# Stopping games under partial information 

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## Plan of presentation:

(1) Dynkin's game-stopping game

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- Deterministic priority-permutation of the player's rang
- Random priority after observation of the state
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(2) Coordination of players behaviour
(3) Selection of a Correlated Rational Equilibria
(4) Further research and remarks


## Description of the considered problems

## Various level of information acquisition

The paper deals with competitive optimal stopping of the discrete time Markov sequence by more than two decision makers when more than one stopping action for the player is allowed (see [Szajowski(2002)], [Ramsey and Szajowski(2001b)]). The decision makers are able to observe the Markov process sequentially to extend their knowledge about the process by enclosing the actual observation. However, the stopping decisions disturb this process and limit knowledge about the states to other players. If the role of the players in the competition is not equal then the information acquisition is strongly dependent of the player's position in the decision process. This is the natural extension for the stopping game models (see also [Ramsey(2007)]). Further, an extension of the concept of correlated strategies in Markov stopping games is also discussed. It follows the discussion from [Ramsey and Szajowski(2008)].

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## Correlated decision of the players

The Nash equilibrium approach to solve nonzero-sum stopping games may give multiple solutions. An arbitrator can suggest the strategy to be applied. This is a form of equilibrium selection. Utilitarian, egalitarian, republican and libertarian concepts of correlated equilibria selection are used. A formalization of the model and a construction of equilibrium for a finite horizon games is given. The examples of such decision problems related to the best choice problem is solved. The model is generalization of the games considered by Szajowski [Szajowski(1994)], [Szajowski(1995)] and Enns \& Ferenstein [Enns and Ferenstein(1987)].

## Description of the models

## Normal form of the game

A Markov chain is observed. K players look for the most profitable state. Each state is available to only one player. Some methods of assignment the state in conflict of interest is given by defining the priority of the players. The normal form of the game is derived. The privileges of players are modeled by division of the unit interval and a sequence of random variables with uniform distribution on it. The strategies of the players are $K$-taples of randomized stopping times. A construction of Nash equilibrium for the game is given.

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## Information acquisition

In the case when more than one player would like to accept the state there are random mechanisms to choose the beneficiary. However, different structures of decision process and access to the observation can appear.
(1) The priority of the players is decided after appears of state but:
(1) The information about accepted state is known to all players or (2) it is hidden to the players which do not accepted the item.
(2) The random assignment of the rights can run before observation of each item by the players. In this case the accepted observation is not known to players with lowest priority. It makes that after the first acceptance some players are better informed than the others.

## Preliminary denotation and remarks

## Markov chain and players

Let $\left(X_{n}, \mathcal{F}_{n}, \mathbf{P}_{x}\right)_{n=0}^{N}$ be a homogeneous Markov process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with state space $(\mathbb{E}, \mathcal{B})$. At each moment $n=1,2, \ldots, N$ the decision makers (henceforth called Player $i, i \in \overline{1, N}$ ) are able to observe the Markov chain sequentially. Each player has his own utility function $g_{i}: \mathbb{E} \rightarrow \Re, i \in \overline{1, K}$, and at moment $n$ each decides separately whether to accept or reject the realization $x_{n}$ of $X_{n}$. We assume the functions $g_{i}$ are measurable and bounded.

## Permutation of players' rangs

## Model of assignments

In multi-person Dynkin's game the role of an arbiter was given to the random process $\xi_{n}$. The simplest model can assume that the players are ordered before the play to avoid the conflict in assignment of presented sequentially states. At each moment the successive state of the process is presented to the players, they decide to stop and accept the state or continue observation. The state is given to the players with highest rang (we adopt here the convention that the player with rang 1 has the highest priority). In this case each stopping decision reduce the number of players in a game. It leads to recursive algorithm of construction the game value and in a consequence to determining the equilibrium (see [Nowak and Szajowski(1998)], [Sakaguchi(1995)] for review of such models investigation).

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## Effective stopping time for player $i$

- Let $P=\{1,2, \ldots, N\}$ be the set of players and $\pi$ a permutation of $P$. It determines the priority $\pi(i)$ of player $i$.


## Extension of basic problem to fix deterministic priority

## Effective stopping time for player $i$

- Let $\left(p_{n}^{i}\right)_{n=1}^{T}$ be the pure stopping strategy. If it is randomized stopping time we can find pure stopping time with respect to an extended filtration. The effective stopping strategy of the player $i$ is following:

$$
\begin{equation*}
\tau_{i}(\overrightarrow{(p)})=\inf \left\{k \geq 1: p_{k}^{i} \prod_{j=1}^{N}\left(1-p_{k}^{j}\right) \mathbb{I}_{\{j: \pi(j)<\pi(i)\}}=1\right\} \tag{1}
\end{equation*}
$$

where $\vec{p}=\left(p^{1}, p^{2}, \ldots, p^{N}\right)$ and each $p^{i}=\left(p_{n}^{i}\right)_{n=1}^{T}$ is adapted to the filtration $\left(\mathcal{F}_{n}^{i}\right)_{n=1}^{T}$. The effective stopping time of the player $i$ is the stopping time with respect to the filtration $\tilde{\mathcal{F}}_{n}^{i}=\sigma\left\{\mathcal{F}_{n}^{i},\left\{\left(p_{k}^{j}\right)_{k=1,\{j: \pi(j)<\pi(i)\}}^{n}\right\}\right\}$.

- The above construction of effective stopping time assures that each player will stop at different moment. It translates the problem of fixed priority optimization problem to the ordinary stopping game with payoffs $G_{i}\left(\tau_{1}, \tau_{2}, \ldots, \tau_{N}\right)=g_{i}\left(X_{\tau_{1}}, X_{\tau_{2}}, \ldots, X_{\tau_{N}}\right)$.


## Dynamic deterministic priority

## Effective stopping time for player $i$

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\tau_{i}(\vec{p})=\inf \left\{k \geq 1: p_{k}^{i} \prod_{j=1}^{N}\left(1-p_{k}^{j}\right) \mathbb{I}_{\left\{j: \pi_{k}(j)<\pi_{k}(i)\right\}}=1\right\} \tag{2}
\end{equation*}
$$

where $\vec{p}=\left(p^{1}, p^{2}, \ldots, p^{N}\right)$ and each $p^{i}=\left(p_{n}^{i}\right)_{n=1}^{T}$ is adapted to the filtration $\left(\mathcal{F}_{n}^{i}\right)_{n=1}^{T}$. The effective stopping time of the player $i$ is the stopping time with respect to the filtration
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## Priority assignment

If players have selected the same moment $n$ to accept $x_{n}$, then a lottery decides which player gets the right (priority) of acceptance. Let $0 \leq \alpha_{n}^{i} \leq 1$ for $n=1,2, \ldots, N, i \in \overline{1, K}$. According to the lottery, at moment $\tau$, if $K$ players would like to accept $x_{\tau}$, then Player $i$ is chosen with probability $\alpha_{\tau}^{i}$. If only players with numbers $\mathcal{S} \subset \overline{1, K}$ compete for the observation $x_{T}$, then the priority of Player $i, i \in \mathcal{S}$, is proportional to $\alpha_{\tau}^{i}$. The players rejected by the lottery may select any other realization $x_{n}$ at a later moment $n$, $\tau<n \leq N$. Once accepted a realization cannot be rejected, once rejected it cannot be reconsidered.

## Randomize stopping times

Let $A_{1}^{i}, A_{2}^{i}, \ldots, A_{N}^{i}$ be i.i.d.r.v. from the uniform distribution on $[0,1]$ and independent of the Markov process $\left(X_{n}, \mathcal{F}_{n}, \mathbf{P}_{x}\right)_{n=0}^{N}$. Let $\mathcal{H}_{n}$ be the $\sigma$-field generated by $\mathcal{F}_{n},\left\{A_{1}^{i}, A_{2}^{i}, \ldots, A_{n}^{i}\right\}$. A randomized Markov time $\tau\left(p^{i}\right)$ for strategy $p^{i}=\left(p_{n}^{i}\right) \in \mathcal{P}^{N, i}$ is defined by $\tau\left(p^{i}\right)=\inf \left\{N \geq n \geq 1: A_{n} \leq p_{n}^{i}\right\}$. We denote by $\mathbf{M}_{i}^{N}, i \in \overline{1, K}$, the sets of all randomized strategies of the $i$-th Player. A $\left\{\mathcal{F}_{n}\right\}$ - Markov time $\tau^{i}$ corresponds to the strategy $p^{i}=\left(p_{n}^{i}\right)$ with $p_{n}^{i}=\mathbb{I}_{\left\{\tau^{i}=n\right\}}$, where $\mathbb{I}_{A}$ is the indicator function for the set $A$.

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Random assignment of priority to a player requires to consider modified strategies with respect to the sets of stopping times. Denote $\mathcal{T}_{k}^{N}=\left\{\tau \in \mathcal{T}^{N}: \tau \geq k\right\}$ and $\mathcal{P}_{k}^{N}=\left\{p \in \mathcal{P}^{N}: p_{j}=0\right.$ for $\left.j=1,2, \ldots, k-1\right\}$. One can define the sets of strategies $\tilde{\mathbf{M}}^{N, i}=\left\{\left(p^{i},\left\{p_{n}^{i}\right\}, \ldots,\left\{\tau_{n}^{i}\right\}\right): p^{i} \in \mathcal{P}^{N, i},\left\{p_{n}^{i}\right\} \in\right.$ $\mathcal{P}_{n}^{N, i}, \ldots, \tau_{n}^{i} \in \mathcal{T}_{n+1}^{N}$ for every $\left.n\right\}$ for Player $i$.

## Construction of random priority mechanism

Let $\xi_{1}, \xi_{2}, \ldots$ be i.i.d.r.v. uniformly distributed on $[0,1]$ and independent of $\bigvee_{n=1}^{N} \mathcal{H}_{n}$ and the lottery be given by the sequence of divisions of the interval $[0,1]$. Denote this divisions by $\mathcal{C}=\left\{\tilde{C}_{n}\right\}_{n=1}^{N}$, where
$\tilde{C}_{n}=\left(C_{n}^{1}, \ldots, C_{n}^{K}\right), C_{n}^{i} \cap C_{n}^{j}=\emptyset$ for $i \neq j, i, j \in \overline{1, K}, \mathrm{~m}\left(C_{n}^{i}\right)=\alpha_{n}^{i}$, and $\bigcup_{j=1}^{K} C_{n}^{j}=[0,1]$ for $n=1,2, \ldots, N$. The lottery is used in such a way that if $\lambda_{i}\left(p^{i}\right)=\lambda_{j}\left(p^{j}\right)=n$ for $i, j \in \mathcal{S}$ and $\lambda_{k}\left(p^{k}\right) \neq n, k \notin \mathcal{S}$, then the r.v. $\xi_{n}$ is simulated with restriction to $\bigcup_{j \in \mathcal{S}} C_{n}^{j}$. Denote $\tilde{\mathcal{H}}_{n}=\sigma\left\{\mathcal{H}_{n}, \xi_{1}, \xi_{2}, \ldots, \xi_{n}\right\}$ and let $\tilde{T}^{N}$ be the set of Markov times with respect to $\left(\tilde{\mathcal{H}}_{n}\right)_{n=0}^{N}$. Every tuple $\left(s^{1}, \ldots, s^{K}\right)$ such that $s^{i} \in \tilde{\mathbf{M}}^{N, i}$ define the effective stopping time for each player.

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## Definition

The Markov times $\tau_{i}(\vec{s})$ for $i \in \overline{1, K}$ are the selection times of Player $i$ when they use strategies $s^{i} \in \tilde{\mathbf{M}}^{N, i}$ and the lottery is defined by partition $\mathcal{C}$.

For each $\vec{s} \in \tilde{\mathbf{M}}^{N, 1} \times \ldots \times \tilde{\mathbf{M}}^{N, K}$ and given $\mathcal{C}$ the payoff function for the $i$-th player is defined as $f_{i}(\vec{s})=g_{i}\left(X_{\tau_{i}(\vec{s})}\right)$. Let
$\tilde{R}_{i}(x, \vec{s})=\mathbf{E}_{x} f_{i}(\vec{s})=\mathbf{E}_{x} g_{i}\left(X_{\tau_{i}(\vec{s})}\right)$ be the expected gain of the $i$-th player, if the players use $s^{i}, i \in \overline{1, K}$. We have defined the game in normal form $\left(\tilde{\mathbf{M}}^{N, 1}, \ldots, \tilde{\mathbf{M}}^{N, K}, \tilde{R}_{1}, \ldots, \tilde{R}_{K}\right)$. This random priority game will be denoted $\mathcal{G}_{r p}$.

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## Definition

A tuple $\vec{s}^{*}$ of strategies such that $s^{i^{*}} \in \tilde{\mathbf{M}}^{N, i}, i=1, \ldots, K$, is called a Nash equilibrium in $\mathcal{G}_{r p}$, if for all $x \in \mathbb{E}$

$$
v_{i}(x)=\tilde{R}_{i}\left(x, \vec{s}^{* *}\right) \geq \tilde{R}_{i}\left(x,\left(s, s^{-i^{*}}\right)\right) \text { for every } s \in \tilde{\mathbf{M}}^{N, i}
$$

where $\left(s, s^{-i^{*}}\right)=\left(s^{1^{*}}, \ldots, s^{i-1^{*}}, s, s^{i+1^{*}}, \ldots, s^{K^{*}}\right)$. The tuple $\left(v_{1}(x), \ldots, v_{K}(x)\right)$ will be called the Nash value.

Remarks on subgames
Let $\mathcal{S} \subset \overline{1, K}$ and $\vec{s}_{\mathcal{S}}=\left(s_{i_{1}}, \ldots, s_{i_{s}}\right)$ when $\mathcal{S}=\left\{i_{1}, \ldots, i_{s}\right\}$. Denote $\tau_{\mathcal{S}}=\tau\left(\vec{s}_{\mathcal{S}}\right)$ and

$$
\left.\mathcal{S}_{R_{i}\left((n, x), \vec{p}_{n+1, N}\right.}\right)=\mathbf{E}_{x}\left[g_{i}\left(X_{\tau \mathcal{S}}\right) \mid \mathcal{F}_{n}\right] .
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For $n=0$ we have ${ }^{\mathcal{S}} R_{i}(x, \vec{p})=\mathbf{E}_{x} g_{i}\left(X_{\tau \mathcal{S}}\right)$.

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\mathcal{S}_{v_{i_{k}}}(x)=\tilde{R}_{i_{k}}\left(x, \vec{s}_{\mathcal{S}}^{*}\right) \geq \tilde{R}_{i_{k}}\left(x,\left(s, s_{\mathcal{S}}^{-i_{k}^{*}}\right)\right) \text { for every } s \in \tilde{\mathbf{M}}^{N, i_{k}}, i_{k} \in \mathcal{S},
$$

where $\left(s, s_{\mathcal{S}}^{-i_{k}{ }^{*}}\right)=\left(s^{i_{1}^{*}}, \ldots, s^{i_{k-1}}{ }^{*}, s, s^{i_{k+1}{ }^{*}}, \ldots, s^{i_{s}^{*}}\right)$. The tuple $\left({ }^{{ }^{V_{i_{1}}}}(x), \ldots,{ }_{V_{i_{s}}}(x)\right)$ will be called the Nash value.

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## Value of the truncated $\mathcal{S}$ players game

On $\left\{\omega: X_{n}(\omega)=y\right\}$ let us denote

$$
v_{i_{k}}^{\mathcal{S}}(n, y)={ }^{\mathcal{S}} R_{i}\left((n, y), \vec{p}_{n+1, N}^{\star}\right)=\mathbf{E}_{x}\left[g_{i}\left(X_{\tau_{\mathcal{S}}^{\star}}\right) \mid \mathcal{F}_{n}\right] .
$$

## Equivalent fictitious game without priority

Sequential behavior of the players
Let $\mathcal{S}$ be the set of the players active in the game. At the beginning $\mathcal{S}=\overline{1, K}$. In the sequences of players decision there is the first moment $n$ at which there are players, let us say $\vec{j}=\left(i_{1}, \ldots, i_{s}\right)$, who declare to stop and accept the state of the process at this moment. Some of them, $r$ th, get the state others not. In fact, the players remaining in the game, $\mathcal{S}^{\prime}=\mathcal{S} \backslash\{r\}$, will play the game with expected payoff dependent on the strategies chosen by the remaining players and the priority scheme.

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## Dynamics of the priority

Let players $\vec{j}=\left(i_{1}, \ldots, i_{s}\right)$ ，where $i_{k} \in \mathcal{S}, k=1, \ldots, s$ ，declare the willing to accept the state of $X_{n}=x$ ．The priority of them is now proportional to $\vec{j}_{\alpha_{n}^{j}}^{j}=\frac{\alpha_{n}^{j}}{\sum_{k=i_{1}}^{i_{k}^{k}} \alpha_{n}^{k}}$ ．
There are $\mathcal{S}$ active players and players $\vec{j}$ declare to stop．At moment $n$ ， for each $\vec{j}, s=0, \ldots,|\mathcal{S}|$ ，the payoff of the player $i_{k}$ is denoted $\mathcal{S}_{R_{i k}}^{\vec{j}}=\mathcal{S}_{R_{i k}}^{\vec{j}}\left((n, x),\left({ }^{\mathcal{S}} \vec{p}_{n}^{\vec{j}}, \mathcal{S}_{\vec{p}}^{n+1, N}\right)\right)$ ．

Construction of $\vec{p}$ is going by the solution of all sequential subgames related to the initial problem. To this end the payoff matrices at state $X_{n}=x$ for any subgame of players $\mathcal{S} \subset \overline{1, K}$ should be constructed and an equilibrium for this matrices should be derived.

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At state $(n, x)$, when there are $\mathcal{S}$ player in the game, after the players decision the set of players for the next stage will change to $\mathcal{S}^{\prime}=\mathcal{S}_{g}=\mathcal{S} \backslash\{g\}$ or it remains unchanged.
In algorithm of the equilibrium construction for the subgame of $\mathcal{S}$ players at moment $n$ let us denote $\vec{j}=\left(i_{1}, \ldots, i_{s}\right)$ when $\mathbb{I}_{\left\{p_{n}^{i_{k}} \leq A_{n}^{i_{k}}\right\}}=1$, for $k=1, \ldots, s$, where $s=|\vec{j}| \leq|\mathcal{S}|$.

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In algorithm of the equilibrium construction for the subgame of $\mathcal{S}$ players at moment $n$ let us denote $\vec{j}=\left(i_{1}, \ldots, i_{s}\right)$ when $\mathbb{I}_{\left\{p_{n}^{i_{k}} \leq A_{n}^{i_{k}}\right\}}=1$, for $k=1, \ldots, s$, where $s=|\vec{j}| \leq|\mathcal{S}|$.

$$
\mathcal{S}^{\prime} R_{i}\left((n, x), \vec{j}^{\mathcal{S}^{\prime}} \vec{p}_{n+1, N}^{\star}\right)= \begin{cases}\mathbf{T}^{\mathcal{S}_{v}}(n, x) & |\vec{j}|=0, \\ g_{i}(x) & |\vec{j}|=1, i_{1}=i, \\ \vec{j} \alpha_{n}^{i} g_{i}(x) & \\ +\sum_{k=1}^{s} \vec{j}_{\alpha_{n}}^{i_{k} \mathcal{S}_{i_{k}}} R_{i}\left((n, x), \vec{p}_{n+1, N}^{\star}\right) & |\vec{j}|=s, i \in \vec{j}, \\ \sum_{k=1}^{s} \vec{j}_{\alpha_{n}}^{i_{k} \mathcal{S}_{i_{k}}} R_{i}\left((n, x), \vec{p}_{n+1, N}^{\star}\right) & |\vec{j}|=s, i \notin \vec{j} .\end{cases}
$$

Various arbitrary decisions of the players at moment $n$ followed by equilibrium behavior on $\overline{n+1, N}$ leads to payoffs ${ }^{\mathcal{S}} R_{i}\left((n, x), \vec{j}, \mathcal{S}^{\prime} \vec{p}_{n+1, N}^{\star}\right)$, where $\left|\mathcal{S} \backslash \mathcal{S}^{\prime}\right| \leq 1$ and the composition of $\mathcal{S}^{\prime}$ depends on the decisions and assignments of priority at $n$. In this way the extensive form of the multiple person stopping game is defined where priorities are included to the payoff structure. The game will be called the competitive multiple stopping without priorities and it is denote $\mathcal{G}_{\text {wp }}$.

Various arbitrary decisions of the players at moment $n$ followed by equilibrium behavior on $\overline{n+1, N}$ leads to payoffs $\mathcal{S}_{R i}\left((n, x), \vec{j}, \mathcal{S}^{\prime} \vec{p}_{n+1, N}^{\star}\right)$, where $\left|\mathcal{S} \backslash \mathcal{S}^{\prime}\right| \leq 1$ and the composition of $\mathcal{S}^{\prime}$ depends on the decisions and assignments of priority at $n$. In this way the extensive form of the multiple person stopping game is defined where priorities are included to the payoff structure. The game will be called the competitive multiple stopping without priorities and it is denote $\mathcal{G}_{\text {wp }}$.

## Theorem

There exists a Nash equilibrium $\left(\vec{p}^{{ }^{*}}, \ldots, \vec{p}^{K^{*}}\right)$ in the game $\mathcal{G}_{\text {wp }}$. The Nash value and equilibrium point can be calculated recursively.

## Correlated stopping times

## Definition

Correlated stopping strategy is a random sequence $\hat{q}=\left\{\left(q_{n}^{\vec{j}}\right)\right\}_{n \in\{\overline{1}, N}^{\vec{j}\{0,1\}^{K}} \vec{q}_{n}$ such that, for each $n$,
(i) $q_{n}^{\vec{j}}$ are adapted to $\mathcal{F}_{n}$ for $\vec{j} \in\{0,1\}^{K}$;
(ii) $\sum_{\vec{j} \in\{0,1\}^{k}} q_{n}^{\vec{j}}=1$ a.s.

The set of all such sequences will be denoted by $\hat{\mathcal{Q}}^{N}$.

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The set of all such sequences will be denoted by $\hat{\mathcal{Q}}^{N}$.

## Implementation

Let $\left\{A_{i}\right\}_{i=1}^{N}$ be a sequence of i.i.d.r.v having $U[0,1]$ and independent of the Markov process $\left(X_{n}, \mathcal{F}_{n}, \mathbf{P}_{x}\right)_{n=0}^{N}$. Denote $\overrightarrow{\mathcal{B}}_{n}$ the partition $\left(B_{n}^{j}\right)^{\vec{j} \in\{0,1\}^{K}}$ of $[0,1]$ such that $\mathrm{m}\left(B_{n}^{\vec{j}}\right)=q_{n}^{\vec{j}}$ and $(1,-i)=\left(j_{1}, \ldots, j_{i-1}, 1, j_{i+1}, \ldots, K\right)$. Correlated stopping times $\left(\lambda^{i}(\hat{q})\right)_{i=1}^{K}$,

$$
\begin{equation*}
\lambda^{i}(\hat{q})=\inf \left\{0 \leq n \leq N: A_{n} \in B_{n}^{(1,-i)}\right\}, \tag{3}
\end{equation*}
$$

are Markov times with respect to the $\sigma$-fields $\mathcal{H}_{n}=\sigma\left\{\mathcal{F}_{n},\left(A_{i}\right)_{i=1}^{n}\right\}$.

## How the correlated strategy works

## The player which does not follow the correlation profile

If Player $i$ departs from the correlation profile $\hat{q}$, then the strategy of the other player is based on the marginal correlated profile $\hat{q}^{-i}$ and the strategy of Player $i$ is defined by $\hat{p}=\left(p_{1}, p_{2}, \ldots, p_{N}\right)$.

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## Effective stopping

Let $\tau^{i}(\hat{p})=\tau^{i}\left(\hat{p}_{i}\right)=\inf \left\{0 \leq n \leq N: A_{n}^{\prime} \leq p_{n}\right\}$, where $\left(A_{n}^{\prime}\right)_{n=1}^{N}$ is a sequence of i.i.d. r.v., $A_{n}^{\prime} \sim \mathrm{U}([0,1])$, independent of $\left(A_{n}\right)_{n=1}^{N}$ and the Markov process $\left(X_{n}, \mathcal{F}_{n}, \mathbf{P}_{x}\right)_{n=0}^{N}$. Let $\lambda(\hat{q})=\lambda^{1}(\hat{q}) \wedge \ldots \wedge \lambda^{K}(\hat{q})$.

- $\bar{G}_{i}(\hat{q})=G_{i}\left(\lambda(\hat{q}), X_{\lambda(\hat{q})}\right)$
- $\bar{G}_{i}\left(\left(\hat{p}_{i}, \hat{q}^{-i}\right)\right)=G_{i}\left(\tau^{i}\left(\hat{p}_{i}\right) \wedge \lambda\left(\hat{q}^{-i}\right), X_{\tau^{i}\left(\hat{p}_{i}\right) \wedge \lambda\left(\hat{q}^{-i}\right)}\right)$
- $\hat{G}_{i}(x, \hat{q})=\mathbf{E}_{x} \bar{G}_{i}(\hat{q})$ and $\hat{G}_{i}\left(x,\left(\hat{p}_{i}, \hat{q}^{-i}\right)\right)=\mathbf{E}_{x} \bar{G}_{i}\left(\left(\hat{p}_{i}, \hat{q}^{-i}\right)\right)$.


## How the correlated strategies work

## Stopping times

Let $q_{n}^{i_{s}}=\sum_{\left(i_{s},-s\right) \in\{0,1\}^{\kappa}} q_{n}^{\vec{j}}$ and define $\vec{q}_{n}^{\vec{j}}=\prod_{s=1}^{K} q_{n}^{i_{s}} \mathbb{I}_{\left\{\vec{j}=\left(i_{i}, \ldots, i_{k}\right)\right\}}$.

- Let $(\widehat{\bar{q}})=\left(\bar{q}_{n}\right)_{n=0}^{N}$ then $\lambda^{i}(\widehat{\bar{q}}), i=1, \ldots, K$ are independent random variables.
- If $q_{n}^{i} \in\{0,1\}$ then $\lambda^{i}(\hat{q}), i=1, \ldots, K$ are pure stopping times.


## How the correlated strategies work

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## Definition

A correlated equilibrium point is $\hat{q}^{*} \in \hat{\mathcal{Q}}^{N}$ if

$$
\begin{equation*}
\hat{G}_{i}\left(x, \hat{q}^{*}\right) \geq \hat{G}_{i}\left(x,\left(\hat{p}_{i}, \hat{q}^{\star-i}\right)\right) \text { for every } x \in \mathbb{E}, \hat{p}_{i} \text { and } i=1,2, \ldots, K . \tag{4}
\end{equation*}
$$

## An algorithmic calculation

## Expected payoffs

Correlated stopping strategy can be presented as $\hat{q}$ and $\hat{\mathcal{Q}}_{n}^{N}=\left\{\hat{q} \in \hat{\mathcal{Q}}^{N}: \lambda(\hat{q}) \geq n\right\}$. For $\hat{q}^{(n)} \in \hat{\mathcal{Q}}_{n}^{N}$ define

$$
\begin{align*}
u_{s}^{(1,-s)}\left(n, x, \hat{q}^{(n+1)}\right) & =\mathbf{E}_{X_{n}} \bar{G}_{s}\left(\hat{q}^{(n)}\right) \mathbb{I}_{\left\{\lambda^{s}\left(\hat{q}^{(n)}\right)=n\right\}} \mathbb{I}_{\left\{X_{n}=x\right\}}  \tag{5}\\
u_{s}^{(0,-s)}\left(n, x, \hat{q}^{(n+1)}\right) & =\mathbf{E}_{X_{n}} \bar{G}_{s}\left(\hat{q}^{(n)}\right) \mathbb{I}_{\left\{\lambda^{s}(\hat{q})>n\right\}} \mathbb{I}_{\left\{X_{n}=x\right\}} .
\end{align*}
$$

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u_{s}^{(0,-s)}\left(n, x, \hat{q}^{(n+1)}\right) & =\mathbf{E}_{X_{n}} \bar{G}_{s}\left(\hat{q}^{(n)}\right) \mathbb{I}_{\left\{\lambda^{s}(\hat{q})>n\right\}} \mathbb{I}_{\left\{X_{n}=x\right\}} .
\end{align*}
$$

## Rational behavior

At each stage $n \in\{0,1, \ldots, N\}$ for a given correlated profile, the players can observe the payoffs in the K-matrix game defined by $\left.\left(u_{i}^{\vec{j}}\right)_{i \in\{\bar{j} \in K}^{\overrightarrow{1}, K}\right\}^{K}$ defined by (5). Based on the concept of a correlated equilibrium for such K-matrix game, one can define rational behaviour at stage $n$.

## A correlated rational strategy

## Definition

$\hat{q}^{*} \in \hat{\mathcal{Q}}^{N}$ is called a correlated rational strategy of $\mathcal{G}_{m}$, if every restriction $\hat{q}^{*^{(n)}}, n=0,1, \ldots, N$, of $\hat{q}^{*}$ to $\mathcal{Q}_{n}^{N}$ fulfills for $s=1, \ldots, K$ :

$$
\begin{aligned}
\sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(1,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \pi_{n}^{*(1,-s)} \geq & \sum_{\vec{j} \in\{0,1\}} u_{s}^{(0,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \pi_{n}^{*(1,-s)}(6) \\
\sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(0,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \pi_{n}^{(0,-s)} \geq & \left.\sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(1,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right)\right)_{n}^{*(0,-s)}(7) \\
& \text { on }\left\{\omega: \lambda^{s}\left(\hat{q}^{*^{(n)}}\right)>n\right\} .
\end{aligned}
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\begin{aligned}
& \sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(1,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \pi_{n}^{*(1,-s)} \geq\left.\sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(0,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right)\right)_{n}^{*(1,-s)}(6) \\
& \sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(0,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \pi_{n}^{(0,-s)} \geq \text { on }\left\{\omega: \lambda^{s}\left(\hat{q}^{*^{(n)}}\right)=n\right\}, \\
&\left.\sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(1,-s)}\left(n, X_{n}, \hat{q}^{*(n+1)}\right)\right)_{n}^{*(0,-s)}(7) \\
& \text { on }\left\{\omega: \lambda^{s}\left(\hat{q}^{*(n)}\right)>n\right\} .
\end{aligned}
$$

## The correlated value of the game

For every $x \in \mathbb{E}$ and $\hat{q}^{*} \in \mathbb{C} \mathbb{E}$ we have the correlated value of the game $\left(\hat{v}_{s}\left(x, \hat{q}^{*}\right)\right)$, where:

$$
\hat{v}_{s}\left(x, \hat{q}^{*}\right)=\mathbf{E}_{x} G_{s}\left(\lambda\left(\hat{q}^{*}\right), X_{\lambda\left(\hat{q}^{*}\right)}\right), s=1,2, \ldots, K .
$$

## Theorem

The set of correlated equilibrium points $\mathbb{C E}$ is not empty.

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Proof.
Define $\hat{G}_{i}^{\mathcal{F}_{n}}(x, \hat{q})=\mathbf{E}_{x}\left[\bar{G}_{i}(\hat{q}) \mid \mathcal{F}_{n}\right]$. From the properties of conditional expectation

$$
\begin{equation*}
\hat{G}_{i}^{\mathcal{F}_{n}}(x, \hat{q})=\mathbf{E}_{X_{n}} \bar{G}_{i}\left(\hat{q}^{(n)}\right)=\hat{G}_{i}\left(X_{n}, \hat{q}\right) . \tag{8}
\end{equation*}
$$

For $n=N$ it is a consequence of CE for K-matrix game (4) and Definition 6. For some correlated strategy $\hat{q}^{*}$ define

$$
\overline{\pi_{n}^{\left(i_{s},-s\right)}}=\frac{\pi_{n}^{*}\left(i_{s},-s\right)}{\sum_{-s \in\{0,1\}^{k-1}} \pi^{*}{ }_{n}^{\left(i_{s},-s\right)}} .
$$

## A correlated equilibrium by a rational strategies

## Proof cd.

Let us assume that the equivalence is established for $n+1, n+2, \ldots, N$. On $\left\{\omega: \lambda^{s}\left(\hat{q}^{*^{(n)}}\right)=n\right\}$ by (8)

$$
\begin{align*}
\hat{G}_{s}\left(X_{n}, \hat{q}^{*(n)}\right) & =\sum_{\vec{j} \in\{0,1\}^{k-1}} u_{s}^{(1,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \overline{\pi_{n}^{(1,-s)}}  \tag{9}\\
\hat{G}_{s}\left(X_{n},\left(\hat{p}_{s}, \hat{q}_{-s}^{*}\right)\right) & =\sum_{\vec{j} \in\{0,1\}^{K-1}} u_{s}^{(0,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \overline{\pi_{n}^{*(1,-s)}}, \tag{10}
\end{align*}
$$

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\end{align*}
$$

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$$
\begin{align*}
\hat{G}_{s}\left(X_{n}, \hat{q}^{*^{(n)}}\right) & =\sum_{\vec{j} \in\{0,1\}^{k-1}} u_{s}^{(0,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \overline{\pi^{*}{ }_{n}^{(0,-s)}}  \tag{11}\\
\hat{G}_{s}\left(X_{n},\left(\hat{p}_{s}, \hat{q}_{-s}^{*}\right)\right) & =\sum_{\vec{j} \in\{0,1\}^{k-1}} u_{s}^{(1,-s)}\left(n, X_{n}, \hat{q}^{*^{(n+1)}}\right) \overline{\pi^{*}}{ }_{n}^{(0,-s)} \tag{12}
\end{align*}
$$

Let $\hat{q}^{*}$ fulfils (4) for $s=1, \ldots, K$, then from (9-12) condition (6) must be satisfied. As a consequence we have on $\left\{\omega: \lambda\left(\hat{q}^{*^{(n)}}\right) \leq n\right\}$ from the correlated equilibrium definition that conditions (6) and (7) are satisfied for every Player.

## Natural restriction narrowing down the set of solutions

## The concepts...

which are used here do not come from the concepts of solution to Nash's problem of cooperative bargaining, but were used by
[Greenwald and Hall(2003)] for computer learning of equilibria in Markov games. Denote $\Pi_{n}^{\bullet}=\left\{\vec{\pi}_{n}:\left(\vec{\pi}_{n}, \hat{q}_{0}^{\left(^{(n+1)}\right.}\right)=\hat{q}^{\star^{(n)}}\right\}$.

## A Stepwise Utilitarian Correlated Equilibrium

$\hat{q}_{U}^{*}=\left\{\left(\tilde{\pi}_{n}^{*}\right)\right\}_{n=0, N}^{\vec{j}\{0,1\}^{k}}$ is an equilibrium $\hat{q} \in \mathbb{C E}$ such that for every $n \leq N$, the sum of the values of the restricted game to the players is maximized, given the equilibrium values calculated for stages $k, n<k \leq N$, i.e. on $\left\{\omega: X_{n}=x\right\}$

$$
\begin{equation*}
\max _{\vec{\pi}_{n} \in \Pi_{n}^{U}} \sum_{\vec{j} \in\{0,1\}^{K}} \sum_{i=1}^{K} \pi_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{U}^{\star(n+1)}\right)=\sum_{\vec{j} \in\{0,1\}^{K}} \sum_{i=1}^{K} \pi_{n}^{* \vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{U}^{\star(n+1)}\right) . \tag{13}
\end{equation*}
$$

## Further classes of restrictions

## A Stepwise Egalitarian Correlated Equilibrium

$\hat{q}_{E}^{*}=\{(\overbrace{n}^{*}{ }_{n}^{\vec{j}})\}_{n=\bar{j}, N}^{\vec{j} \in\{0,1\}^{K}}$ is an equilibrium $\hat{q} \in \mathbb{C E}$ such that for every $n \leq N$, the minimum value is maximized, given the equilibrium values calculated for stages $k$, $n<k \leq N$, on $\left\{\omega: X_{n}=x\right\}$.

$$
\begin{equation*}
\max _{\vec{\pi}_{n} \in \Pi_{n}^{E}} \min _{i \in \overline{1, K}} \sum_{\vec{j} \in K} \pi_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{E}^{\star(n+1)}\right)=\min _{i \in 1, K} \sum_{\vec{j} \in K} *_{n}^{\vec{j}_{n}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{E}^{\star(n+1)}\right) \text { on }\left\{\omega: X_{n}=x\right\} \tag{14}
\end{equation*}
$$

## Further classes of restrictions

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$$
\begin{equation*}
\max _{\vec{\pi}_{n} \in \Pi_{n}^{E}} \min _{i \in \overline{1, K}} \sum_{\vec{j} \in K} \pi_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{E}^{\star(n+1)}\right)=\min _{i \in \overline{1, K}} \sum_{\vec{j} \in K} \pi_{n}^{* \vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{E}^{\star}{ }^{(n+1)}\right) \text { on }\left\{\omega: X_{n}=x\right\} \tag{14}
\end{equation*}
$$

## A Stepwise Republican Correlated Equilibrium

$\hat{q}_{R}^{*}=\left\{\left(\tau_{n}^{*}\right)\right\}_{n=\bar{j}}^{\vec{j} \in\{0,1\}^{K}}$ is an equilibrium $\hat{q}=\in \mathbb{C} \mathbb{E}$ such that for every $n \leq N$ the maximum value of the restricted game is maximized given the equilibrium values calculated for stages $k, n<k \leq N$ on $\left\{\omega: X_{n}=x\right\}$.

$$
\begin{equation*}
\max _{\vec{\pi}_{n} \in \Pi_{n}^{R}} \max _{i \in 1, K} \sum_{\vec{j} \in B} \pi_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{R}^{\star(n+1)}\right)=\max _{i \in 1, K} \sum_{\vec{j} \in B} \pi_{n}^{* \vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{a}_{R}^{\star^{(n+1)}}\right) . \tag{15}
\end{equation*}
$$

## A Stepwise Libertarian $i$ Correlated Equilibrium

$\hat{q}_{L}^{*}=\{(\overbrace{n}^{*})\}_{n=\overline{0, N}}^{\vec{j} \in\{0,1\}^{K}}$ is an equilibrium $\hat{q}=\in \mathbb{C} \mathbb{E}$ such that for every $n \leq N$ the value of the restricted game to Player $i$ is maximized, given the equilibrium values calculated for stages $k, n<k \leq N$ on $\left\{\omega: X_{n}=x\right\}$

$$
\begin{equation*}
\max _{\vec{\pi}_{n} \in \Pi_{n}^{L}} \sum_{\vec{j} \in K} \pi_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{R}^{\star(n+1)}\right)=\sum_{\vec{j} \in K} \hat{\pi}_{n}^{*}{ }_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{R}^{\star(n+1)}\right) . \tag{16}
\end{equation*}
$$

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$$
\begin{equation*}
\max _{\vec{\pi}_{n} \in \Pi_{n}^{L}} \sum_{\vec{j} \in K} \pi_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{R}^{\star(n+1)}\right)=\sum_{\vec{j} \in K} \stackrel{*}{j}_{n}^{\vec{j}} u_{i}^{\vec{j}}\left(n, x, \hat{q}_{R}^{\star^{(n+1)}}\right) . \tag{16}
\end{equation*}
$$

## Theorem

The set of correlated equilibrium points satisfying any one of the given criteria above is not empty.

## Remarks on equilibria selection

Using these criteria, the appropriate correlated equilibria can be defined by recursively solving a set of linear programming problems. In the case of libertarian and utilitarian equilibria, the linear objective functions are given by equation (16) and (13), respectively. The constraints are those used in Definition 6. Since, the feasible set of this linear programming problem is non-empty (a correlated rational strategy always exists), such solutions always exist.

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(1) The maximization of the value of the game to Player 1 subject to the constraints from Definition 6, together with the constraint that the value of the game to Player 1 is at least the value of the game to Player 2.
(2) The maximization of the value of the game to Player 2 subject to the constraints from Definition 6, together with the constraint that the value of the game to Player 2 is at least the value of the game to Player 1.

In order to find a correlated strategy satisfying the republican criterion, it suffices to choose an appropriate solution from the solutions to these two problems (maximizing the maximum value). The union of the feasible sets of these two linear programming problems is the set of correlated rational strategies.

## Remarks on equilibria selection

An analogical procedure using two linear programming problems can be used to find a correlated equilibrium satisfying the egalitarian criterion. When the objective function is defined by the maximization of the value of the game to Player $i$, then the additional constraint in the linear programming problem is that the value of the game to Player $i$ is not greater than the value of the game to the other Player.

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An analogical procedure using two linear programming problems can be used to find a correlated equilibrium satisfying the egalitarian criterion. When the objective function is defined by the maximization of the value of the game to Player $i$, then the additional constraint in the linear programming problem is that the value of the game to Player $i$ is not greater than the value of the game to the other Player.

## Note 1:

In order to define a correlated equilibrium satisfying one of the four given criteria, it is always possible to concentrate any non-trivial correlations on the two pairs of actions $(c, s)$ and $(s, c)$. Such correlated equilibria have the desirable property that an external judge is not required. The players can correlate their actions by firstly agreeing on the strategy to be used and then jointly observing the results of an appropriate randomization to correlate their strategies.

## Remarks on equilibria selection cont.

## Note 2:

Here, we have considered correlated equilibria in dynamic games as a sequence of correlated equilibria in appropriately defined matrix games. One may treat such a sequence of correlated equilibrium as a single, global correlated equilibrium. One can randomize over various global correlated equilibria, hence obtaining another form of correlated equilibrium. E.g., at the start of a game it is possible to choose, based on toss of a coin, the Libertarian 1 or Libertarian 2 equilibrium. In the case of symmetric games of this form ( $\alpha=0.5$ ), it can be seen that the resulting correlation has the very desirable properties of being both utilitarian and egalitarian.

Competitive Staff Selection Problem

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## Description of the competitive staff selection

[Baston and Garnaev(2005)] have been proposed the following model of the staff selection competition in the case of two departments. The heads of the two departments together interview the applicants in turn and make their decisions on one applicant before interviewing any others. If a candidate is rejected by both departmental heads, the candidate cannot be considered for either post at a later date. When both heads decide to make an offer, they consider the following possibilities.
(1) The departments are equally attractive, so that an applicant has no preference between them;
(2) One department can offer better prospects to applicants, who will always choose that department.

Correlated equilibria: further research

## Correlated Equilibria in Competitive Staff Selection Problem

## Modeling the staff selection process

## Modeling the staff selection process

## DM point of view

There are precisely $N$ applicants and that each applicant has "a level of expertise" which is random. The interview process enables the directors to observe these levels of expertise, which form a sequence of i.i.d random variables from a continuous distribution. The aim of each DM is to choose the applicant with the highest expected level of expertise. If no appointment is made to a department from these $N$ applicants, then the department will suffer from a shortfall of expertise. Game 2 has one Nash equilibrium, which can be used as the solution to the problem. Game 1 has many Nash equilibria. This raises the question of equilibrium selection.

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## Choosing the model

[Baston and $\operatorname{Garnaev}(2005)$ ] interpreted such a variety of Nash equilibria solutions as a way of modeling different dynamics within the organization, which can result in various outcomes during the conscription process. If one departmental head is aggressive and one passive, we might expect a different outcome to the one in which both are of a similar temperament. When both have a similar temperament one expects a symmetric strategy and value, but when they have different temperaments one should expect an asymmetric equilibrium and value. The different character of heads is modelled by the notion of a Stackleberg leader. Also, the difference in the level of complication of equilibria might also be an argument justifying this approach to equilibrium selection. It is shown that these non-symmetric equilibria have the advantage that the players use pure strategies, whereas at the symmetric equilibrium, the players are called upon to employ specific actions with complicated probabilities.

## General remark on staff selection problem

## Staff selection vs. BCP

This staff selection problem is closely related to the best choice problem (BCP). There are some potential real applications of decision theory which strengthen the motivation of the BCP (the one decision maker problem). One group of such problems are models of many important business decisions, such as choosing a venture partner, adopting technological innovation, or hiring an employee using a sequential decision framework (see [Stein et al.(2003)Stein, Seale, and Rapoport], [Chun(2000)]). Others are an experimental investigations of the "secretary problem", which compare the optimal policy from the mathematical model with behaviour of human beings (see [Seale and Rapoport(1997)]). We have not found any such investigation for BCP games. It could be that the theoretical results are not complete enough to start applied and experimental research.

## General remark on staff selection problem

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## On BCP

In spite of the long history of BCP and its generalisations presented in review papers by [Ferguson(1989)] and [Samuels(1991)], there are also competitive versions, on which researchers' attention has been focused (see [Sakaguchi(1995)] for review papers). The concept of equal priority of the players in the selection process in a model of a non-zero-sum game related to BCP was introduced by [Fushimi(1981)].
[Szajowski(1994)] extended this model to permit random priority. [Ramsey and Szajowski(2001a), Ramsey and Szajowski(2005)] considered a mathematical model of competitive selection with random priority and random acceptance of the offer (uncertain employment) by candidates.

## Games with no candidate preferences

## One candidate case

If an applicant with expertise $\xi_{i}=x$ is chosen, the department gains $x$. The candidates have i.i.d. expertise $\xi_{i}$ with uniform distribution on $[0,1]$ by assumption. The cost of not selecting an applicant is $c$.
If there is only one candidate, then the selection process will end with value $d=\frac{1}{2} \mathbf{E} \xi_{1}-\frac{1}{2} c=\frac{1-2 c}{4}$ to both players (both want to select and the probability of winning is $\frac{1}{2}$ for both of them). Denote $b=\max \left\{0, \frac{1-2 c}{4}\right\}$.

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## Correlated equilibria of the two stage game

When there are two candidates, then we have a two stage game. The payoff bimatrix $M_{2}(x)$ is of the form (see [Baston and Garnaev(2005)]):

$$
M_{2}(x)=\begin{array}{cc}
s \\
f
\end{array}\left(\begin{array}{cc}
\mathrm{s} & \mathrm{f}  \tag{17}\\
\left(\frac{\left(x+\frac{1}{2}\right)}{2}, \frac{\left(x+\frac{1}{2}\right)}{2}\right) & \left(x, \frac{1}{2}\right) \\
\left(\frac{1}{2}, x\right)^{2} & (d, d)
\end{array}\right)
$$

The game has one pure Nash equlilibrium, $(s, s)$, for $x \geq \frac{1}{2}$ and $(f, f)$ for $x \leq b$. For $x \in\left[b, \frac{1}{2}\right]$ there are two asymmetric pure Nash equilibria and one symmetric Nash equilibrium in mixed strategies. Without extra assumptions it is not clear which equilibrium should be played.

## Polytope of Nash equilibria

## Vertices of the correlated equilibrium polytope

We will use an extensive communication device to construct correlated equilibria. Usually the set of correlated equilibria contain the convex hull of Nash equilibria. Natural selection criteria can be proposed. The possibility of preplay communication and use of an arbitrator solve the solution selection problem. The players just specify the criterion. Such criteria are formulated in (13)-(16). The set of solutions which fulfil one of these points are not empty.
For $M_{2}(x)$, when $x \in\left[b, \frac{1}{2}\right]$ the set of correlated equilibria is a polytope with five vertices (see [Peeters and Potters(1999)]). Let us denote $\alpha=\frac{1}{2} \frac{x-\frac{1}{2}}{d-x}, \gamma=2 \frac{d-x}{x-\frac{1}{2}}$ and $\mu=\left(\mu_{s s}, \mu_{f f}, \mu_{f s}, \mu_{s f}\right)$. Table 1 shows the polytope of correlated equilibria for the considered game.

| $\mu$ | $\mu_{s s}$ | $\mu_{f f}$ | $\mu_{f s}$ | $\mu_{s f}$ |
| :--- | :---: | :---: | :---: | :---: |
| $\mu_{C}^{*}(\alpha, \gamma)$ | 0 | 0 | 1 | 0 |
| $\mu_{D}^{*}(\alpha, \gamma)$ | 0 | 0 | 0 | 1 |
| $\mu_{E}^{*}(\alpha, \gamma)$ | $\frac{\gamma}{1+\gamma+\alpha \gamma}$ | 0 | $\frac{1}{1+\gamma+\alpha \gamma}$ | $\frac{\alpha \gamma}{1+\gamma+\alpha \gamma}$ |
| $\mu_{F}^{*}(\alpha, \gamma)$ | 0 | $\frac{\alpha}{1+\alpha+\alpha \gamma}$ | $\frac{1}{1+\alpha+\alpha \gamma}$ | $\frac{\alpha \gamma}{1+\alpha+\alpha \gamma}$ |
| $\mu_{G}^{*}(\alpha, \gamma)$ | $\frac{\gamma}{(1+\alpha)(1+\gamma)}$ | $\frac{\alpha}{(1+\alpha)(1+\gamma)}$ | $\frac{1}{(1+\alpha)(1+\gamma)}$ | $\frac{\alpha \gamma}{(1+\alpha)(1+\gamma)}$ |

Table: The five vertices of the correlated equilibrium polytope.

## Stackleberg solution

## (C)

The values of the game to the players at vertex $C$ are

$$
\begin{align*}
& v_{1}^{(C)}=\int_{0}^{b} b d x+\int_{b}^{\frac{1}{2}} \frac{1}{2} d x+\frac{1}{2} \int_{\frac{1}{2}}^{1}\left(x+\frac{1}{2}\right) d x=b^{2}-\frac{1}{2} b+\frac{9}{16}(18) \\
& v_{2}^{(C)}=\int_{0}^{b} b d x+\int_{b}^{\frac{1}{2}} x d x+\frac{1}{2} \int_{\frac{1}{2}}^{1}\left(x+\frac{1}{2}\right) d x=\frac{1}{2} b^{2}+\frac{7}{16} \tag{19}
\end{align*}
$$

When Player 1 takes the role of Stackleberg leader his expected gain is $v_{1}^{(C)}$, while the Stackleberg follower has $v_{2}^{(C)}$ (see [Baston and Garnaev(2005)]).

## Stackleberg solution

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$$
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& v_{2}^{(C)}=\int_{0}^{b} b d x+\int_{b}^{\frac{1}{2}} x d x+\frac{1}{2} \int_{\frac{1}{2}}^{1}\left(x+\frac{1}{2}\right) d x=\frac{1}{2} b^{2}+\frac{7}{16} \tag{19}
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$$

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[Baston and Garnaev(2005)]).

## (D)

The values at vertex $D$ can be obtained from those at vertex $C$, because matrix $M_{2}(x)$ is symmetric: $v_{1}^{(D)}=v_{2}^{(C)}$ and $v_{2}^{(D)}=v_{1}^{(C)}$.

## Stackleberg solution

## (E)

The expected gain of the players at correlated equilibrium $E$ given the expertise of the candidate $x \in\left[b, \frac{1}{2}\right]$ is of the form.

$$
\begin{align*}
& w_{1}^{(E)}=\left(x+\frac{1}{2}\right) \frac{x-\frac{1}{2}}{2\left(d-\frac{1}{2}\right)}+\frac{1}{2}\left(x+\frac{1}{2}\right) \frac{d-x}{d-\frac{1}{2}}=\frac{1}{2}\left(x+\frac{1}{2}\right)  \tag{20}\\
& w_{2}^{(E)}=\frac{1}{2}\left(x+\frac{1}{2}\right) . \tag{21}
\end{align*}
$$

The value of the two-stage game to the players at vertex $E$ is

$$
\begin{equation*}
v_{1}^{(E)}=v_{2}^{(E)}=\int_{0}^{b} b d x+\frac{1}{2} \int_{b}^{1}\left(x+\frac{1}{2}\right) d x=\frac{3}{4} b^{2}-\frac{1}{4} b+\frac{1}{2} \tag{22}
\end{equation*}
$$

The values at these three vertices are such that $v_{1}^{(D)}<v_{1}^{(E)}<v_{1}^{(C)}$.

## Stackleberg solution

## (F)

This correlated equilibrium is of the form: $\mu_{s s}=0$ and

$$
\mu_{f f}=\frac{x-\frac{1}{2}}{4 d-3 x-\frac{1}{2}} ; \mu_{f s}=\mu_{s f}=\frac{2(d-x)}{4 d-3 x-\frac{1}{2}}
$$

The expected gain of the players at correlated equilibrium $F$ given $x \in\left[b, \frac{1}{2}\right]$ is

$$
w_{1}^{(F)}=\frac{1}{2}\left(x+\frac{1}{2}\right)+\frac{\left(x-\frac{1}{2}\right)\left(d-\frac{x}{2}-\frac{1}{4}\right)}{4 d-3 x-\frac{1}{2}} \leq \frac{1}{2}\left(x+\frac{1}{2}\right)
$$

The value of the two-stage game to the players at vertex $F$ is

$$
v_{1}^{(F)}=v_{2}^{(F)}=v_{1}^{(E)}+\int_{b}^{\frac{1}{2}} \frac{\left(x-\frac{1}{2}\right)\left(d-\frac{x}{2}-\frac{1}{4}\right)}{4 d-3 x-\frac{1}{2}} d x<v_{1}^{(E)}
$$

(G)

This correlated equilibrium (the Nash equilibrium in mixed strategies) is of the form:

$$
\mu_{s s}=\frac{4(d-x)^{2}}{\left(2 d-x-\frac{1}{2}\right)^{2}} ; \quad \mu_{f f}=\frac{\left(x-\frac{1}{2}\right)^{2}}{\left(2 d-x-\frac{1}{2}\right)^{2}} ; \mu_{f s}=\mu_{s f}=\frac{2(d-x)\left(x-\frac{1}{2}\right)}{\left(2 d-x-\frac{1}{2}\right)^{2}}
$$

The expected gain of the players at correlated equilibrium $G$ given $x \in\left[b, \frac{1}{2}\right]$ is

$$
w_{1}^{(G)}=w_{2}^{(G)}=\frac{1}{2}\left(x+\frac{1}{2}\right)+\frac{\left(x-\frac{1}{2}\right)^{2}\left[d-\frac{1}{2}\left(x+\frac{1}{2}\right)\right]}{\left(2 d-x-\frac{1}{2}\right)^{2}} \leq \frac{1}{2}\left(x+\frac{1}{2}\right) .
$$

The value of the two-stage game to the players at vertex $G$ is

$$
v_{1}^{(G)}=v_{2}^{(G)}=v_{1}^{(E)}+\int_{b}^{\frac{1}{2}} \frac{\left(x-\frac{1}{2}\right)^{2}\left[d-\frac{1}{2}\left(x+\frac{1}{2}\right)\right]}{\left(2 d-x-\frac{1}{2}\right)^{2}} d x<v_{1}^{(E)}
$$

## The selection of equilibria process

## The republican or eqalitarian

To apply the selection criteria on the set of correlated equilibria of the two stage game we define a linear programming problem, in which the objective function is defined by the criterion and the feasible set is the set of vectors $\mu$ defining a correlated equilibrium. The republican or egalitarian solution are solutions of the appropriate two linear programming problems:

1) Maximise $v_{1}$ given the equilibrium constraints and the constraint $v_{1} \leq v_{2}$ when the egalitarian condition is used or $v_{1} \geq v_{2}$ when the republican condition is used.
2) Maximise $v_{2}$ given the equilibrium constraints and the constraint $v_{2} \leq v_{1}$ when the egalitarian condition is used or $v_{2} \geq v_{1}$ when the republican condition is used.

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2) Maximise $v_{2}$ given the equilibrium constraints and the constraint $v_{2} \leq v_{1}$ when the egalitarian condition is used or $v_{2} \geq v_{1}$ when the republican condition is used.

From the symmetry of the game the hyperplane $\mu_{f s}-\mu_{s f}=0$ splits the set of correlated equilibria into the two feasible sets for these problems and $\mu=\left(0,0, \frac{1}{2}, \frac{1}{2}\right)$, the vertex $H$, becomes a vertex of the feasible set in each of the problems. This vertex replaces vertex $C$ or vertex $D$ depending on the additional constraint.

$$
v_{1}^{(H)}=v_{2}^{(H)}=\frac{v_{1}^{(C)}+v_{1}^{(D)}}{2}=v_{1}^{(E)}
$$

## The selection of equilibria process

## Libertarian equilibria

From the above calculation it follows that the maximal game value for the first player is guaranteed at vertex $(f, s)$ and for the second player at $(s, f)$. It means that $\delta_{L 1}^{\star}=(f, s)=C$ is the libertarian 1 and $\delta_{L 2}^{\star}=(s, f)=D$ is the libertarian 2 correlated equilibrium. In relation to the solutions presented by Baston and Garnaev, the libertarian $i$ equilibrium corresponds to the Stackleberg solution at which Player $i$ takes the role of the Stackleberg leader.

## Republican equilibria

Let us denote $V^{\delta}=\max _{i \in\{1,2\}} v_{i}^{\delta}$. Similar consideration of the vertices as made in the case of egalitarian equilibria leads to conclusion that the republican equilibria are $\delta_{R}^{\star} \in\{C, D\}$ and $V^{\delta_{R}^{\star}}=v_{1}^{(C)}=v_{2}^{(D)}$.

## The selection of equilibria process

## Egalitarian equilibria

Let us denote $v^{\delta}=\min _{i \in\{1,2\}} v_{i}^{\delta}$. We are looking for $\delta_{E}^{\star}$ such that $v^{\delta}{ }_{E}^{\star}=\max _{\delta} v^{\delta}$. For $\delta \in\{E, F, G, H\}$ we have $v_{1}^{\delta}=v_{2}^{\delta}, v_{1}^{(F)}<v_{1}^{(E)}=v_{1}^{(H)}$ and $v_{1}^{(G)}<v_{1}^{(E)}$. For $\delta \in\{C, D\}$ the minimal values are $v^{(C)}=v_{2}^{(C)}$ and $v^{(D)}=v_{1}^{(D)}$. Moreover, $v_{2}^{(C)}=v_{1}^{(D)}<v_{1}^{(E)}$. Therefore $E$ and $H$ define egalitarian equilibria and $v^{\delta_{E}^{\star}}=v_{1}^{(E)}$. It follows that any linear combination of these equilibria $p E+(1-p) H$, where $p \in[0,1]$ defines an egalitarian equilibrium. It should be noted that $H$ is an intuitively pleasing solution, since it corresponds to a solution in which the players observe the toss of a coin and if heads appears Player 1 acts as the Stackleberg leader, otherwise Player 2 plays this role. This is one of the solutions considered by Baston and Garnaev. At any of the other solutions the arbitrator must send signals to each of the players separately in order to obtain the appropriate correlation. It should be noted that the value of the game to the players is independent of the egalitarian solution adopted.

## Multi-stage game

## Utilitarian equilibria

Let us denote $v_{+}^{\delta}=v_{1}^{\delta}+v_{2}^{\delta}$. We have $v_{+}^{(C)}=v_{+}^{(D)}=\frac{3}{2} b^{2}-\frac{b}{2}+1=2 v_{1}^{(E)}$. Since $2 b \leq x+\frac{1}{2}$, it follows that $v_{+}^{(C)}>v_{+}^{(F)}$ and $v_{+}^{(C)}>v_{+}^{(G)}$. Hence, $C, D$ and $E$ are utilitarian equilibria. It follows that any linear combination $p C+q D+r E$ ( $p, q, r \geq 0, p+q+r=1$ ) defines a utilitarian equilibrium.
$v_{+}^{\delta_{U}^{\star}}=v_{+}^{(C)}=v_{+}^{(D)}=v_{+}^{(E)}$. It should be noted that $H$ is a linear comibination of these three vertices with $p=q=\frac{1}{2}, r=0$. Also, the value of the game to the players is dependent on the utilitarian equilibrium played.

## Multi-stage game

## Construction of equilibria

We define correlated equilibria by recursion as a series of correlated equilibria in the appropriately defined matrix games. The correlated strategy used when both players are deciding whether to accept or reject the $n$-th last candidate is given by $\mu_{n}=\left(\mu_{n, s s}, \mu_{n, f f}, \mu_{n, f s}, \mu_{n, s f}\right)$. The game played on observing the $n$-th last candidate is given by

$$
M_{n}(x)=\begin{gathered}
\mathrm{s} \\
f
\end{gathered}\left(\begin{array}{cc}
\left(\frac{x+u_{n-1}}{2}, \frac{x+u_{n-1}}{2}\right) & \mathrm{f} \\
\left(u_{n-1}, x\right) & \left(x, u_{n-1}\right) \\
\left(v_{n-1}^{\pi}, w_{n-1}^{\pi}\right)
\end{array}\right),
$$

where $u_{n}$ is the optimal expected reward of a lone searcher with $n$ candidates remaining (see [Baston and Garnaev(2005)]) and $v_{n}^{\pi}, w_{n}^{\pi}$ are the values of the $n$-stage game to Players 1 and 2, respectively, when the equilibrium $\pi$ is played. From the form of the payoff matrix it can be seen that $(s, s)$ is the unique Nash equilibrium when $x>u_{n-1}$. Similarly, $(f, f)$ is the unique Nash equilibrium when $x<\min \left\{v_{n-1}^{\pi}, w_{n-1}^{\pi}\right\}$.

## Comparison of equilibria

## Libertarian equilibria

First we consider $N=3$. From the calculations made for $N=2$, it follows that $v_{2}^{L 1}>w_{2}^{L 1}$. Considering the payoff matrix $(f, s)$ is the unique Nash equilibrium for $v_{2}^{L 1}<x<w_{2}^{L 1}$ and both $(f, s)$ and $(s, f)$ are pure Nash equilibrium for $v_{2}^{L 1}<x<u_{2}$. In this interval there is also an equilibrium in mixed strategies. We only need to consider equilibrium selection for $\nu_{2}^{L 1}<x<u_{2}$. Since the payoff matrix is now longer symmetric, the vertices of the polytope defining the set of correlated equilibrium are of a different form. Since $(f, s)$ is a Nash equilibrium, $\mu_{3}=(0,0,1,0)$ is a vertex of this polytope. For $v_{2}^{L_{1}^{1}}<x<u_{2}$, it can be seen that $u_{2}$ is the maximal payoff in the payoff matrix. It follows that $\mu_{3}$ is the vertex that strictly maximises the expected payoff of Player 1 and thus uniquely defines the libertarian 1 equilibrium. It follows that $v_{3}^{L 1}>w_{3}^{L 1}$ and hence $M_{4}(x)$ is of a similar form to $M_{3}(x)$.

## Comparison of equilibria cont.

## Egalitarian equilibrium

It will be shown by induction that for $N \geq 3$ an egalitarian equilibrium is of the same form as for $N=2$. Suppose that $v_{n-1}^{E}=w_{n-1}^{E}$. The coordinates of the vertices of the polytope describing the set of correlated equilibria is of the form given in Table 1 with $\alpha=\frac{u_{n-1}-x}{2\left(x-v_{n-1}\right)}$ and $\gamma=\frac{2\left(x-v_{n-1}\right)}{u_{n-1}-x}$. Considering the values of the game at these vertices when $x \in\left[v_{n-1}, u_{n-1}\right]$, the egalitarian criterion is satisfied at vertices $E$ and $H$. It follows that $v_{n}^{E}=w_{n}^{E}$ and any linear combination of $E$ and $H$ defines an egalitarian equilibrium. Since $v_{2}^{E}=v_{2}^{E}$ it follows by induction that an egalitarian equilibrium is of the required form.

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## Republican equilibria

Suppose libertarian 1 is taken to be the republican equilibrium for the last 2 stages. For $N=3$ the calculations are similar to the calculations made for the libertarian 1 equilibrium. It can be shown that the libertarian 1 equilibrium again maximises the maximum value. Using an iterative argument, it can be shown that the libertarian 1 equilibrium is a republican equilibrium. By the symmetry of the game it follows that the libertarian 2 equilibrium is also a republican equilibrium.

## Comparison of equilibria cont.

## Utilitarian equilibria

Unfortunately, the value function of a utilitarian equilibrium for $N=2$ is not uniquely defined. In order to find a "globally optimal" utilitarian equilibrium, we cannot use simple recursion. From the form of the payoff matrix it can be seen that when $\max \left\{v_{n-1}, w_{n-1}\right\}<x<u_{n-1}$ the maximum sum of payoffs is $x+u_{n-1}$. This is obtained when at least one of the players accepts the candidate. Such a payoff is attainable at a correlated equilibrium, since $(f, s)$ and $(s, f)$ are correlated equilibrium. It follows from the definition of a utilitarian equilibrium that $\mu_{n, f f}=0$ when $\max \left\{v_{n-1}, w_{n-1}\right\}<x<u_{n-1}$.

Correlated equilibria: further research

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Correlated equilibria: further research

## Theorem

The libertarian equilibria are the only globally optimal utilitarian equilibria for $N \geq 3$ (ignoring strategies whose actions differ from those defined by one of these strategies on a set with probability measure zero).

