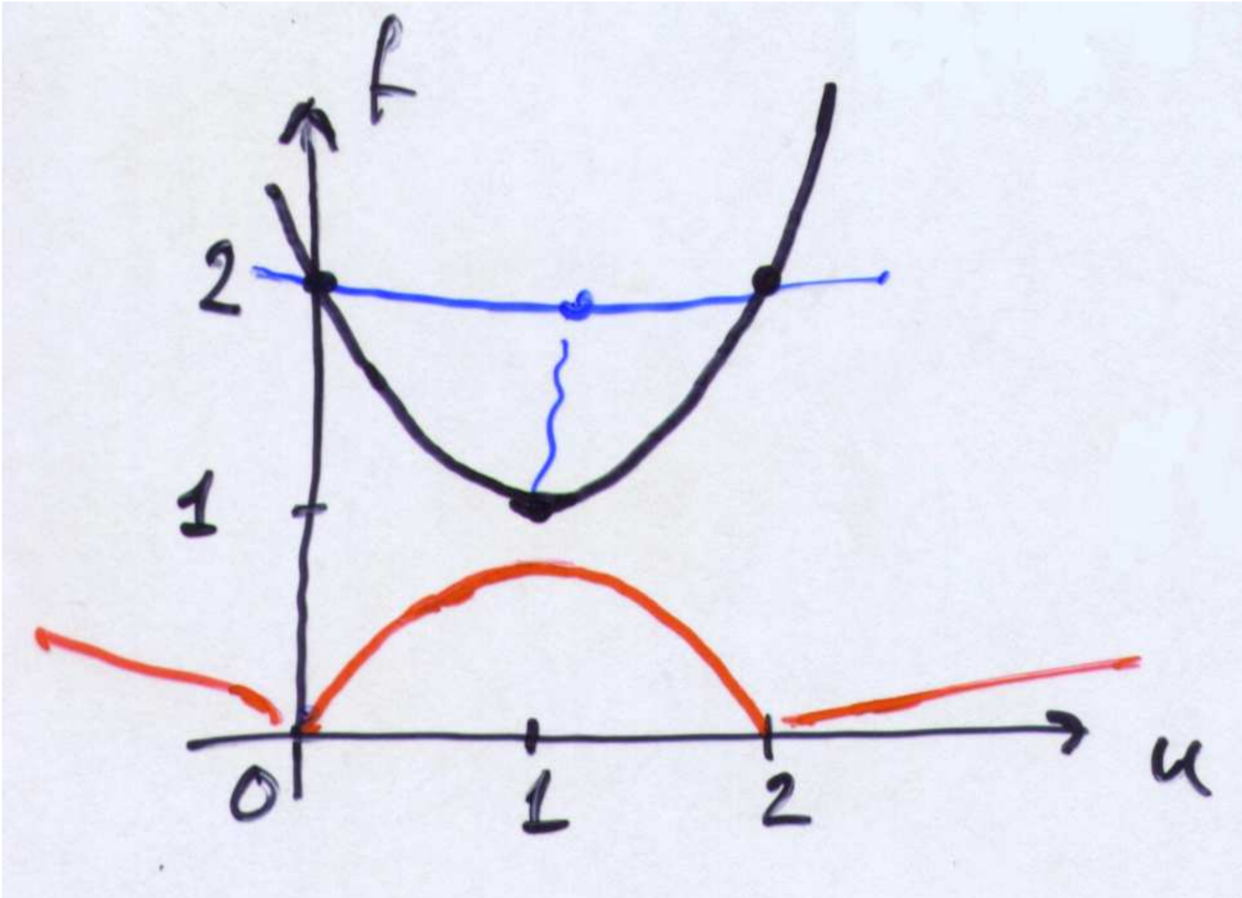
$$\min_u (f(u) + \gamma(u)), \quad \max_{\gamma(\cdot)} \gamma(u).$$

subject to $\gamma(\cdot) \geqslant 0$ and $\gamma(0) = 0$.

Example If $f(u) = (u-1)^2 + 1$, then the bank will choose

$$\gamma(u) = \begin{cases} (f(0) - f(u))(1-\epsilon), & \text{if } 0 \leqslant u \leqslant 2; \\ \text{nonnegative} & \text{elsewhere,} \end{cases}$$

where ϵ is a small positive number.



f
2
1
0
1
2
u

If one wants to adhere to the nondecreasing property of γ, then the previous γ could be replaced by

$$\gamma(u) = \begin{cases} (f(0) - f(u))(1-\epsilon), & \text{if } 0 \leqslant u \leqslant 1; \\ 1 - \epsilon + (1-u)^2, & \text{if } u \geqslant 1, \end{cases}$$

and for negative u: $\gamma(u) = \gamma(-u)$, without altering the results.

Relationship to the design of optimal contracts.

"It is a remarkable feature of these problems (i.e. contract design games) that the leader always takes all, pushing the follower to zero utility."

See e.g. Bernard Salanie, *The economics of contracts*, MIT Press, 5th printing, 2002.

More than one Leader and/or Follower

The stage: $\overline{P} = 2$ producers of electricity, and $\overline{M} = 2$ consumers. Players P_1 and P_2 are the Followers (consumers) and P_3 and P_4 the Leaders (producers). Decision variables are $u_i,\ i = 1, 2, 3, 4$. The Leaders announce $u_i = \gamma_i(u_1, u_2),\ i = 3, 4$. The format of cost functions is:

$$J_1(u_3, u_4, u_1),\ J_2(u_3, u_4, u_2),\ J_3(u_3, u_1, u_2),\ J_4(u_4, u_1, u_2).$$

Leaders, and also Followers, play Nash among themselves.

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Special case: One leader, two followers

The cost functions are

$$J_1 = u_1^2 - u_1 u_3 + 2u_3^2, \; J_2 = u_2^2 - 2u_2 u_3 + 5u_3^2,$$

$$J_3 = u_3^2 + 2u_1 u_3 + 5u_2 u_3 + u_1^2 + u_2^2 + 4u_3^2.$$

The leader will try to obtain his team minimum by choosing the coefficients α_i in

$$u_3 = \gamma_3(u_1, u_2) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3$$

The absolute team minimum for P_3 is obtained for $u_1 = -8/25$, $u_2 = -20/25$ and $u_3 = 8/25$, resulting in $J_3(u_1, u_3) = 928/625$.

Consider the constant level curve $J_1(u_1, u_3) = 928/625$. This curve determines u_3 as a function of u_1. By taking the total derivative of $J_1(u_1, u_3) = 928/625$ with respect to u_1 one obtains $\frac{\partial u_3}{\partial u_1} = \frac{3}{5}$ in the team minimum. By considering the constant level curve $J_2(u_2, u_3)$ through the same point, one obtains similarly $\frac{\partial u_3}{\partial u_2} = \frac{7}{15}$. Hence, if a linear γ_3 function exists, it is

$$u_3 = \gamma_3(u_1, u_2) = \frac{3}{5}u_1 + \frac{7}{15}u_2 + \frac{332}{375}.$$

with $\alpha_1 = \frac{3}{5}$, $\alpha_2 = \frac{7}{15}$. Second order conditions are fulfilled. The adage "Divide and conquer" seems to be applicable here.

For the cost functions

$$J_1 = u_1^2 - u_1 u_3,\ J_2 = u_2^2 - 2u_2 u_3,$$

$$J_3 = u_3^2 + 2u_1 u_3 + 5u_2 u_3 + u_1^2 + u_2^2 + 4u_3^2$$

such a linear strategy for the leader does not exist (nonlinear strategies do exist).

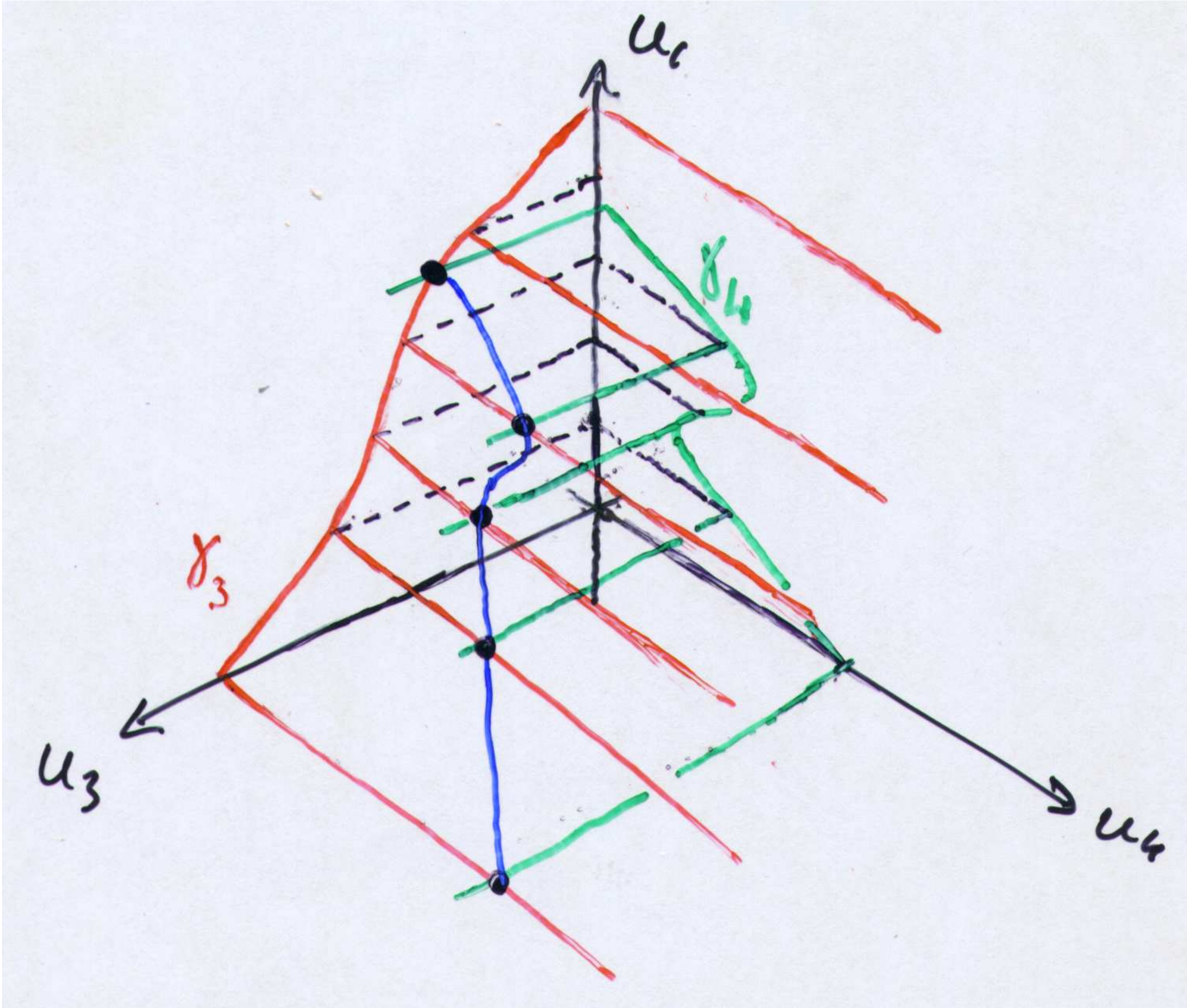
Special case: Two leaders, one follower

The cost functions are

$$J_1 = u_1^2 + u_3^2 + u_4^2,$$

$$J_3 = (u_1 - 1)^2 + (u_3 - 1)^2,$$

$$J_4 = (u_1 - 2)^2 + (u_4 - 1)^2.$$



u_1
γ_4
γ_3
u_3
u_4

Suppose that the two leaders will choose their functions γ_i as

$$u_3 = \gamma_3(u_1) = \alpha_1 u_1 + \alpha_2,\ u_4 = \gamma_4(u_1) = \beta_1 u_1 + \beta_2.$$

In the three dimensional (u_1, u_2, u_3) space these two planes have a line of intersection and the follower will choose the minimum value of J_1 on this line:

$$u_1 = -\frac{\alpha_1\alpha_2 + \beta_1\beta_2}{1 + \alpha_1^2 + \beta_1^2}.$$

The leaders will realize this and choose the coefficients according to

$$\frac{\partial J_3(\alpha_1, \alpha_2, \beta_1, \beta_2)}{\partial \alpha_i} = 0,\ \frac{\partial J_4(\alpha_1, \alpha_2, \beta_1, \beta_2)}{\partial \beta_i} = 0.$$

A solution is

$$\alpha_1 = -5,\ \alpha_2 = 10,\ \beta_1 = -2,\ \beta_2 = 5,$$

with corresponding $u_1 = 2,\ u_3 = 0,\ u_4 = 1$. Second order conditions are fulfilled. Note that this solution corresponds to absolute minimum of P_4.
Another solution corresponds to the absolute minimum of P_3.

However, solution is only locally optimal. If the leaders would choose their optimal strategies alternately, then

$$\begin{aligned}
u_4 &= \gamma_4(u_1) = -2u_1 + 5, \\
u_3 &= \gamma_3(u_1) = +5u_1 - 4, \\
u_4 &= \gamma_4(u_1) = -32u_1 + 65, \\
u_3 &= \gamma_3(u_1) = +1055u_1 - 1054, \\
u_4 &= \gamma_4(u_1) = -1114082u_1 + 2228165,
\end{aligned}$$

et cetera.

Theorem *If* $u_{1,J_3} \neq u_{1,J_4}$, *a Nash solution between the leaders does not exist (irrespectively of class of* γ_i *functions).*

Here (u_{1,J_3}, u_{3,J_3}) refers to the pair (u_1, u_3) that minimizes J_3 and similarly (u_{1,J_4}, u_{4,J_4}) minimizes J_4. The adage about "two captains on a ship" seems to be applicable here.

Addition of constraints

If one adds the constraints $-1 \leqslant u_i \leqslant +3$, $i = 1, 2, 3$, to previous example, then

$$\gamma_3(u_1) = \gamma_4(u_1) = \begin{cases} 3 & \text{if} \quad u_1 \neq 0, \\ 0 & \text{if} \quad u_1 = 0. \end{cases}$$

is a stable Nash solution, which, however, leads to the team minimum of the follower. Follower is "laughing third party".

Problem statement in continuous time

The main problem to be considered is

$$
\begin{aligned}
\dot{x} &= f(x,u),\ x(0)=x_0, \\
\min_u J_{\mathrm{F}}^{\mathrm{c}} &= \min_u \left(q(x(T)) + \int_0^T g(x,u)\mathrm{d}t + \int_0^T \gamma(u(t))\mathrm{d}t \right), \\
\max_{\gamma(\cdot)} J_{\mathrm{L}} &= \max_{\gamma(\cdot)} \int_0^T \gamma(u(t))\mathrm{d}t.
\end{aligned}
$$

The function γ is up to the choice of the leader subject to the restriction

$$\gamma(0)=0,\ \ \gamma(\cdot)\geqslant 0,\ \ \gamma(u)=\gamma(-u).$$

Occasionally we will also require that γ is nondecreasing with respect to $|u|$. An assumption is that γ does not depend on time or state explicitly (such a dependence would of course give more flexibility).
Even in the "conventional" Stackelberg dynamic games already many information structures exist (open- versus closed -loop for the leader; leader can announce his decisions with time running or everything at the beginning of the game).

Dynamic example 1

The model is

$$\dot{x} = u, \;\; x(0) = 1,$$

with x and u scalar. The follower chooses u such as to minimize

$$J_{\mathrm{F}} = (\frac{1}{2}\int_0^1 u^2(t)\mathrm{d}t + \frac{1}{2}x^2(1) + \int_0^1 \gamma(u(t))\mathrm{d}t).$$

(Change of notation! ${}^J_{\mathrm{F}}\mathrm{c} \rightarrow J_{\mathrm{F}}$.) The leader wants to choose $\gamma(\cdot)$ such as to $\max_{\gamma(\cdot)} J_{\mathrm{L}}$, where

$$J_{\mathrm{L}} = \int_0^1 \gamma(u(t))\mathrm{d}t.$$

With $\gamma \equiv 0$, the solution for the follower is obtained via

$$H = \lambda u + \frac{1}{2}u^2 \rightarrow u^* = -\lambda,$$

from which

$$u^*(t) = -\frac{1}{2},\ x^*(t) = 1 - \frac{1}{2}t,\ J_{\mathrm{F}}(u = u^*) = \frac{1}{4},\ J_{\mathrm{F}}(u = 0) = \frac{1}{2}.$$

A candidate for γ^* might be $\gamma(u) = -(\frac{1}{2} - \epsilon)u(1 + u)$ on the interval $[0, 1]$ and $\gamma(u) \geqslant 0$ elsewhere, with ϵ an arbitrarily small positive number. This yields

$$J_{\mathrm{F}} = \frac{3}{8} - \frac{1}{4}\epsilon,\ J_{\mathrm{L}} = \frac{1}{8} - \frac{1}{4}\epsilon.$$

The bank can do better, however, even with a quadratic γ. Let us try

$$\gamma(u) = \frac{1}{2}\beta u^2 + \alpha u.$$

This leads to $\beta = -1 + \epsilon$, where ϵ is an arbitrarily small positive number, and $\alpha = -\frac{2}{3} + \frac{2}{9}\epsilon$,

$$J_{\mathrm{F}} = \frac{4}{9} - \frac{1}{27}\epsilon,\ J_{\mathrm{L}} = \frac{1}{6} - \frac{1}{18}\epsilon,$$

One might be tempted to think that the bank can get its transaction costs arbitrarily close to $\frac{1}{4}$ by means of the following non quadratic choice:

$$\gamma(u) = \begin{cases} 0, & \text{if } u = 0; \\ \delta - \epsilon, & \text{if } u \neq 0, \end{cases}$$

with probably $\delta = \frac{1}{4}$ as from

$$J_{\mathrm{F}}^{\gamma\equiv 0}(u=0) - J_{\mathrm{F}}^{\gamma\equiv 0}(u=u^*) = \frac{1}{2} - \frac{1}{4}.$$

This is not true, since the solution now becomes, apart from ϵ terms, $J_{\mathrm{F}} = \frac{3}{8}$, $J_{\mathrm{L}} = \frac{1}{8}$.

Discretization

Here we consider a discretized version of the continuous time problem. The model is

$$x_1 = x_0 + \frac{1}{2}u_1 = 1 + \frac{1}{2}u_1,\ x_2 = x_1 + \frac{1}{2}u_2 = 1 + \frac{1}{2}(u_1 + u_2),$$

and the criteria are

$$J_{\mathrm{F}}(u_1, u_2) = \frac{1}{4}(u_1^2 + u_2^2) + \frac{1}{2}(1 + \frac{1}{2}(u_1 + u_2))^2 + \frac{1}{2}(\gamma(u_1) + \gamma(u_2)),$$

$$J_{\mathrm{L}} = \frac{1}{2}(\gamma(u_1) + \gamma(u_2)).$$

The leader wants to maximize $\delta = \frac{1}{2}(\gamma(\alpha) + \gamma(\beta))$, with $\alpha = \beta$, subject to

$$J_{\mathrm{F}}(\alpha, \beta) \leqslant J_{\mathrm{F}}(0, \beta) \quad \to \quad J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta) + \tfrac{1}{2}\delta \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0, \beta) \quad \to$$
$$\tfrac{1}{2}\delta \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0, \beta) - J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta);$$
$$J_{\mathrm{F}}(\alpha, \beta) \leqslant J_{\mathrm{F}}(\alpha, 0) \quad \to \quad J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta) + \tfrac{1}{2}\delta \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, 0) \quad \to$$
$$\tfrac{1}{2}\delta \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, 0) - J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta);$$
$$J_{\mathrm{F}}(\alpha, \beta) \leqslant J_{\mathrm{F}}(0, 0) \quad \to \quad J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta) + \delta \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0, 0) \quad \to$$
$$\delta \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0, 0) - J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta),$$

for suitably chosen $\alpha = \beta \neq 0$. The maximal δ is obtained for $\alpha = \beta = -\frac{2}{5}$ and thus $\delta = \frac{1}{5}$.

The function γ will be chosen as

$$\frac{1}{2}\gamma(u) = -\frac{3}{8}u^2 - \frac{2}{5}u + \frac{1}{2}\epsilon(u^2 + \frac{4}{5}u).$$

The first part at this right-hand side makes $J_{\mathrm{F}}(u, -\frac{2}{5}) + \frac{1}{2}\gamma(u)$ a constant for varying u; the second part, the term with ϵ yields a unique minimum for $u = -\frac{2}{5}$ and hence this u will be the follower's choice.

Unfortunately this is not the whole story. The Hessian of

$$J_{\mathrm{F}}(u_1,u_2)=\frac{1}{4}(u_1^2+u_2^2)+\frac{1}{2}(1+\frac{1}{2}(u_1+u_2))^2+\sum_{i=1}^{2}(-\frac{3}{8}u_i^2-\frac{2}{5}u_i+\frac{1}{2}\epsilon(u_i^2+\frac{4}{5}u_i))$$

equals

$$\begin{pmatrix} \epsilon & \frac{1}{4} \\ \frac{1}{4} & \epsilon \end{pmatrix}$$

and is not positive definite for $\epsilon < \frac{1}{4}$. For such ϵ's the situation can be remedied by choosing a different $\gamma(u)$ on the interval $u < -\frac{2}{5}$.

If we now repeat this analysis with smaller time steps, i.e. the model becomes

$$x_i = x_{i-1} + \frac{1}{N} u_i,\ i = 1, 2, \ldots, N,\ x_0 = 1,$$

then

$$u_i^* = -\frac{N}{3N-1},\ i = 1, 2, \ldots, N,$$

$$\frac{1}{N}\gamma(u) = \frac{1}{2}[-\frac{1}{N}u^2 - \frac{1}{N^2}u^2 - \frac{4}{3N-1}u] + \frac{\epsilon}{N}(u_1 + \frac{2N}{3N-1}u).$$

For $N \to \infty$ this leads to $u^*(t) = -\frac{1}{3}$, and $\gamma(u) = -\frac{1}{2}u^2 - \frac{2}{3}$, exactly the same result as obtained for the quadratic γ approach, at least for the non ϵ-terms. There are subtleties with respect to second order conditions.

The Hessian equals

$$\begin{pmatrix} \frac{2\epsilon}{N} & \frac{1}{N^2} & \cdots & \frac{1}{N^2} \\ \frac{1}{N^2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{N^2} \\ \frac{1}{N^2} & \cdots & \frac{1}{N^2} & \frac{2\epsilon}{N} \end{pmatrix}.$$

For $N > \frac{1}{2\epsilon}$ all eigenvalues lie in the right half plane. For $N \leqslant \frac{1}{2\epsilon}$, however, the Hessian is not positive definite. In the latter case, one can use the following trick (used earlier) i.e. for $-\frac{N}{3N-1} \leqslant u \leqslant 0$, $\gamma(u)$ is as above, and for $u < -\frac{N}{3N-1}$ we choose it as a decreasing function.

Dynamic example 2

The starting point is the dynamic model

$$\dot{x} = u, \;\; x(0) = 1,$$

with criterion

$$\min_{u} \frac{1}{2} \int_{0}^{1} (x^2 + u^2)\mathrm{d}t + \frac{1}{2} x^2(1).$$

An essential difference with the problem of the previous section is that the optimal control is not constant anymore: $u^*(t) = -e^{-t}$ which leads to the minimal value $J_{\mathrm{F}}^* = \frac{1}{2}$.

Discretization

Discretization with two time steps. The model is as before. The criterion for the leader also remains the same, but the criterion for the follower is now defined as

$$
\begin{aligned}
J_{\mathrm{F}} &= \frac{1}{4}(u_1^2 + u_2^2 + x_0^2 + x_1^2) + \frac{1}{2}x_2^2 + \frac{1}{2}(\gamma(u_1) + \gamma(u_2)) = \\
&= \frac{1}{4}(u_1^2 + u_2^2 + 1 + (1 + \frac{1}{2}u_1)^2) + \frac{1}{2}(1 + \frac{1}{2}(u_1 + u_2))^2 + \\
&\quad \frac{1}{2}(\gamma(u_1) + \gamma(u_2)) \\
&= \frac{7}{16}u_1^2 + \frac{3}{8}u_2^2 + 1 + \frac{3}{4}u_1 + \frac{1}{2}u_2 + \frac{1}{4}u_1 u_2 + \frac{1}{2}(\gamma(u_1) + \gamma(u_2)).
\end{aligned}
$$

The best the leader can hope for is the maximum value of $\frac{1}{2}(\delta_1 + \delta_2)$, where $\delta_1 = \gamma(u_1)$ and $\delta_2 = \gamma(u_2)$, for which not only the following inequalities must hold for a suitable choice of α and β:

$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta) + \frac{1}{2}(\delta_1 + \delta_2) \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0, \beta) + \frac{1}{2}\delta_2;$$

$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta) + \frac{1}{2}(\delta_1 + \delta_2) \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, 0) + \frac{1}{2}\delta_1;$$

$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha, \beta) + \frac{1}{2}(\delta_1 + \delta_2) \leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0, 0),$$

but also

$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha,\beta)+\tfrac{1}{2}(\delta_1+\delta_2)\leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\beta,\beta)+\delta_2;$$
$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha,\beta)+\tfrac{1}{2}(\delta_1+\delta_2)\leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha,\alpha)+\delta_1;$$
$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha,\beta)+\tfrac{1}{2}(\delta_1+\delta_2)\leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(0,\alpha)+\tfrac{1}{2}\delta_1;$$
$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha,\beta)+\tfrac{1}{2}(\delta_1+\delta_2)\leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\beta,0)+\tfrac{1}{2}\delta_2;$$
$$J_{\mathrm{F}}^{\gamma\equiv 0}(\alpha,\beta)+\tfrac{1}{2}(\delta_1+\delta_2)\leqslant J_{\mathrm{F}}^{\gamma\equiv 0}(\beta,\alpha)+\tfrac{1}{2}(\delta_1+\delta_2).$$

The analysis becomes unwieldy, the more so for the discretization in $N > 2$ time steps. See my JOTA-paper for results in this direction.

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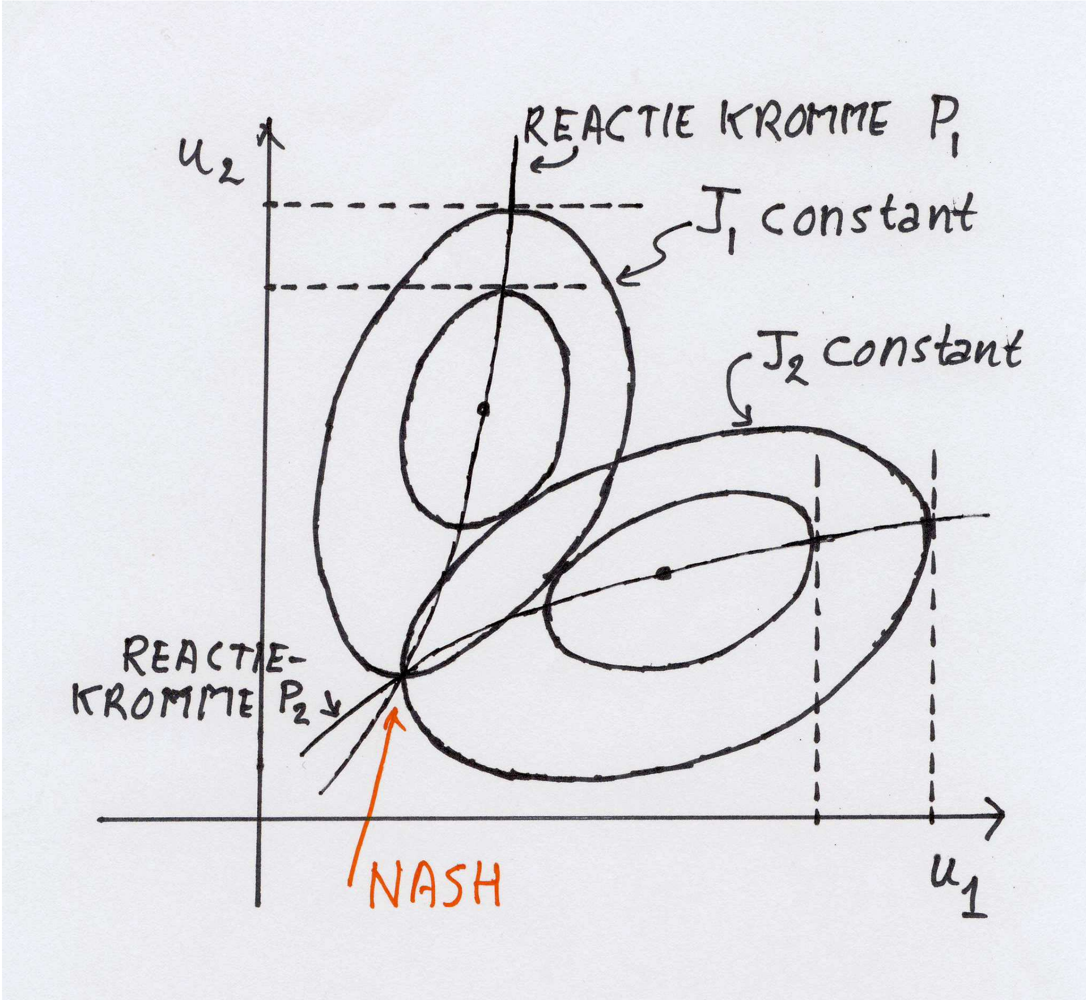

REACTIE KROMME P1
J1 constant
J2 constant
u2
REACTIE-
KROMME P2
NASH
u1

Consistent Conjectural Variations (CCV).

Players are supposed to react to each other according to (yet unknown) functions

$$u_1 = \psi_1(u_2); \; u_2 = \psi_2(u_1).$$

Equilibrium conditions are ("double sided Stackelberg")

$$\frac{\partial J_1}{\partial u_1} + \frac{\partial J_1}{\partial u_2}\frac{d\psi_2}{du_1} = 0,$$

$$\frac{\partial J_2}{\partial u_2} + \frac{\partial J_2}{\partial u_1}\frac{d\psi_1}{du_2} = 0.$$

The first of these equations yields u_1 as a function of u_2: $u_1 = \mu_1(u_2)$. The second equation yields similarly $u_2 = \mu_2(u_1)$. The equilibrium is determined by $\psi_i(u_j) \equiv \mu_i(u_j), \; i,j = 1,2, \; i \neq j$.

The functions ψ_i satisfy

$$\frac{\partial J_1(\psi_1(u_2),u_2)}{\partial u_1}+\frac{\partial J_1(\psi_1(u_2),u_2)}{\partial u_2}\frac{\partial \psi_2(\psi_1(u_2))}{\partial u_1}=0,\ \forall u_2\in U_2,$$

$$\frac{\partial J_2(u_1,\psi_2(u_1))}{\partial u_2}+\frac{\partial J_2(u_1,\psi_2(u_1))}{\partial u_1}\frac{\partial \psi_1(\psi_2(U_1))}{\partial u_2}=0,\forall u_1\in U_1.$$

Besides, we need second order conditions to be satisfied and, moreover,

$$U_1\supset\psi_1(U_2),\ U_2\supset\psi_2(U_1).$$

The equilibrium solution (u_1^*,u_2^*) is defined as the (or: a) solution of the coupled set of equations

$$u_1=\psi_1(u_2),\ u_2=\psi_2(u_1).$$

Example 1

$$J_1 = (u_1 - 84)(-12\frac{1}{2}u_1 + 21u_2 + 756),$$

$$J_2 = (u_2 - 50)(25u_1 - 50u_2 + 560).$$

(Players are maximizing.) Assumption: linear reaction curves of the form

$$u_1 = \psi_1(u_2) = \alpha_2^1 u_2 + \beta^1,\ u_2 = \psi_2(u_1) = \alpha_1^2 u_1 + \beta^2.$$

Two sets of solutions, only one of them satisfies the second order conditions:

$$\alpha_2^1 = \frac{6}{5},\ \alpha_1^2 = \frac{5}{14},\ \beta^1 = \frac{336}{5},\ \beta^2 = \frac{312}{14}.$$

Both for $U_1 = U_2 = \mathbf{R}$ and $U_1 = U_2 = \mathbf{R}^+/\{0\}$ the inclusions hold:

$$U_1 \supset \frac{6}{5}U_2 + \frac{336}{5}, \; U_2 \supset \frac{5}{14}U_1 + \frac{312}{14}.$$

The equilibrium solution is $u_1^* = 164\frac{2}{5}$, $u_2^* = 81$. It turns out that the CCV solution is better for both players than the Nash solution (always true?).

If we would change the parameters in this quadratic game, the following results are possible:

- no affine reaction curves exist;
- two sets of affine reaction curves exist, but for only one of them the second order conditions hold;
- two sets of affine reaction curves exist for which all conditions hold.

We will construct formal power series ψ_i such that

$$-25\psi_1(u_2)+21u_2+1806+21(\psi_1(u_2)-84)\frac{d\psi_2(\psi_1(u_2))}{du_1} \equiv 0, \ \forall u_2 \in U_2,$$

$$-100\psi_2(u_1)+25u_1+3060+25(\psi_2(u_1)-50)\frac{d\psi_1(\psi_2(u_1))}{du_2} \equiv 0, \ \forall u_1 \in U_1,$$

and, in addition, that $(\tilde{u}_1, \tilde{u}_2)$, an arbitrary point in $U_1 \times U_2$, is the point of intersection of the two reaction curves. If such ψ_i functions exist, then every point in $U_1 \times U_2$ can be the solution to the game problem. We make translations $u_i \longrightarrow q_i$ and $\psi_i \longrightarrow \varphi_i$ as follows:

$$u_i = \tilde{u}_i+q_i, \ i = 1, 2, \ \tilde{u}_2+\varphi_2(q_1) = \psi_2(\tilde{u}_1+q_1), \ \tilde{u}_1+\varphi_1(q_2) = \psi_1(\tilde{u}_2+q_2),$$

and construct formal power series

$$\varphi_1(q_2) = \sum_{k=1}^{\infty} a_k q_2^k, \ \varphi_2(q_1) = \sum_{k=1}^{\infty} b_k q_1^k.$$

Setting constant terms in the ψ-equations equal to zero leads to

$$-25\tilde{u}_1 + 21\tilde{u}_2 + 1806 + 21(\tilde{u}_1 - 84)a_1 = 0,$$

$$-100\tilde{u}_2 + 25\tilde{u}_1 + 3060 + 25(\tilde{u}_2 - 50)b_1 = 0,$$

and the same with the linear terms:

$$-25b_1 + 21 + 21(\tilde{u}_1 - 84)2a_2b_1 + 21b_1a_1 = 0,$$

$$-100a_1 + 25 + 25(\tilde{u}_2 - 50)2b_2a_1 + 25a_1b_1 = 0,$$

et cetera. The N-th order terms determine a_{N+1} and b_{N+1} (linear equations). Hence polynomial reaction curves will not exist in general (exception if $\tilde{u}_1 = 164\frac{2}{5}, \tilde{u}_2 = 81$, then $a_1 = \frac{5}{14}, b_1 = \frac{6}{5}$ and $a_i = b_i = 0$ for $i > 1$). Positive radii of convergence, second order conditions?

Example 2

$$J_1(u_1,u_2) = -(u_1-1)^2-(u_2-b)^2-2mu_1u_2,\ J_2(u_1,u_2) = J_1(u_2,u_1),$$

b and m are parameters. We only consider $U_1 = U_2 \stackrel{\text{def}}{=} U$ and identical reaction curves $\psi_1 = \psi_2 \stackrel{\text{def}}{=} \psi$. The function ψ satisfies

$$\{\psi(u) - 1 + mu\} + \{u - b + m\psi(u)\}\frac{d\psi(\psi(u))}{du} \equiv 0,\ \forall u \in U.$$

As "initial condition" we have $\psi(\tilde{u}) = \tilde{u}$, the solution to the game, where $\tilde{u} \in U$ (but otherwise arbitrary). New translated variable and function, defined by

$q \stackrel{\text{def}}{=} u - \tilde{u},\ \varphi(q) \stackrel{\text{def}}{=} \psi(u) - \tilde{u} = \psi(q + \tilde{u}) - \tilde{u}$, lead to

$$\{\tilde{u}+\varphi(q)-1+m(q+\tilde{u})\}+\{q+\tilde{u}-b+m(\tilde{u}+\varphi(q))\}\frac{d\varphi(\varphi(q))}{dq} \equiv 0.$$

Substitution of the formal power series $\varphi(q) = \sum_{k=1}^{\infty} c_k q^k$ leads to

$$\tilde{u} - 1 + m\tilde{u} + (\tilde{u} - b + m\tilde{u})c_1 = 0,$$

$$c_1 + m + (1 + mc_1)c_1 + (\tilde{u} - b + m\tilde{u})2c_2c_1 = 0,$$

et cetera. Assumption: a positive radius of convergence for $\varphi(q)$. Then the second order condition and the inclusion relation ($U \supset \psi(U)$, where U is a small open subset around $\tilde{u}$) are satisfied if

$$2c_1^2 + 3mc_1 - \frac{m}{c_1} > 0,\ |\tilde{u} - 1 + m\tilde{u}| < |\tilde{u} - b + m\tilde{u}|.$$

A linear φ-function only exists for specific $\tilde{u}$-values.

Existence of solution

Consider

$$\frac{d\varphi(\varphi(q))}{dq} = f(\varphi(q), q), \ \varphi(0) = 0.$$

Under the assumptions

- f is C^2 on the square $[-\nu, +\nu] \times [-\nu, +\nu]$, $\nu > 0$,
- on this square it satisfies $|\frac{\partial f}{\partial q}| < K, \ 0 < M_1 \leqslant f \leqslant M_2 < \infty$, where K, M_1, M_2 are fixed positive numbers,

a solution φ exists on the interval $[-\delta, +\delta]$, where $\delta = \min\{\frac{M_1}{2KM_2}, \frac{\nu}{M_2}, \nu, M_1\nu\}$. If $M_1 > 1$ then the solution is unique, if $M_2 < 1$, then the solution depends on arbitrary function. Cases $M_1 < 1$ and $M_2 > 1$ have not been investigated on uniqueness issues. Proof is by means of a suitably chosen contraction operator.

Continuation example 2. Take $m = -1, b = -2,\ U = (\bar{u} - \epsilon, \bar{u} + \epsilon)$ with $\bar{u}$ arbitrary and ϵ sufficiently small. Application of existence theorem leads to: every point $\tilde{u} \in U$ is an equilibrium point. Therefore a continuum of CCV equilibria exists.

Conclusions

- The surface has been scratched only.
- From a mathematical point of view, the problems with composed functions are challenging.
- From an economic point of view, more realistic cost functions must be considered (e.g. concave utility functions w.r.t. consumption).

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