# A few Remarks on "Fixed-Width Output Analysis for Markov Chain Monte Carlo" by Jones et al.

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**Abstract**

The aim of this note is to relax assumptions and simplify proofs in results given by Jones et al. in the recent paper "Fixed-Width Output Analysis for Markov Chain Monte Carlo."

KEY WORDS: Markov chain, regeneration, geometric ergodicity, batch means, regenerative simulation.

In the sequel we refer to the setting and notation introduced in [5] where the following lemma is stated and used repeatedly.

**Lemma 1** (Lemma 1 of [5]). *Let $X$ be a Harris ergodic Markov chain on $\mathsf{X}$ with invariant distribution $\pi$ and suppose that $g : \mathsf{X} \to \mathbb{R}$ is a Borel function.*

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*Assume $X$ is geometrically ergodic and the minorization condition holds, i.e. there exists a function $s : \mathsf{X} \to [0,1]$, for which $E_\pi s > 0$ and a probability measure $Q$ such that*

$$P(x, A) \geq s(x)Q(A) \qquad \text{for all } x \in \mathsf{X} \text{ and } A \in \mathcal{B}(\mathsf{X}). \tag{1}$$

*Then for every integer $p \geq 1$,*

*a. If $E_\pi|g|^{2^{(p-1)}+\delta} < \infty$ for some $0 < \delta < 1$ then $E_Q N_1^p < \infty$ and $E_Q S_1^p < \infty$.*

*b. If $E_\pi|g|^{2^p+\delta} < \infty$ for some $0 < \delta < 1$ then $E_Q N_1^p < \infty$ and $E_Q S_1^{p+\delta} < \infty$,*

*where $N_r = \tau_r - \tau_{r-1}$, $S_r = \sum_{i=\tau_{r-1}}^{\tau_r - 1} g(X_i)$, and $0 = \tau_0 < \tau_1 < \dots$ are the regenerations times of the chain (see Section 2.1 of [5] for detailed definitions).*

The lemma generalizes the main theoretical result of [4] and is also of independent interest. However, the following stronger result holds true.

**Lemma 2.** *Under the assumptions of Lemma 1, if $E_\pi|g|^{p+\delta} < \infty$ for some $p > 0$ and $\delta > 0$, then $E_Q N_1^p < \infty$ and $E_Q S_1^p < \infty$.*

*Proof.* It is enough to show that $E_\pi S_1^p < \infty$, since the remaining part of the original proof is valid under the relaxed assumption. To this end first note that

$$C := \left( \left( E_\pi |g(X_i)|^{p+\delta} \right)^{\frac{p}{p+\delta}} \right)^{1/p} < \infty. \tag{2}$$

For $p \geq 1$ we use first the triangle inequality in $L^p$, then Hölder inequality, then (2) and finally Corollary A.1 of [5].

$$
\begin{aligned}
(E_\pi S_1^p)^{1/p} &\leq \left[E_\pi \left(\sum_{i=0}^{\tau_1 - 1} |g(X_i)|\right)^p\right]^{1/p} \\
&= \left[E_\pi \left(\sum_{i=0}^{\infty} \mathbf{1}(i \leq \tau_1 - 1)|g(X_i)|\right)^p\right]^{1/p} \\
&\leq \sum_{i=0}^{\infty} \left[E_\pi \mathbf{1}(i \leq \tau_1 - 1)|g(X_i)|^p\right]^{1/p} \\
&\leq \sum_{i=0}^{\infty} \left[(E_\pi \mathbf{1}(i \leq \tau_1 - 1))^{\frac{\delta}{p+\delta}} \left(E_\pi |g(X_i)|^{p+\delta}\right)^{\frac{p}{p+\delta}}\right]^{1/p} \\
&= C \sum_{i=0}^{\infty} (Pr_\pi(\tau_1 \geq i+1))^{\frac{\delta}{p(p+\delta)}} < \infty. \qquad (3)
\end{aligned}
$$

For $0 < p < 1$ we use the fact $x^p$ is concave and then proceed similarly as in (3) to obtain

$$
\begin{aligned}
E_\pi S_1^p &\leq E_\pi \left(\sum_{i=0}^{\infty} \mathbf{1}(i \leq \tau_1 - 1)|g(X_i)|\right)^p \\
&\leq \sum_{i=0}^{\infty} E_\pi \mathbf{1}(i \leq \tau_1 - 1)|g(X_i)|^p \\
&\leq C^p \sum_{i=0}^{\infty} (Pr_\pi(\tau_1 \geq i+1))^{\frac{\delta}{(p+\delta)}} < \infty.
\end{aligned}
$$

□

*Remark.* Without additional restrictions $E_\pi|g|^p < \infty$ does not imply $E_Q S_1^p < \infty$, so Lemma 2 can not be improved. To see this note that Theorem 17.2.2 of [6] combined with the presumption that in the setting of Lemma 1 $E_\pi|g|^p < \infty$ implies $E_Q S_1^p < \infty$ yields the Central Limit Theorem for normalized sums of $g(X_i)$ for geometrically ergodic Markov chains assuming only $E_\pi g^2 < \infty$. This however is not enough for the CLT, Bradley in [2] and also Häggström in [3] provide counterexamples. Hence to obtain the implication $E_\pi|g|^p <$

$\infty \Rightarrow E_Q S_1^p < \infty$, one needs stronger assumptions, e.g. uniform ergodicity is enough, as proved in [1].

Lemma 2 allows us to restate results from section 3.2 of [5] with relaxed assumptions. In particular in Lemma 2 and in Proposition 3 therein it is enough to assume $E_\pi|g|^{2+\delta+\varepsilon} < \infty$ for some $\delta > 0$ and some $\varepsilon > 0$, instead of $E_\pi|g|^{4+\delta} < \infty$ for some $\delta > 0$. Modifications of the proofs in [5] are straightforward. Hence we have

**Lemma 3** (Part b of Lemma 2 of [5])**.** *Let $X$ be a Harris ergodic Markov chain with invariant distribution $\pi$. If $X$ is geometrically ergodic, (1) holds and $E_\pi|g|^{2+\delta+\varepsilon} < \infty$ for some $\delta > 0$ and some $\varepsilon > 0$, then there exists a constant $0 < \sigma_g < \infty$, and a sufficiently large probability space such that*

$$\left|\sum_{i=1}^{n} g(X_i) - nE_\pi g - \sigma_g B(n)\right| = O(\gamma(n))$$

*with probability 1 as $n \to \infty$, where $\gamma(n) = n^\alpha \log n$, $\alpha = 1/(2+\delta)$, and $B = \{B(t), t \geq 0\}$ denotes a standard Brownian motion.*

**Proposition 4** (Proposition 3 of [5])**.** *Let $X$ be a Harris ergodic Markov chain with invariant distribution $\pi$. Further, suppose $X$ is geometrically ergodic, (1) holds and $E_\pi|g|^{2+\delta+\varepsilon} < \infty$ for some $\delta > 0$ and some $\varepsilon > 0$. If*

1. *$a_n \to \infty$, as $n \to \infty$,*
2. *$b_n \to \infty$ and $b_n/n \to 0$ as $n \to \infty$,*
3. *$b_n^{-1} n^{2\alpha}[\log n]^3 \to 0$ as $n \to \infty$, where $\alpha = 1/(2+\delta)$,*
4. *there exists a constant $c \geq 1$, such that $\sum_{n=1}^{\infty}(b_n/n)^c < \infty$,*

*Then $\hat{\sigma}^2_{BM} \to \sigma^2_g$ w.p.1 as $n \to \infty$.*

*Concluding Remark.* Compare the foregoing result with Proposition 1 of [5] to see that both methods described by Jones et al., i.e. regenerative simulation (RS) and batch means (CBM), provide strongly consistent estimators of $\sigma_g^2$ under the same assumption for the target function $g$.

## References


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