How Hard is Competition for Rank?

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April 2010

Easy and hard problems

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To prove problem P is hard, take a problem H that is already believed to be hard, and "efficiently encode" instances of H in terms of P so that the answer to P tells you the answer to H ...

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Given a graph G of indegree/outdegree at most 1, and a vertex of degree 1, find another vertex of degree 1.The catch is, G 's edges are represented by boolean circuits that take any pair of endpoints in $\{0,1\}^n$ and output whether an edge is present between them.

Overview

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This talk

- Some intuition on the hardness of unrestricted NE
- A class of games that appears to be “realistic” for which we so far have some positive results

The "Dragons' Den" Game

Two entrepreneurs, Alice and Bob, want to raise £100,000 from a venture capitalist. Each of them may decide to spend £2,000 on image consulting. Alice has a better business idea, and the only way Bob will receive the investment is if he buys the image consulting and Alice does not.

Question: which of them will buy the image consulting?

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look for *mixed (randomised) strategies*; the problem becomes: *compute the probabilities*

		Bob	
		don't spend 5000	spend 5000
Alice	spend 5000	0 45	-5 45
	don't spend 5000	0 50	45 0

Numbers are multiples of £5,000; assume it is worth £50,000 to win the investment.

"Incentive direction" of the players

Bob

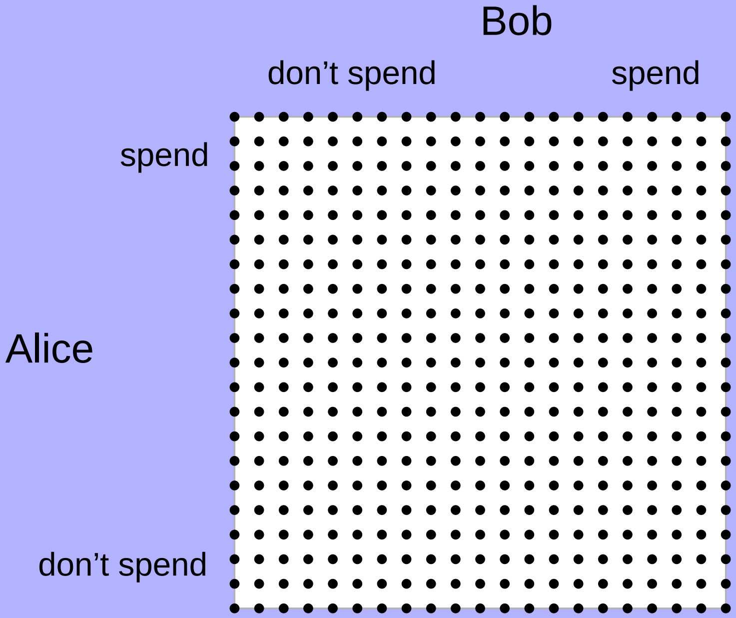
don't spend spend

Alice

spend

don't spend

“Incentive direction” of the players



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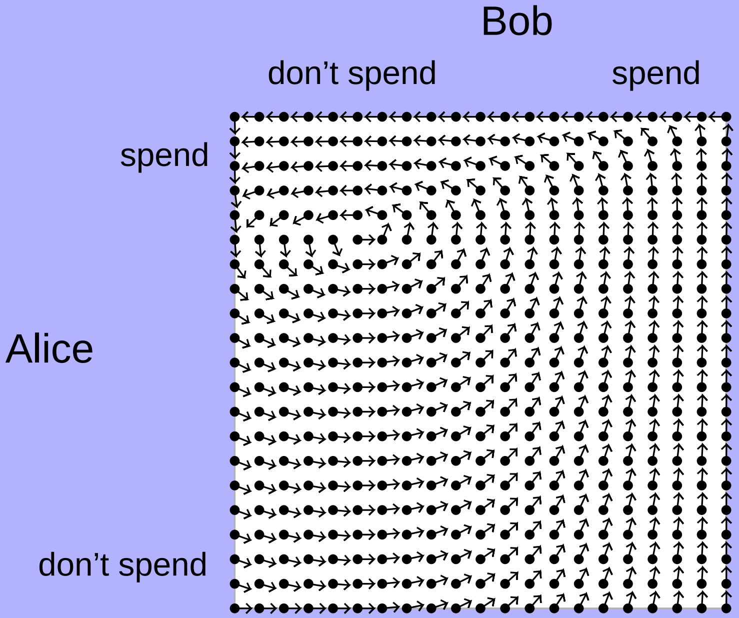

Nash equilibrium

Brouwer's fixpoint theorem: continuous functions from a compact domain to itself, have fixpoints. A *non-constructive* proof.

L.E.J. Brouwer (1881-1966)

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But, how to compute the probabilities? We would like an "efficient algorithm". **Next: how search for NE relates to search on large graphs**

John Forbes Nash

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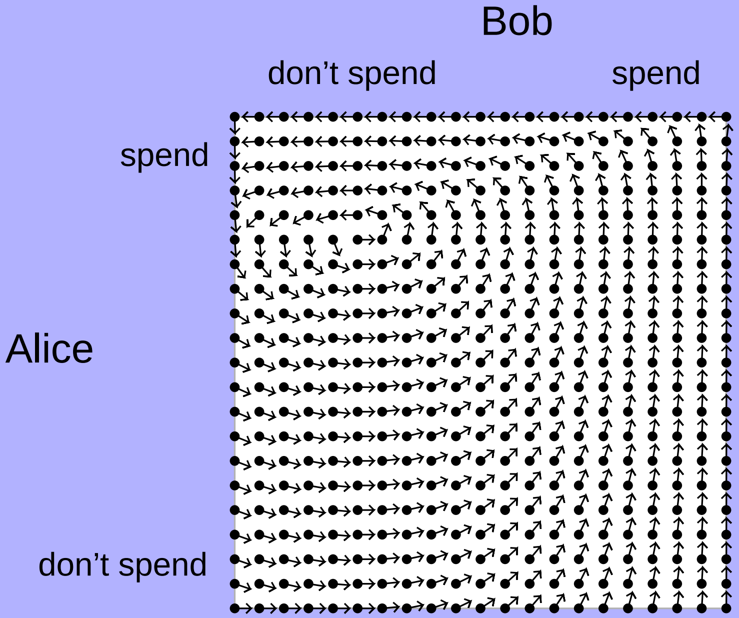
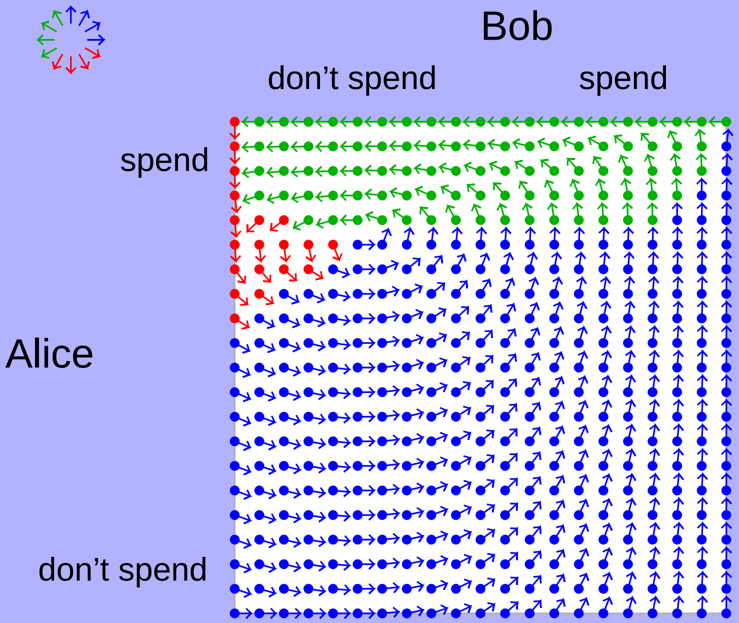
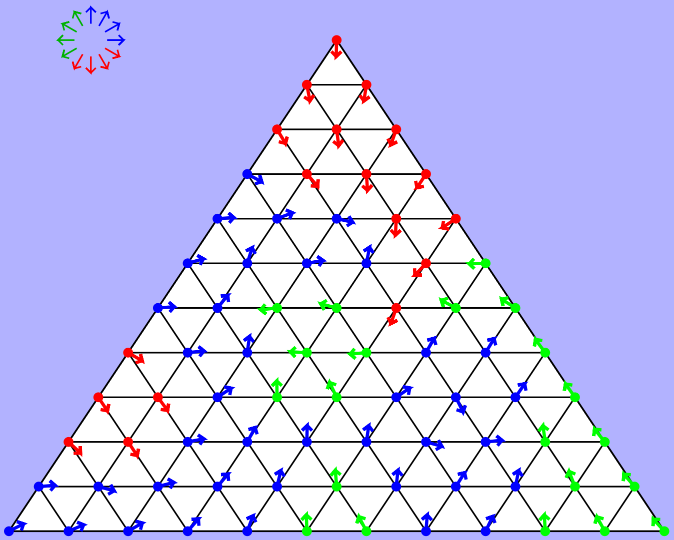


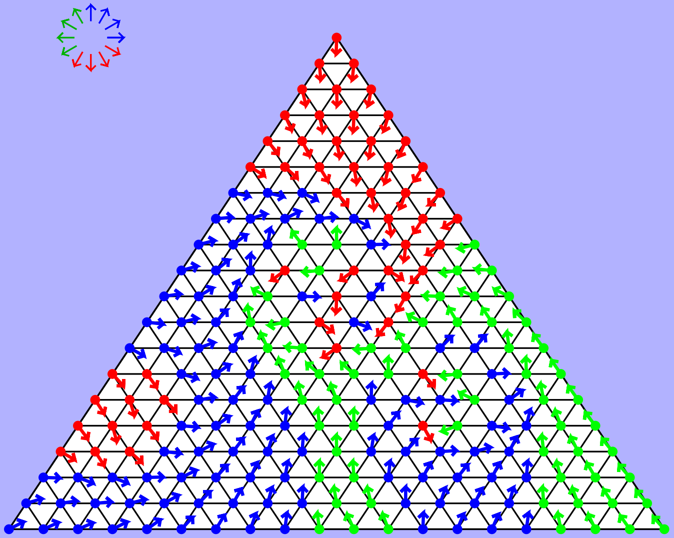
“Incentive direction”, colour-coded



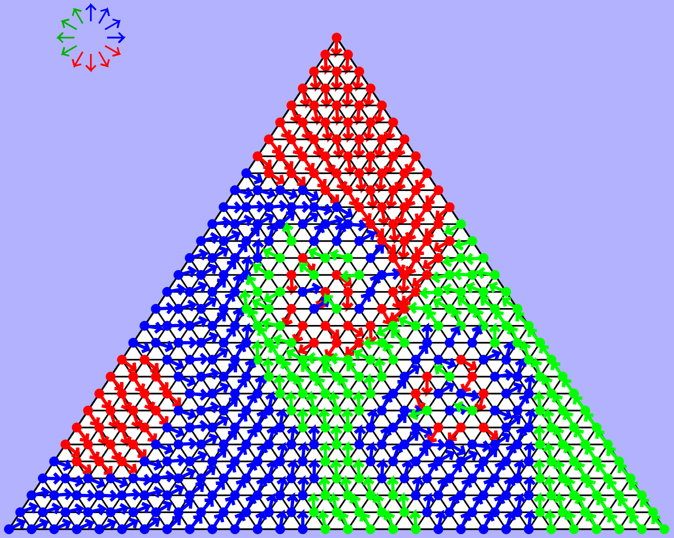
Now, pretend this triangle is high-dimension domain



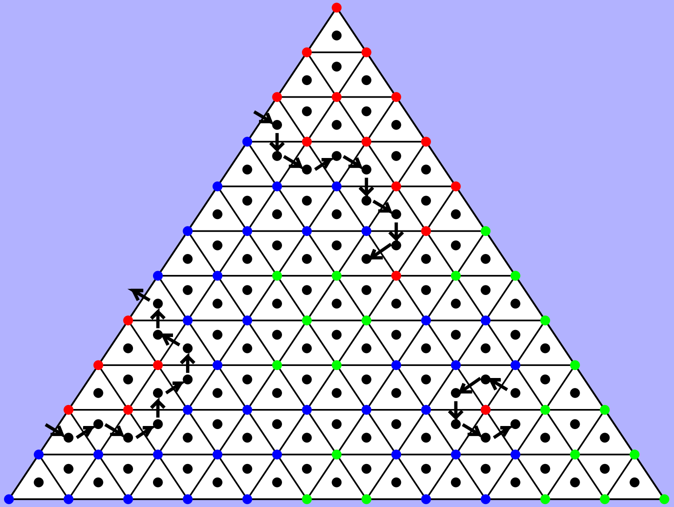
Search for “trichromatic triangles” at higher resolution...



...converges to Brouwer fixpoint



The corresponding graph



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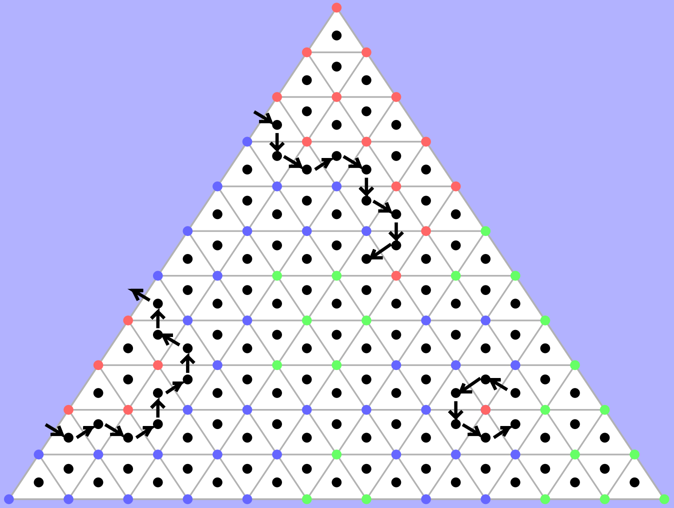

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  Basically, solving a game is equivalent to finding your way around a very large graph, one that allows efficient local exploration and consists of long paths.

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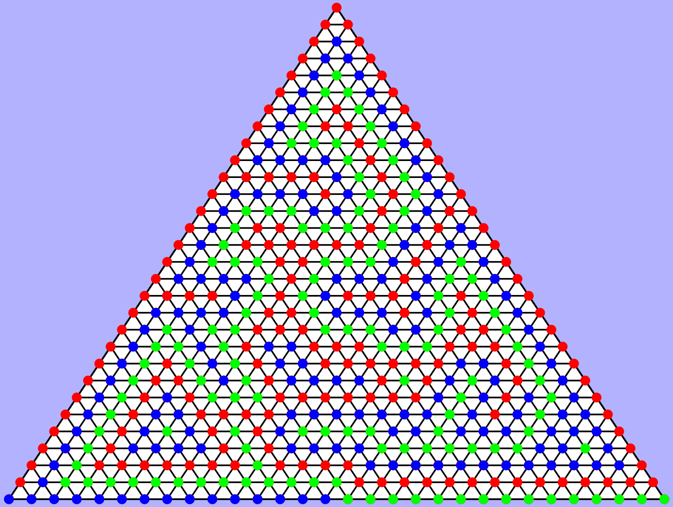
  Basically, solving a game is equivalent to finding your way around a very large graph, one that allows efficient local exploration and consists of long paths.
- 2-players (Chen, Deng and Teng '06); 2-players, 0/1-valued payoffs (Abbott, Kane and Valiant '05)

How to make a hard case of the problem



Chen and Deng ('06, '09): 2D-Brouwer

coming back to “Dragons’ Den”

(Current work with colleagues at Liverpool)

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What if there are

- more than 2 competitors?

coming back to “Dragons’ Den”

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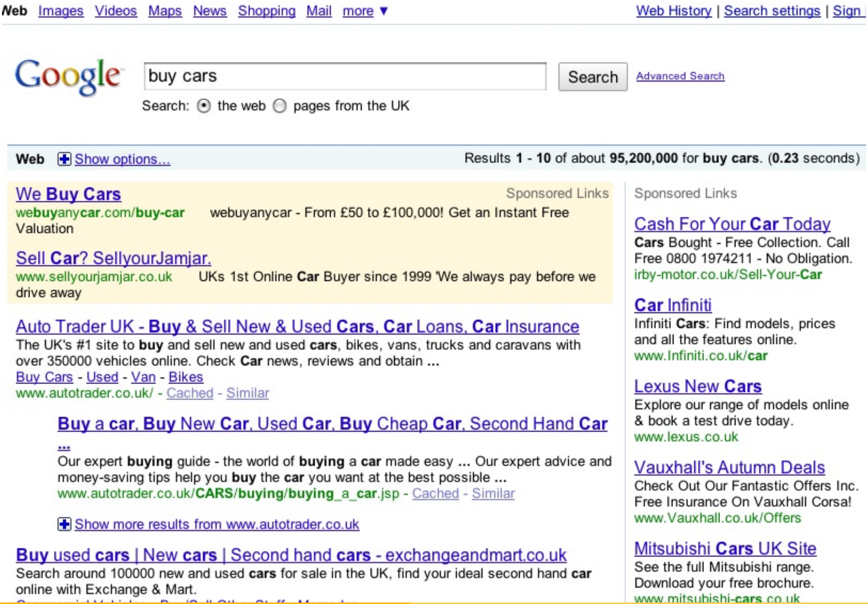
What if there are

- more than 2 competitors?
- many choices per competitor?
- more than one “prize” for winning?

Players compete for *rank*.

Competition for rank



Competition for rank

- Who's Top Ten for their Subject 2010
- University League Table
- Methodology & Notes
- Universities by region
- University League Table 2009
- University League Table 2008

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www.cem.ac.uk

University League Table 2010

Published April 30th 2009

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Compare | Clear

To compare 2 or more universities, select the box next to the name and click Compare.
To create your own ranking see below

Rank	Institution	Student Satisfaction	Research Assesment	Entry Standards	Student-Staff Ratio	Academic Services Spend	Facilities Spend	Good Honours	Graduate Prospects
1	The original Complete University Guide ranking	4.1	3.0	523.9	10.9	2,639	587	91.1	82.3
2	Cambridge	4.2	3.0	538.5	11.6	1,755	657	87.0	85.5
3	Imperial College	3.7	2.9	489.4	10.3	3,036	575	68.5	88.4
4	London School of Economics	3.8	3.0	483.0	14.1	1,391	265	76.0	90.6
5	Durham	4.0	2.7	459.0	15.2	1,036	682	77.5	78.3
6	Warwick	3.9	2.8	463.0	13.5	1,630	354	79.7	79.2
7	St Andrews	4.2	2.7	467.9	12.3	1,152	357	85.1	77.8

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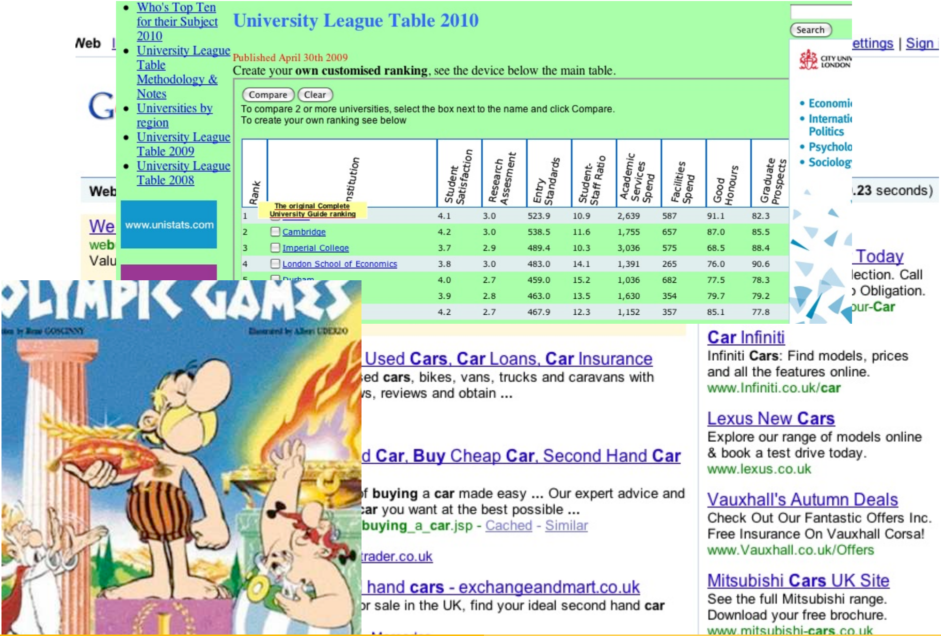
Competition for rank



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Britain's quality of life worse than former Communist countries

Britain's has fallen to 25th position on a list of best places in the world to live.

By Myra Butterworth, Personal Finance Correspondent
Published: 6:54PM GMT 06 Jan 2010

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Our poor climate, soaring unemployment and congested roads means we are now ranked behind

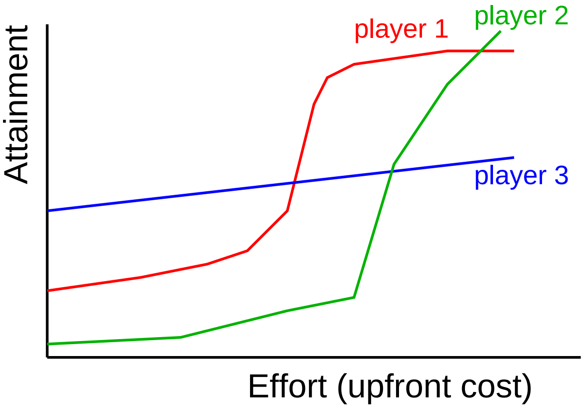
Some background on ranking games

"Ranking games" (Brandt, Fisher, Harrenstein and Shoham)

each combination of strategies results in a ranking of the players; every player has a monotonically decreasing function from rank to utility.

Problem: unrestricted ranking games are still hard: a 3-player ranking game can easily encode an unrestricted 2-player 0/1 game. (as noted earlier, hard to solve)
Our idea: assume strategies are correlated with "competitiveness"

The model



Each player has his own function from effort to performance.

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If a player plays a_j and wins the k -th prize, his overall utility is $u_k - c_j$.

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Observation

We can concisely represent games with many players/strategies, in contrast with unrestricted ranking games.

Some results

We can pre-process a d -player game so as to assume that $u_1 = 1$, $u_d = 0$; all costs c_j^i lie in range $[0, 1]$; costs and returns are *strictly* monotonic in j , else we would have dominated actions; each player's weakest action has cost 0.

Theorem

Suppose there is just one prize ($u_1 > 1$; $u_j = 0$ for $j > 1$). Suppose ties are impossible (if all r_j^i -values are distinct, or equivalently there is a tie-breaking rule). Then there is just one player who gets positive payoff (all others get zero); namely the player who has the strongest action.

Some results

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- Finally, we found precisely one player who can get positive payoff.

□

What if the strongest action has cost 1? What about > 1 prizes?

Some results

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(So, that's like 2-player normal-form games! Is that interesting?)

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- d-players, n actions, where d is constant: Approximate NE can be found in poly-in-n time by brute-force approach.
- FPTAS for d players, 1 prize (in the paper, done for just 2 players) Dynamic programming approach

Linear-prize ranking games

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So, we have reduced the game to a zero-sum polymatrix game, which is known to be solvable in poly-time (Daskalakis and Papadimitriou '09).

Conclusions and further work

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- focus on "natural" types of more tractable games
- For these games, continue by looking for *decentralised* algorithms (a solution is implausible if it needs to be found centrally and then handed out to the players).
- Another direction: weaken the objective – "approximate equilibria" replace "no incentive for a player to change" with "only a small incentive to change" — an interesting and challenging problem, both for centralised and decentralised algorithms!