(Monetary) Measures of Risk

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Qn: What is risk?

- Risk is about uncertainty;
- but also about loss or gain.

Example

Consider

Investment A *Invest \$ 1000. One month later receive \$1005;*

Investment B *Invest \$1000. One month later receive \$1011 with probability* 0.5 *and \$1000 with probability* 0.5

Which investment is riskier?
Most respondents would say Investment B.

Example

Now consider

Investment A *Invest \$ 1000. One month later receive \$1005;*

Investment C *Invest \$1000. One month later receive \$1005 with probability* 0.5 *and \$1016 with probability* 0.5

Now which investment is riskier? According to the actuaries and the trustees of the Universities Superannuation Scheme (the defined benefit pension scheme for British academics; assets £42 bn), the answer is Investment C (he exaggerated slightly).

Why?
Because they use variance as their measure of risk!

Recall: $\mathbb{E}X$ is the expectation of a random variable X: the "average". The variance of X, $Var(X)$, is $\mathbb{E}[(X - \mathbb{E}X)^2]$: the expected squared deviation from the average. Thus variance is a measure of variability, but it is *centred* around the average so it pays no attention to average levels.

Clearly we need some measure of risk which also respects the relative level or desirability of (random) outcomes.

Utility theory is an attempt to deal with ideas of risk while respecting preferences.

The idea is that "rational agents" have a utility function, u, and will seek to maximise the expected value of their utility:

$$X \text{ is preferred to } Y \text{ if and only if } \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)].$$

The von Neuman-Morgenstern Theorem (aka the Expected Utility Theorem) states that if an agent's preferences for outcomes satisfy the following four conditions:

Completeness Given two outcomes A and B, the agent prefers A, written $A\, B$, or prefers B ($B > A$) or is indifferent ($A \sim B$);

Transitivity If $A \geq B$ and $B \geq C$ then $A \geq C$. [Note $A \geq B$ means that either the agent prefers A to B or the agent is indifferent between A and B];

Independence ApB denote a lottery which gives outcome A with probability p and outcome B with probability $1 - p$. If $A \geq B$ then $ApC \geq BpC$ for any outcome C;

Continuity If $A \geq B \geq C$, then there is a value of p such that $ApC \sim B$.

It's a great theorem but there are some problems:

1. There is no quantification of risk;
2. the probabilities involved are assumed to be objective and completely determined so if you are unsure about them your uncertainty needs to be included in your utility;
3. outcomes are supposed to be "states of the world", so you need to include everything in an outcome when calculating your utility;
4. utilities are only defined up to an arbitrary scale and origin. So, for example, u and $2u + 3$ are different functions but represent the same utilities/preferences. This makes it hard to compare or combine different agents' utilities.

So let's move to exclusively monetary measures of risk. By far the most popular one-used in accounting, banking regulation and many other areas, is Value At Risk, usually denoted VaR or V@R. If X is the (random) amount you will hold then $VaR_p(X)$ is (minus) the 100pth percentile of the distribution of X i.e

$$\mathbb{P}(X \leq -VaR_p(X)) = p$$

or "the probability that your deficit will be at least $VaR_p(X)$ is p".

Basel II, the international banking accords which currently control (amongst much else) the way investment banks value and reserve for their trading book, mandated the use of the 99% VaR as the reserving requirement for the trading book and enforced backtesting of these estimates (the probabilities have to be estimated and then tested against past performance).

There's a problem. Whilst VaR is relatively conceptually simple and reasonably easy to 'backtest', it's an *incoherent* measure of risk.

A coherent (monetary) risk measure, ρ, on (discounted) final monetary values is one which has the following four properties:

Positivity If $X \leq 0$ then $\rho(X) \geq 0$ (f X is definitely a loss then it has non-negative risk).

Translation invariance If c is a constant then $\rho(X - c) = \rho(X) + c$ (a guaranteed additional loss of 5 increases the risk by 5).

Positive scaling If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$ (twice the loss, twice the risk; these two are what makes it a monetary measure)

Subadditivity If X and Y are two monetary outcomes then $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (risk pooling reduces the risk).

You can think of $\rho(X)$ as the minimal addition to the final monetary value to make it have acceptable monetary risk (to you).

Magically (to me anyway), under some mild but technical conditions we can characterise all coherent risk measures as the outcomes of "scenario analysis"

We call a collection $\mathcal{Q}$ of probability measures a collection of "scenarios".

Example

I'm about to spin a coin: I have just two probability measures, $\mathbb{P}$ *and* $\mathbb{Q}$*. Under* $\mathbb{P}$*, the coin spin is fair so* $\mathbb{P}(Head) = 0.5$*. On the other hand,* $\mathbb{Q}(Head) = 0.7$*. You bet £1 that the coin comes up Head. I pay you £c. If I want zero risk, then scenario analysis tells me to set the maximum expected loss (under the two probability measures) to zero. So I set the maximum of 0.5c-0.5 and 0.7c-0.3 to 0. So I set c=3/7. Conversely, if you bet on tails I will set* $c = 1$*.*

Formally, the characterisation is that, under mild technical conditions, ρ may be expressed by

$$\rho(X) = \max_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}_{\mathbb{P}}[-X],$$

for a suitable collection of scenarios, $\mathcal{Q}$.

- So why is VaR incoherent?
- Obviously, because it's not a coherent risk measure!
- But how does it fail?

VaR fails the risk-pooling/subadditivity test. It may be the case that $VaR_p(X+Y) > Var_p(X) + Var_p(Y)$.

Example

Suppose X is a uniform random variable on the interval [-10,90] and $Y = X + 4$ if $-10 \le X \le 86$, while $Y = X - 1000$ if $86 < X \le 90$. Take p=5% .
Obviously $VaR_{0.05}(X) = 5$ (since $\mathbb{P}(X \le -5) = 0.05$).

Similarly, $VaR_{0.05}(Y) = 5$, since $Y \le -5$ if and only if either $-10 \le X \le -9$ or $90 \ge X > 86$.

However, $\mathbb{P}(X + Y \le -10) = \mathbb{P}(X > 86) + \mathbb{P}(X \le -7) = .07$, so

$$VaR_{0.05}(X + Y) > 10.$$

This is bad! A bank could reduce its capital reserve requirements by splitting up its risks.

• So is there a *coherent* risk measure which looks quite like VaR? The answer is 'yes' and it's called *expected shortfall* (and many other things).

Given a random final amount X, the expected shortfall at level p, is the conditional expected loss given that it's at least the VaR:

$$ES_p(X) = -\mathbb{E}[X|X \leq -VaR_p(X)].$$

Clearly this is bigger than VaR, since it's the expectation given that the loss is at least the VaR.

Sticking with our previous example:
$ES_{0.05}(X) = \mathbb{E}[-X|X \leq -5] = 7.5$,

$ES_{0.05}(Y) = \mathbb{E}[-Y|Y \leq -5] = \mathbb{E}[-Y|X \leq -9 \text{ or } X > 86] = 731.5$,

and

$$ES_{0.05}(X+Y) = 668.6 < 7.5 + 731.5$$

Basel III (to be implemented from 2019) proposes to replace VaR by expected shortfall! This is quite controversial and has produced a lot of protest.

Question: how do we see that ES really is a coherent risk measure?

For the sophisticates, it goes like this: let $\mathcal{Q}$ be all the probability measures which have a density, $f_{\mathbb{Q}}$, with respect to $\mathbb{P}$ which is bounded by $1/p$. Then

$$ES_p(X) = \max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X] = \max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{P}}[-f_{\mathbb{Q}} X].$$

[1] FT article (Oct 31, 2013)
http://www.ft.com/cms/s/0/ecd4a130-421d-11e3-9d3c-00144feabdc0.html

[2] BCBS (2012). Consultative Document May 2012. Fundamental review of the trading book. Basel Committee on Banking Supervision. Basel: Bank for International Settlements.

[3] BCBS (2013). Consultative Document October 2013.Fundamental review of the trading book: A revised market risk framework.

[4] Carlo Acerbi, Dirk Tasche (2001): Expected Shortfall: a natural coherent alternative to Value at Risk

[5] Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, David Heath (1998):Coherent measures of risk

[6] Jon Danielsson, Kevin James, Marcela Valenzuela, and Ilknur Zer (2014): Model Risk of Risk Models

[7] Johanna F. Ziegel (2014): Coherence and elicitability