THE EMM CONDITIONS IN A GENERAL MODEL FOR INTEREST RATES.

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Abstract. Assuming that the forward rates f_t^u are semimartingales, we give conditions on their components for the discounted bond prices to be martingales. To achieve this we give sufficient conditions for the integrated processes $\bar{f}_t^u = \int_0^u f_t^v dv$ to be semimartingales and identify their various components. We recover the no-arbitrage conditions in well-known models in the literature, and finally, we formulate a new random field model for interest rates and give its EMM (no-arbitrage) condition.

1. Introduction

We consider a model for bond markets based on the forward rates, in which forward rates are semimartingales. Heath, Jarrow and Morton (HJM) (1990,1992) considered diffusion models for the forward rates, Shirakawa (1991) introduced jumps into the model, and more general jump diffusion models were introduced by Björk, Kabanov and Runggaldier (BKR) (1997) and Björk, Di Masi, Kabanov and Runggaldier (BDMKR) (1997). Kennedy (1994) considered a Gaussian Markov Field model for interest rates. A large part of these papers devoted attention to finding no-arbitrage conditions in the models introduced. The semimartingale model is the most general one if pricing is done by the principle of no-arbitrage, and includes all of the above models as particular cases. The concept of "no-arbitrage" or "no-free lunch" is appealing from the financial point of view, the models should not allow for "free lunch". The "no-arbitrage" concept is easy to define in simple models of finitely many assets in discrete time, and the result known as the First Fundamental Theorem of asset pricing asserts that "no-arbitrage" is equivalent to

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the existence of the Equivalent Martingale Measure (the EMM property), the measure under which the discounted security prices are martingales. It is harder to define the concept of "no-arbitrage" in more general models, for example in BKR (1997) and BDMKR (1997), this concept was defined by using measure-valued portfolios and function-valued processes, to reflect the fact that in bond markets there are infinitely many (continuum) traded securities (bonds parametrized by their maturity). In continuous time the EMM is an essential assumption of the models, see e.g. Harrison and Pliska (1981), Shiryaev (1999). Stricker (1990), Delbaen and Schachermayer (1994) showed that in the models with finitely many assets "no-free lunch with bounded risk" is equivalent to the EMM. They also showed that the EMM property implies no-arbitrage. The semimartingale model is the most general one with respect to no-arbitrage pricing, since if the security price process is not a semimartingale then the market model admits arbitrage opportunities, Schachermayer (1993); on the other hand, if the security price process is locally bounded and satisfies the no-free lunch with vanishing risk property for simple integrands, then it must be a semimartingale, Delbaen and Schachermayer (1994).

We do not go into definitions of "no-arbitrage" and its relation to the EMM (the reader can find a list of results in Shiryaev (1999)), but rather assume the EMM property and find its implications on the coefficients in the model. Often in practice and in the literature the EMM conditions are called the "no-arbitrage" conditions, we shall also use these terms interchangeably.

After giving the general result Theorem 2.3, we show how the EMM (no-arbitrage) conditions in well-known models in the literature (HJM, Kennedy Gaussian field, BDMKR) are recovered from our condition. Finally, we formulate a new random field model for interest rates and give its EMM condition.

To obtain the EMM conditions one needs to work with integrated semimartingales of forward rates $\bar{f}_t^u = \int_0^u f_t^v dv$. To this end we give conditions for these processes to be semimartingales and identify their various components. This contribution of the paper is of independent interest (in some of the literature the semimartingale property of the integrated semimartingales was simply assumed).

2. Semimartingale model for forward rates

We assume the existence of a continuum of bonds $P(t,T)$, $0 \le t \le T$, where T is a fixed horizon. We also assume the existence of a jointly measurable collection of infinitesimal forward rates (implied by the bond at t with delivery at u), $\breve{f}_t^u = \breve{f}(t,u)$, $0 \le t \le u \le T$.

Forward rates and bonds are related by the identity

$$P(t,u) = \exp\left\{ -\int_t^u \breve{f}(t,v)dv \right\}. \tag{2.1}$$

The bond market is said to have the EMM property if there exists a measure $Q \sim P$ such that for each $u \le T$ the discounted bond price process

$$S_t^u = S(t,u) = \exp\left\{ -\int_0^t \breve{f}(s,s)ds \right\} P(t,u) \tag{2.2}$$

is a Q-martingale. As usual, we have assumed the existence of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the "usual conditions".

We observe that the EMM condition can be restated as

$$S(t,u) = \mathbb{E}_Q\left[\exp\left(-\int_0^u \breve{f}(v,v)dv \right) \middle| \mathcal{F}_t \right]. \tag{2.3}$$

As Musiela (1993) points out, the "HJM" parametrisation of the forward rate curve is problematic in the sense that its domain of definition is not rectangular. Musiela deals with this problem by introducing the time to maturity as an independent variable. We suggest extending the definition of $\breve{f}$ by letting, for all t and all u,

$$f(t,u) = \breve{f}(t \wedge u, u). \tag{2.4}$$

Now (2.2) can be rewritten as

$$S_t^u = \exp\left\{ -\int_0^u f(t,v)dv \right\}. \tag{2.5}$$

It is intuitively clear that by applying the logarithmic transformation to S_t^u and then differentiating, we obtain that, under the EMM condition, f_t^u, as a process in t, must be a semimartingale. Before we embrace this model, we show that, under the EMM condition, the "discounted forward rate" process is a Q-martingale; as a consequence, we establish that f_t^u, must be a semimartingale. More precisely, let

$$\phi_t^u = \phi(t,u) = f_t^u S_t^u. \tag{2.6}$$

ϕ_t^u represents today's "value" of the infinitesimal forward rate available at t for a contract straddling u.

THEOREM 2.1. *Assume that $\phi_t^u \geq 0$, for all t and u, then the market has the EMM property if and only if there exists $Q \sim P$ such that*

(1) *for Lebesgue almost all $u \in [0,T]$, ϕ_t^u, as a process in t, is a Q-martingale;*
(2) *for all $t, u \in [0,T]$, $\int_0^u \phi(t,v)dv$ is Q-integrable.*

Proof of Theorem 2.1. First we observe that $S(t,u) = \exp\left\{-\int_0^u f(t,v)dv\right\}$, as a function of u, is Lebesgue almost everywhere differentiable, with

$$\lim_{\varepsilon \to 0} \frac{S(t,u+\varepsilon) - S(t,u)}{\varepsilon} = -f(t,u)S(t,u) = -\phi(t,u) \quad (\text{Leb } a.e.)$$

which, since $S(t,0) = 1$, translates to

$$S(t,u) = 1 - \int_0^u \phi(t,v)dv.$$

Assuming (1) and (2), and using the conditional Fubini Theorem, we obtain

$$\begin{aligned} \mathbb{E}_Q\left[S(t,u)|\mathcal{F}_s\right] &= \mathbb{E}_Q\left[1 - \int_0^u \phi(t,v)dv \middle| \mathcal{F}_s\right] \\ &= 1 - \int_0^u \mathbb{E}_Q[\phi(t,v)|\mathcal{F}_s]dv = 1 - \int_0^u \phi(s,v)dv = S(s,u). \end{aligned}$$

Conversely, if S^u is a Q-martingale, then by the conditional Fatou Lemma,

$$\begin{aligned} \phi(s,u) &= -\lim_{\varepsilon \to 0} \frac{S(s,u+\varepsilon) - S(s,u)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left\{\mathbb{E}_Q\left[S(t,u)|\mathcal{F}_s\right] - \mathbb{E}_Q\left[S(t,u+\varepsilon)|\mathcal{F}_s\right]\right\} \\ &\geq \mathbb{E}_Q\left[\lim_{\varepsilon \to 0} \frac{S(t,u) - S(t,u+\varepsilon)}{\varepsilon} \middle| \mathcal{F}_s\right] = \mathbb{E}_Q[\phi(t,u)|\mathcal{F}_s], \end{aligned}$$

and $\phi(t,u)$ is a Q-supermartingale.

On the other hand,

$$
\begin{aligned}
S(0,u) &= \mathbb{E}_Q[S(u,u)] \\
&= \mathbb{E}_Q\left[1-\int_0^u \phi(u,v)dv\right] \\
&= 1-\int_0^u \mathbb{E}_Q[\phi(u,v)]dv \\
&\geq 1-\int_0^u \phi(0,v)dv = S(0,u) \qquad (2.7)
\end{aligned}
$$

so we must have equality in (2.7), from which we conclude that

$$\mathbb{E}_Q[\phi(u,v)] = \phi(0,v), \quad (\text{Leb } a.e.)$$

which, together with the fact that $\phi(t,u)$ is a Q-supermartingale, implies that $\phi(t,u)$ is a Q-martingale. □

Remark The positivity requirement in Theorem 2.1 can clearly be relaxed to the assumption that the forward rates are, locally in t and uniformly in u, bounded below.

COROLLARY 2.1. *If the market has the EMM property then*

$$\phi_t^u = \mathbb{E}_Q\left[f(u,u)\exp\left(-\int_0^u f(v,v)dv\right)\middle|\mathcal{F}_t\right].$$

COROLLARY 2.2. *If the market has the EMM property then S_t^u, as a process in t, is a semimartingale.*

Proof of Corollary 2.2. It follows from Theorem 2.1 that, under the EMM condition, $f_t^u = \dfrac{\phi_t^u}{S_t^u}$ is a semimartingale (with respect to Q, and therefore with respect to P). □

Next we write the canonical representation under Q of the semimartingale f_t^u, as a process in t, for each $u \leq T$,

$$f^u = f_0^u + M^u + B^u + h(x) * (\mu^u - \nu^u) + (x - h(x)) * \mu^u, \qquad (2.8)$$

where M^u is a continuous local martingale, B^u is a predictable process of finite variation, $h(x)$ is a bounded function with compact support satisfying $h(x) = x$ in a neighbourhood of zero (e.g. $h(x) = xI_{|x|\le 1}$), μ^u is the jump measure of f^u, i.e.

$$\mu^u(dt,dx) = \sum_{\{s:\Delta f^u_s \neq 0\}} \varepsilon_{(s,\Delta f^u_s)}(dt,dx),$$

ν^u is the compensator of μ^u, and where we have used the notation

$$(J * \mu)_t = (J(x) * \mu)_t = \int_0^t \int_{\mathbb{R}} J(s,x)\mu(ds,dx),$$

(see Liptser and Shiryaev p. 189, Jacod and Shiryaev p. 84).

As suggested by (2.1) and (2.2), the semimartingales f^u will be integrated with respect to u. For this to occur we shall first assume that all processes arising in (2.8) are integrable in u. We shall not list the required assumptions here since in order to carry out the calculations needed in this paper, more stringent conditions will be imposed on these processes. These conditions will be stated in the results below.

Thus assuming that they exist, we introduce the following integrated processes: $\bar{f}^u_t = \int_0^u f^v_t dv$, $\bar{B}^u_t = \int_0^u B^v_t dv$, $\bar{M}^u_t = \int_0^u M^v_t dv$, $\bar{\mu}^u(dt,dx) = \int_0^u \mu^v(dt,dx)dv$ (i.e. $\bar{\mu}^u([0,t]\times A) = \int_0^u \mu^v([0,t]\times A)dv$), $\bar{\nu}^u(dt,dx) = \int_0^u \nu^v(dt,dx)dv$. As a general rule, a "bar" will signify an integration in u.

Also let $m^u(dt,dx)$ be the jump measure of $\bar{f}^u_t$, and $n^u(dt,dx)$ its compensator.

The following theorem gives sufficient conditions under which $\bar{f}^u$ is a semimartingale, as well as identifies its various components.

THEOREM 2.2. *Assume that*

(M) $\int_0^T \langle M^u, M^u\rangle_T \, du < +\infty$,

(FV) $\int_0^T A^u_T du < +\infty$, *where* A^u *denotes the total variation process of* B^u,

(J1) $\int_0^T \left(\int_0^t \int_{\mathbb{R}} h(x)^2 \nu^u(ds,dx)\right) du$ *is locally integrable,*

(J2) $\int_0^T \left(\int_0^T \int_{\mathbb{R}} |x - h(x)|\mu^u(dt,dx)\right) du < +\infty$.

Then

$$\bar{f}^u = \bar{f}^u_0 + \bar{M}^u + \bar{B}^u + \int_0^u h * (\mu^v - \nu^v)dv + (x - h(x)) * \bar{\mu}^u, \tag{2.9}$$

where $\bar{M}^u$ is a continuous locally square integrable martingale with sharp bracket

$$\left\langle \bar{M}^u, \bar{M}^u \right\rangle_t = \int_0^u \int_0^u \langle M^v, M^w \rangle_t \, dv dw,$$

$\bar{B}^u$ is predictable and of finite variation, and $\int_0^u h * (\mu^v - \nu^v) dv$ is a purely discontinuous local martingale. Furthermore, $\bar{M}^u$ is the continuous martingale part of the semimartingale $\bar{f}^u$ the jumps of which are given by

$$\Delta \bar{f}_t^u = \int_0^u \Delta f_t^v dv.$$

Proof of Theorem 2.2. Propositions 4.2 and 4.6 of the Appendix deal with the first part of the theorem. Here we show the decomposition of $\bar{f}^u$. Since $X^v = h*(\mu^v - \nu^v)$ is a purely discontinuous local martingale, its square bracket is

$$\begin{aligned}
[X^v, X^v]_t &= \sum_{s \leq t} \left[h\left(\Delta f_s^v\right) - \int_{\mathbb{R}} h(x) \nu^v(\{s\} \times dx) \right]^2 \\
&\leq 2 \sum_{s \leq t} \left[h\left(\Delta f_s^v\right)^2 + \left(\int_{\mathbb{R}} h(x) \nu^v(\{s\} \times dx) \right)^2 \right] \\
&\leq 2 \sum_{s \leq t} \left[h\left(\Delta f_s^v\right)^2 + \int_{\mathbb{R}} h(x)^2 \nu^v(\{s\} \times dx) \right] \\
&\leq 2 \left\{ \int_0^t \int_{\mathbb{R}} h(x)^2 \mu^v(ds, dx) + \int_0^t \int_{\mathbb{R}} h(x)^2 \nu^v(ds, dx) \right\},
\end{aligned}$$

where we have used the fact that $\nu^v(\{s\} \times \mathbb{R}) \leq 1$.

Furthermore the local integrability of $\int_0^u \left(\int_0^t \int_{\mathbb{R}} h(x)^2 \mu^v(dt, dx) \right) dv$ is equivalent to that of $\int_0^u \left(\int_0^t \int_{\mathbb{R}} h(x)^2 \nu^v(dt, dx) \right) dv$ and (J1) implies that of the process in t $\int_0^u [X^v, X^v]_t \, dv$. Proposition 4.7 now concludes that $\int_0^u X_t^v dv$ is a purely discontinuous local martingale with jumps given by $\int_0^u \Delta X_t^v dv$.

Finally since $\Delta B_t^v = \int_{\mathbb{R}} h(x) \nu^v(\{t\} \times dx)$, $\Delta \bar{f}_t^u = \int_0^u \int_{\mathbb{R}} h(x) \nu^v(\{t\} \times dx) dv + \int_0^u \left(h(\Delta f_t^v) - \int_{\mathbb{R}} h(x) \nu^v(\{t\} \times dx) \right) dv + \int_0^u \left(\Delta f_t^v - h(\Delta f_t^v) \right) dv = \int_0^u \Delta f_t^v dv$. $\square$

COROLLARY 2.3. *One may replace in the previous theorem (J1) by either*

(J1′) $\int_0^T \left(\int_0^T \int_{\mathbb{R}} h(x)^2 \nu^u(dt,dx)\right) du < +\infty$, *or*

(J1″) $\int_0^T \left(\int_0^T \int_{\mathbb{R}} |h(x)| \nu^u(dt,dx)\right) du$ *is locally integrable.*

Proof of Corollary 2.3. Define the sequence of stopping times

$$T_n = \inf\{t, \int_0^T \left(\int_0^t \int_{\mathbb{R}} h(x)^2 \nu^u(dt,dx)\right) du > n\}.$$

Then by monotone convergence,

$$\begin{aligned}
&\int_0^T \left(\int_0^{T_n} \int_{\mathbb{R}} h(x)^2 \nu^u(dt,dx)\right) du \\
&= \lim_{\varepsilon\downarrow 0} \int_0^T \left(\int_0^{T_n-\varepsilon} \int_{\mathbb{R}} h(x)^2 \nu^u(dt,dx)\right) du + \int_0^T \int_{\mathbb{R}} h(x)^2 \nu^u(\{T_n\} \times dx) du \\
&\leq n + T \sup_x h(x)^2
\end{aligned}$$

and (J1) follows. (J1″) clearly implies (J1) since $h(x)^2 \leq \text{const} \times |h(x)|$. □

THEOREM 2.3. *Assume (M), (FV), (J1), (J2) and*

(D) $(e^{-x} + x - 1) * n^T$ *is locally integrable.*

Then the discounted bond price process S_t^u *given in (2.2) is a Q-martingale (i.e. the bond market has the EMM property) if and only if*

$$\begin{aligned}
&\frac{1}{2}\int_0^u \int_0^u \langle M^v, M^w\rangle_t \, dvdw \\
&= \int_0^u B_t^v dv + \int_0^u ((x - h(x)) * \nu^v)_t \, dv - \left((e^{-x} + x - 1) * n^u\right)_t .
\end{aligned} \tag{2.10}$$

Before we prove the above theorem, we make the following remarks.

Remarks

(1) Condition (J1″) gives a meaning to the, otherwise undefined, processes $h*\mu^u$ and $h * \nu^u$. As a consequence, representation (2.8) simplifies to

$$f^u = f_0^u + M^u + V^u + x * \mu^u, \tag{2.11}$$

where M^u is a continuous local martingale, V^u is a *continuous* process of finite variation, and μ^u is the jump measure of f^u. The two representations then are related by

$$V_t^u = B_t^u - (h * \nu^u)_t, \tag{2.12}$$

and condition (2.10) becomes

$$\begin{aligned} &\frac{1}{2}\int_0^u \int_0^u \langle M^v, M^w\rangle_t \, dvdw \\ &= \int_0^u V_t^v dv + \int_0^u (x * \nu^v)_t \, dv - \left((e^{-x} + x - 1) * n^u\right)_t \end{aligned} \tag{2.13}$$

Furthermore $x * \bar{\mu}^u = x * \bar{m}^u$ and $x * \bar{\nu}^u = x * \bar{n}^u$ which leads to the following simplification of (2.13):

$$\frac{1}{2}\int_0^u \int_0^u \langle M^v, M^w\rangle_t \, dvdw = \int_0^u V_t^v dv - \left((e^{-x} - 1) * n^u\right)_t. \tag{2.14}$$

(2) If $\bar{f}^u$ is continuous ($n^u = 0$), then differentiating (2.10) in u gives

$$\int_0^u \langle M^u, M^v\rangle_t \, dv = B_t^u + ((x - h(x)) * \nu^u)_t. \tag{2.15}$$

(3) If the f^u are continuous, then V_t^u is simply B_t^u, and condition (2.10) becomes (by differentiating)

$$B_t^u = V_t^u = \int_0^u \langle M^u, M^v\rangle_t \, dv. \tag{2.16}$$

(4) It is worth commenting on how and why the proposed model is different from those of HJM or BDMKR. In fact to take full advantage of our model, the filtration used has to be large enough to contain infinitely many Brownian motions. Indeed, if the filtration in use reduces to being generated by only finitely many Brownian motions, then our model would be, thanks to the predictable representation property, no different to HJM or BDMKR. For example, our model allows for processes driven by a Brownian sheet rather than an n-dimensional Brownian motion. Likewise, the underlying jump process allowed by our model could be as rich (information-wise) as a Poisson sheet (or infinitely many independent Poisson processes).

(5) As Cont (1998) points out, the HJM model can be criticized for being an infinite-dimensional diffusion driven by a finite number of independent noises. The same criticism applies to the BDMKR model in that it is an infinite-dimensional jump-diffusion process driven by a finite number of

independent noises. Cont then suggests modelling the forward curves by an infinite-dimensional diffusion driven by an infinite-dimensional Brownian motion. The latter is defined by a family $\mathcal{W}_t$ of random linear functionals on a Hilbert space H (say $L^2([0,+\infty[))$ satisfying

(a) $\forall\psi \in H$, $\frac{1}{|\psi|}\mathcal{W}_t(\psi)$ is a one-dimensional Brownian motion;

(b) $\forall\psi,\psi' \in H$, if $\psi\perp\psi'$, then the Brownian motions $\mathcal{W}_t(\psi)$ and $\mathcal{W}_t(\psi')$ are independent.

We emphasize here that the Brownian sheet W introduced in Example 3.4 encompasses such an infinite-dimensional noise. One has only to set

$$\mathcal{W}_t(\psi) = \int_0^{+\infty} \psi(u)W(t,du).$$

Proof of Theorem 2.3. $\bar{f}_t^u$ can be written as

$$\bar{f}^u = \bar{f}_0^u + \bar{M}^u + \bar{B}^u + h * (\bar{\mu}^u - \bar{\nu}^u) + (x - h(x)) * \bar{\mu}^u, \tag{2.17}$$

where $\bar{M}^u$ is a continuous locally square integrable martingale, $\bar{B}^u$ is predictable and locally integrable, $h * (\bar{\mu}^u - \bar{\nu}^u)$ is a purely discontinuous local martingale, and $\bar{\nu}^u$ compensates $\bar{\mu}^u$. Note that $\bar{M}^u$ is the continuous martingale part of the semimartingale $\bar{f}^u$.

Combining (2.2) with (2.1) and (2.4), leads to

$$S_t^u = S_0^u \exp\left(-\int_0^u f(t,v)dv\right) = S_0^u \exp\big(-\bar{f}_t^u\big). \tag{2.18}$$

By Itô's formula

$$dS_t^u = -S_{t-}^u\Big(d\bar{f}_t^u - \frac{1}{2}d\left\langle \bar{M}^u, \bar{M}^u\right\rangle_t - \Big[e^{-\Delta\bar{f}_t^u} + \Delta\bar{f}_t^u - 1\Big]\Big).$$

Hence S^u is the stochastic exponential of the process

$$dX_t^u = -d\bar{f}_t^u + \frac{1}{2}d\left\langle \bar{M}^u, \bar{M}^u\right\rangle_t + d\left((e^{-x} + x - 1) * m^u\right)_t,$$

and as such is a local martingale if and only if X^u itself is a local martingale.

Using (2.17), we can see that this happens if and only if

$$d\bar{B}_t^u - \frac{1}{2}d\left\langle \bar{M}^u, \bar{M}^u\right\rangle_t + d\big((x-h(x)) * \bar{\mu}^u\big)_t - d\left((e^{-x} + x - 1) * m^u\right)_t \tag{2.19}$$

is a local martingale.

Making use of (J2) and (D), we see that, for S^u to be a local martingale, it is necessary and sufficient that the process

$$d\bar{B}^u_t - \frac{1}{2} d\left\langle \bar{M}^u, \bar{M}^u \right\rangle_t + d\big((x - h(x)) * \bar{\nu}^u\big)_t - d\big((e^{-x} + x - 1) * n^u\big)_t \tag{2.20}$$

is a local martingale. But this process is predictable and of finite variation, therefore it must be zero, and we obtain condition (2.10). $\square$

3. Applications

This section deals with existent models (Examples 4.1 to 3.3) by recovering the results concerned from our general approach. It also presents a new family of models (Example 3.6) which we call "Gaussian and Poisson random fields model". For clarity of the exposition, we shall introduce this model step by step. First, we study a basic Gaussian random field model (Example 3.4). Next, we look at the basic Poisson random field model (Example 3.5). Then combine the two for a study of the general model.

Example 3.1. (HJM model) In the HJM model it is assumed that the forward rates satisfy a diffusion-type stochastic differential equation in t for any $t \le u \le T$,

$$df(t,u) = \alpha(t,u)dt + \sigma(t,u)dW(t), \tag{3.1}$$

where $W(t)$ is the standard Brownian motion. Using convention (2.4), we extend this model to all $t, u \in [0,T]$ by setting $\alpha(t,u) = \sigma(t,u) = 0$, for $t > u$.

It is clear that (3.1) is a particular case of (2.8) with the martingale and finite-variation parts as given below and nil jump part;

$$M^u_t = \int_0^t \sigma(s,u)dW(s) \qquad B^u_t = \int_0^t \alpha(s,u)ds. \tag{3.2}$$

Now

$$\langle M^u, M^v \rangle_t = \int_0^t \sigma(s,u)\sigma(s,v)ds,$$

and our condition (M) becomes

$$\int_0^T \int_0^u \sigma(t,u)^2 dt du = \int_0^T \langle M^u, M^u \rangle_T \, du < +\infty.$$

While this is more restrictive than condition C1 of HJM (p. 80) which only requires finiteness (not integrability) of $\int_0^u \sigma(t,u)^2 dt$, it makes HJM's unpleasant condition

C3 (p. 82) redundant. Theorem 2.10 now states that a necessary and sufficient condition for the existence of an equivalent martingale measure is, for all $t \leq u$,

$$\int_0^t \alpha(s,u)ds = \int_0^u \int_0^t \sigma(s,u)\sigma(s,v)dsdv.$$

Differentiating in t, we recover the no-arbitrage condition for this model derived by Heath, Jarrow and Morton (1992):

$$\alpha(t,u) = \sigma(t,u)\int_t^u \sigma(t,v)dv. \tag{3.3}$$

Example 3.2. (Gaussian random field model, Kennedy 1994.) Consider the Gaussian random field model for forward rates

$$f(t,u) = f(0,u) + V_t^u + M_t^u. \tag{3.4}$$

Here $M_t^u = M([0,t]\times[0,u])$ where M is a Gaussian random measure with covariance function $C(s,t,u,v)$. Also, for each u, the V_t^u's are deterministic, continuous and of finite variation. Note that the M_t^u's are continuous Gaussian martingales such that $E(M_s^u M_t^v) = Cov(M_s^u, M_t^v) = C(s,t,u,v)$ which is a function of $(s\wedge t, u\wedge v)$.

In this case the $f(t,u)$ form a family of continuous semi-martingales.

Now

$$\langle M^u, M^v\rangle_t = E(M_t^u M_t^v) = C(t,t,u,v).$$

Recall that $f_t^u = f_{t\wedge u}^u$, and we recover from our condition (2.15) Kennedy's (1994) no-arbitrage condition,

$$V_t^u = \int_0^u C(t\wedge v, t\wedge v, u, v)dv. \tag{3.5}$$

Example 3.3. (Jump-diffusion BKR, BDMKR model) In this model it is assumed that the forward rates satisfy a jump-diffusion stochastic differential equation (see BDMKR assumption 5.1) in t for any $u \leq T$, $(t \leq u)$

$$df(t,u) = \alpha(t,u)dt + \sigma(t,u)dW(t) + \int_{\mathbb{R}} \delta(t,x,u)(\lambda(dt,dx) - l(dt,dx)), \tag{3.6}$$

where $W(t)$ is the standard Brownian motion, $\lambda(dt,dx)$ is a jump measure of a semimartingale, $l(dt,dx)$ is its *continuous* compensator, and the coefficients $\alpha(t,u)$,

$\sigma(t,u)$ and $\delta(t,x,u)$ are continuous in u and predictable in t, satisfying the following conditions for all $t \le u \le T$:

$$\begin{aligned}
&\int_0^u \left(\int_t^u |\alpha(s,v)| dv \right) ds < \infty \\
&\int_0^u \left(\int_t^u |\sigma(s,v)|^2 dv \right) ds < \infty \qquad (3.7)\\
&\int_0^u \int_{\mathbb{R}} \left(\int_t^u |\delta(s,x,v)|^2 dv \right) l(ds,dx) < \infty.
\end{aligned}$$

Although this model may seem similar to equation (2.8), there are fundamental differences which we underline next. First note that $\delta(t,x,u)\lambda(dt,dx)$ is not the jump measure of f_t^u; it may not even be integer-valued and therefore not a jump measure. This model (3.6) generalizes the HJM model by adding a jump component, but in both cases the processes f_t^u are driven by the same Brownian motion $W(t)$ and the dependence on u is modelled by taking the diffusion coefficient $\sigma(t,u)$ to be a function of u. Similarly, in BKR and in BDMKR the jumps of all processes f_t^u are modelled by the same underlying jump process with jump measure $\lambda(dt,dx)$ and the dependence on u is modelled by taking the jump coefficient $\delta(t,x,u)$ to be a function of u as well. Our model, on the other hand, allows the driving processes (the continuous part as well as the jump part) to depend on u.

Using convention (2.4), we can extend this model to all $t, u \in [0,T]$ by setting $\alpha(t,u) = \sigma(t,u) = \delta(t,x,u) = 0$, for $t > u$.

First we identify the various parameters appearing in our model in terms of those appearing in this model. Let ξ_t be the underlying jump process with jump measure $\lambda(dt,dx)$, then $\Delta f_t^u = 1_{(\Delta\xi_t \neq 0)}\delta(t,\Delta\xi_t,u)$, and

$$\begin{aligned}
\mu^u([0,t]\times A) &= \sum_{s\le t} 1_A(\Delta f_s^u) 1_{(\Delta f_s^u \neq 0)} \\
&= \sum_{s\le t} 1_A(\delta(s,\Delta\xi_s,u)) 1_{(\Delta\xi_s\neq 0)} = \int_0^t \int_{\mathbb{R}} 1_A(\delta(s,x,u))\lambda(ds,dx). \qquad (3.8)
\end{aligned}$$

Also $\sum_{s\le t} |\Delta f_s^u| = \int_0^t \int_{\mathbb{R}} |\delta(s,x,u)|\lambda(ds,dx)$, so that the integrability of $\delta(s,x,u)$ with respect to $\lambda(ds,dx)$, assumed by (3.6), implies that the jumps of f^u are summable and therefore that the simplified model of (2.11) applies. Now the semimartingale $f_t^u - \sum_{s\le t} \Delta f_s^u$ is continuous and therefore special. Identifying the continuous martingale parts and the continuous finite-variation parts, we find that

$$M_t^u = \int_0^t \sigma(s,u)dW(s) \text{ and } V_t^u = \int_0^t \alpha(s,u)ds - \int_0^t \int_{\mathbb{R}} \delta(s,x,u)l(ds,dx).$$

The final step in our identification procedure is the specification of the compensator ν^u. For any predictable $J(t,x) \ge 0$,

$$\begin{aligned}
\mathrm{E}\left[\int_0^t \int_{\mathbb{R}} J(s,x)\nu^u(ds,dx)\right] &= \mathrm{E}\left[\int_0^t \int_{\mathbb{R}} J(s,x)\mu^u(ds,dx)\right] \\
&= \mathrm{E}\left[\int_0^t \int_{\mathbb{R}} J(s,\delta(s,x,u))\lambda(ds,dx)\right] \\
&= \mathrm{E}\left[\int_0^t \int_{\mathbb{R}} J(s,\delta(s,x,u))l(ds,dx)\right],
\end{aligned}$$

so that

$$\nu^u([0,t]\times A) = \int_0^t \int_{\mathbb{R}} 1_A(\delta(s,x,u))l(ds,dx). \tag{3.9}$$

We will also need the jump measure of the integrated process. The fact that the processes f^u are driven by the same Brownian motion (i.e. $W(t)$), for the martingale part, the same jump process (i.e. ξ_t), for the jump part, and the same process of finite variation (i.e. t and $l(dt,dx)$), and taking into account the integrability conditions given in (3.7), we can write with the convention that a "bar" signifies an integration of that quantity with respect to u, that

$$d\bar{f}(t,u) = \bar{\alpha}(t,u)dt + \bar{\sigma}(t,u)dW(t) + \int_{\mathbb{R}} \bar{\delta}(t,x,u)(\lambda(dt,dx) - l(dt,dx)). \tag{3.10}$$

This automatically provides us with the jumps of $\bar{f}^u$,

$$\Delta\bar{f}_t^u = 1_{(\Delta\xi_t \neq 0)}\bar{\delta}(t,\Delta\xi_t,u),$$

and ultimately with the jump measure m^u,

$$m^u([0,t]\times A) = \int_0^t \int_{\mathbb{R}} 1_A(\bar{\delta}(s,x,u))\lambda(ds,dx),$$

and its compensator n^u,

$$n^u([0,t]\times A) = \int_0^t \int_{\mathbb{R}} 1_A(\bar{\delta}(s,x,u))l(ds,dx).$$

Before we are able to apply Theorem 2.3 we need to check that conditions (M), (FV), (J1) and (J2) are satisfied. Conditions (FV) and (M) are nothing but respectively the first and second parts of 3.7. Using (3.9), we rewrite the third part of (3.7) as $\int_t^u \left(\int_0^u \int_{\mathbb{R}} x^2 \nu^v(ds,dx)\right) dv < +\infty$. (J1′) immediately follows.

Thus condition (2.10) becomes

$$\frac{1}{2}\int_0^t \bar{\sigma}^2(s,u)ds = \int_0^t \bar{\alpha}(s,u)ds - \int_0^t \int_{\mathbb{R}} \left(e^{-\bar{\delta}(s,x,u)} + \bar{\delta}(s,x,u) - 1\right) l(ds,dx),$$

which is the condition obtained by BDMKR (see BDMKR equation (5.15) and note that D corresponds to $-\bar{\delta}$, and a to $\frac{1}{2}\bar{\sigma}^2 - \bar{\alpha}$). Furthermore, in the particular case where $l(dt,dx) = k_t(dx)dt$, then differentiation yields the condition

$$\frac{1}{2}\bar{\sigma}^2(t,u) = \bar{\alpha}(t,u) - \int_{\mathbb{R}} \left(e^{-\bar{\delta}(t,x,u)} + \bar{\delta}(t,x,u) - 1\right) k_t(dx),$$

which is condition (5.16) in BMDKR. Further, differentiating with respect to u we obtain condition (see also (24) of BKR)

$$\alpha(t,u) = \sigma(t,u)\bar{\sigma}(t,u) + \int_{\mathbb{R}} \delta(t,x,u) \left(1 - e^{-\bar{\delta}(t,x,u)}\right) k_t(dx).$$

Example 3.4. (Basic Gaussian Random field model.)

Assume that

$$f_t^u = f_0^u + V_t^u + \int_0^t \int_0^u \sigma(s,v) W(ds,dv), \tag{3.11}$$

where $\sigma(t,u)$ is deterministic, $W(t,u)$ is a Brownian sheet, and for any fixed u, V_t^u is finite-variation and continuous.

W is a Gaussian random measure on $\mathbb{R}_+^2$ and thus, for any pair (A,B) of disjoint Borel sets in $\mathbb{R}_+^2$, $W(A)$ and $W(B)$ are independent zero-mean Gaussian random variables with variances Leb(A) and Leb(B).

We shall impose the following assumption on $\sigma(t,u)$:

$$\int_0^T \int_0^T \int_0^u \sigma(s,v)^2 dv ds du < +\infty.$$

As we will see later, this is nothing but condition (M). We shall also assume that condition (FV) is satisfied. That is, if A^u is the total-variation process of V^u, then

$$\int_0^T A_T^u du < +\infty.$$

σ being deterministic, $M_t^v = \int_0^t \int_0^v \sigma(s,w)W(ds,dw)$ is a Gaussian martingale with variance $\int_0^t \int_0^v \sigma(s,w)^2 ds dw$. Furthermore, for $v < v'$, $M_t^{v'}$ can be written as the sum of the two independent random variables M_t^v and $\int_0^t \int_v^{v'} \sigma(s,w)W(ds,dw)$. It follows that

$$\left\langle M^v, M^{v'} \right\rangle_t = \left\langle M^{v\wedge v'}, M^{v\wedge v'} \right\rangle_t = \int_0^t \int_0^{v\wedge v'} \sigma(s,w)^2 dw ds,$$

and the EMM condition for this model is given by

$$\frac{1}{2}\int_0^u \int_0^u \int_0^t \int_0^{v\wedge v'} \sigma(s,w)^2 dw ds dv dv' = \int_0^u V_t^v dv.$$

Differentiating with respect to u, we obtain the following EMM condition:

$$\int_0^u \int_0^t \int_0^v \sigma(s,w)^2 dw ds dv = V_t^u. \tag{3.12}$$

Note that Kennedy's condition (see Example 4.2) can be recovered from the above by noting that the covariance function $C(s,t,u,v) = \int_0^{s\wedge t} \int_0^{u\wedge v} \sigma(s',v')^2 ds' dv'$.

The no-arbitrage condition in the SPDE model (Hamza and Klebaner (1995)) can also be easily recovered from (3.12).

Example 3.5. (Basic Poisson Random field model.)
Assume that

$$f_t^u = f_0^u + V_t^u + \int_0^t \int_0^u \lambda(s,v)N(ds,dv), \tag{3.13}$$

where $N(t,u)$ is a Poisson sheet, $\lambda(s,v)$ is deterministic, and for any fixed u, V_t^u is of finite variation, continuous and satisfies condition (FV).

As for the Brownian sheet, N is a Poisson random measure on $\mathbb{R}_+^2$ such that, for any pair (A,B) of disjoint Borel sets in $\mathbb{R}_+^2$, $N(A)$ and $N(B)$ are independent Poisson random variables with means $\mathrm{Leb}(A)$ and $\mathrm{Leb}(B)$.

We shall impose the following assumption on $\lambda(s,v)$:

$$\int_0^T \int_0^T \int_0^u |\lambda(s,v)| dv ds du < +\infty.$$

As we will see later, this is nothing but conditions (J1$''$) and (J2) combined.

In the sequel, u is fixed. Let $(T_1, U_1), (T_2, U_2), \dots$ be the random points defining the restriction of the Poisson sheet N to $[0,u] \times [0,u]$. The numbering of this sequence will be such that $T_n < T_{n+1}$. Let, for $v \le u$,

$$N_t^v = N([0,t] \times [0,v]), \quad Z_t^v = \int_0^t \int_0^v \lambda(s,w) N(ds,dw) = \sum_n \lambda(T_n, U_n) 1_{T_n \le t} 1_{U_n \le u}.$$

Then, with

$$U(t) = U_n, \text{ for } T_n \le t < T_{n+1},$$

we can write that

$$\Delta f_t^v = \Delta Z_t^v = \lambda(t, U(t)) \Delta N_t^v = \lambda(t, U(t)) 1_{U(t) \le v} \Delta N_t^u.$$

Note that if $\Delta N_t^v = 1$ then $\Delta N_t^u = 1$ for each $u \ge v$. Since Z_t^v (as a process in t) has independent increments, the compensator of its jump measure is deterministic. This statement is obtained by a monotone-class argument identical to that of Jacod and Shiryaev p. 71. In the computation below, we use the fact that N_s^v, as a process in v, is Poisson, and therefore that the conditional law of $U(s)$ given the entire path of N_θ^u (as a process in θ), is uniform on the interval $[0,u]$.

$$\begin{aligned}
\nu^v([0,t] \times A) &= \mathrm{E}[\mu^v([0,t] \times A)] = \sum_{s \le t} \mathbb{P}[\Delta Z_s^v \in A^*] \qquad\qquad (A^* = A \setminus \{0\}) \\
&= \sum_{s \le t} \mathbb{P}[\lambda(s, U(s)) 1_{U(s) \le v} \in A^*, \Delta N_s^u \ne 0] \\
&= \sum_{s \le t} \mathrm{E}\left[\mathbb{P}[\lambda(s, U(s)) 1_{U(s) \le v} \in A^* | N_\theta^u, \theta \ge 0] 1_{\Delta N_s^u \ne 0}\right] \\
&= \sum_{s \le t} \mathrm{E}\left[\frac{1}{u} \int_0^u 1_{w \le v} 1_{A^*}(\lambda(s,w)) dw 1_{\Delta N_s^u \ne 0}\right] \\
&= \frac{1}{u} \int_0^v \mathrm{E}\left[\sum_{s \le t} 1_{A^*}(\lambda(s,w)) \Delta N_s^u\right] dw \\
&= \frac{1}{u} \int_0^v \mathrm{E}\left[\int_0^t 1_{A^*}(\lambda(s,w)) dN_s^u\right] dw \\
&= \frac{1}{u} \int_0^v \int_0^t 1_{A^*}(\lambda(s,w)) u ds dw \\
&= \int_0^t \int_0^v 1_{A^*}(\lambda(s,w)) dw ds.
\end{aligned}$$

It follows that for any ϕ,

$$(\phi * \nu^v)_t = \int_0^t \int_0^v \phi^*(s, \lambda(s,w))dwds, \tag{3.14}$$

where $\phi^*(s,x) = \phi(s,x)1_{x\neq 0}$. Note that $\phi^*(s,x) = \phi(s,x)$ if $\phi(s,0) = 0$.

We now turn to the jump measure of the integrated process and notice that, $\bar{V}^u$ being continuous, the jumps of $\bar{f}^u$ are those of $\bar{Z}^u$, which we now evaluate.

$$\bar{Z}_t^u = \int_0^u \left(\int_0^t \int_0^v \lambda(s,w)N(ds,dw) \right) dv = \int_0^t \int_0^u (u-w)\lambda(s,w)N(ds,dw).$$

Its jump measure can thus be obtained by simply replacing $\lambda(s,w)$ by $(u-w)\lambda(s,w)$ in (3.14). We find that for any deterministic ϕ,

$$(\phi * n^u)_t = \int_0^t \int_0^u \phi^*(s,(u-w)\lambda(s,w))dwds. \tag{3.15}$$

Applying (2.13), we find the following EMM condition:

$$\begin{aligned}
\int_0^u V_t^v dv + \int_0^u (x * \nu^v)_t dv &= \left((e^{-x} + x - 1) * n^u \right)_t \\
&= \int_0^t \int_0^u \left(e^{-(u-w)\lambda(s,w)} + (u-w)\lambda(s,w) - 1 \right) dwds \\
&= \int_0^t \int_0^u e^{-(u-w)\lambda(s,w)} dwds + \int_0^u (x * \nu^v)_t dv - tu,
\end{aligned}$$

that is

$$\int_0^u V_t^v dv = \int_0^t \int_0^u e^{-(u-w)\lambda(s,w)} dwds - tu.$$

Differentiating with respect to u gives

$$V_t^v = -\int_0^t \int_0^u \lambda(s,v)e^{-(u-v)\lambda(s,v)} dvds. \tag{3.16}$$

Example 3.6. (Gaussian and Poisson Random field model.)
Assume that

$$f_t^u = f_0^u + V_t^u + \int_0^t H_s^u \int_0^u \sigma(s,v)W(ds,dv) + \int_0^t K_s \int_0^u \lambda(s,v)N(ds,dv), \tag{3.17}$$

where $\sigma(s,v)$ and $\lambda(s,v)$ are deterministic, $W(t,u)$ is a Brownian sheet, $N(t,u)$ is a Poisson sheet, and for any fixed u, V_t^u is finite variation and continuous, and H_t^u and K_t are predictable.

THEOREM 3.1. *The EMM condition for this model is given by*

(3.18)
$$\int_0^u \int_0^t H_s^u H_s^v \int_0^v \sigma(s,w)^2 dw ds dv = V_t^u + \int_0^t K_s \int_0^u \lambda(s,v) e^{-(u-v)K_s\lambda(s,v)} dv ds.$$

When there is no continuous martingale part ($\sigma = 0$ or $H = 0$), (3.18) simplifies to

$$V_t^u = -\int_0^t K_s \int_0^u \lambda(s,v) e^{-(u-v)K_s\lambda(s,v)} dv ds.$$

In the absence of jumps ($\lambda = 0$), (3.18) simplifies to

$$\int_0^u \int_0^t H_s^u H_s^v \int_0^v \sigma(s,w)^2 dw ds dv = V_t^u.$$

Proof of Theorem 3.1. Let $L_t^u = \int_0^t \int_0^u \sigma(s,v) W(ds,dv)$, so that

$$M_t^u = \int_0^t H_s^u dL_s^u.$$

The properties of the Gaussian martingale L_t^u were discussed in Example 3.4. They imply that

(3.19)
$$\left\langle M^u, M^{u'} \right\rangle_t = \int_0^t H_s^u H_s^{u'} \int_0^{u \wedge u'} \sigma(s,v)^2 dv ds.$$

The jump part of f_t^v can be expressed in terms of the process

$$Z_t^v = \int_0^t \int_0^v \lambda(s,w) N(ds,dw)$$

introduced in Example 3.5 as

$$Y_t^v = \int_0^t K_s dZ_s^v.$$

It immediately follows, that with the notations of Example 3.5, the jumps of f_t^v are given by

$$\Delta f_t^v = \Delta Y_t^v = K_t \Delta Z_t^v = K_t \lambda(t, U(t)) 1_{U(t) \le v} \Delta N_t^u.$$

The compensators ν^u and n^u are then found to be

$$\begin{aligned} \nu^u([0,t] \times A) &= \int_0^t \int_0^u 1_{A^*}\left(K_s \lambda(s,v)\right) dv ds, \\ n^u([0,t] \times A) &= \int_0^t \int_0^u 1_{A^*}\left((u-v) K_s \lambda(s,v)\right) dv ds. \end{aligned}$$

Now the result is obtained by differentiating (2.14). □

4. Appendix

In the sequel, we assume given a stochastic basis $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which M_t^v and B_t^v are families, indexed by $v \leq u$, of local martingales and processes of finite variation respectively. Let A_t^v be the total variation process of B_t^v and suppose that for almost every $v \leq u$, $B_0^v = 0$. We also assume given a family of random measures μ^v with compensators ν^v. Results on a finite horizon $[0, T]$ immediately follow by stopping at T.

PROPOSITION 4.1. *Suppose that B_t^v is a family of increasing processes for which $\int_0^u B_t^v dv < +\infty$. Then $\bar{B}_t^u = \int_0^u B_t^v dv$ is an increasing process and $\Delta \bar{B}_t^u = \int_0^u \Delta B_t^v dv$.*

Proof of Proposition 4.1. By monotone convergence we see that $\bar{B}_{t-}^u = \int_0^u B_{t-}^v dv$. The result immediately follows. □

PROPOSITION 4.2. *Suppose that $\int_0^u A_t^v dv < +\infty$. Then $\bar{B}_t^u = \int_0^u B_t^v dv$ is a process of finite variation and $\Delta \bar{B}_t^u = \int_0^u \Delta B_t^v dv$. If for almost every $v \leq u$, B^v is predictable then so is $\bar{B}^u$.*

Furthermore, if $\int_0^u A_t^v dv$ is locally integrable and C_t^v denotes the compensator (or dual predictable projection) of B_t^v, then the compensator of $\bar{B}_t^u$ is $\bar{C}_t^u = \int_0^u C_t^v dv$.

Proof of Proposition 4.2. $B_t^v = (A_t^v + B_t^v) - A_t^v$ is a measurable decomposition of B_t^v as a difference of two increasing processes. Since $|B_t^v| \leq A_t^v$, the integrability condition on V_t^v implies that $\bar{B}_t^u = \int_0^u B_t^v dv = \int_0^u (A_t^v + B_t^v) dv - \int_0^u A_t^v dv$ is a process of finite variation as a difference of two increasing processes. The jumps of $\bar{B}_t^u$ are obtained by applying proposition 4.1 to $A_t^v + B_t^v$ and A_t^v.

Since the C_t^v's are predictable, then the same goes for the approximating sums of $\bar{C}_t^u$. Consequently $\bar{C}_t^u$, as a limit of predictable processes, is predictable itself.

Because the total variation of C^v is smaller than the compensator of A^v, the local integrability of $\int_0^u A_t^v dv$ and the above imply that $\bar{C}_t^u$ is of finite variation. The rest follows by Fubini's theorem. □

PROPOSITION 4.3. *$\bar{\nu}^u(dt,dx) = \int_0^u \nu^v(dt,dx)dv$ is the compensator of $\bar{\mu}^u(dt,dx)$.*

Proof of Proposition 4.3 Since a point-wise limit of predictable processes is predictable, $\bar{\nu}^u(dt,dx) = \int_0^u \nu^v(dt,dx)dv$ is predictable. Let $H(t,x) \geq 0$ be predictable. Then, by Fubini's theorem and the definition of compensators,

$$\begin{aligned}
\mathrm{E}\Big(\int_0^\infty \int_{\mathbb{R}} H(t,x)\bar{\mu}^u(dt,dx)\Big) &= \mathrm{E}\Big(\int_0^u \Big(\int_0^\infty \int_{\mathbb{R}} H(t,x)\mu^v(dt,dx)\Big)dv\Big) \\
&= \int_0^u \mathrm{E}\Big(\int_0^\infty \int_{\mathbb{R}} H(t,x)\mu^v(dt,dx)\Big)dv \\
&= \int_0^u \mathrm{E}\Big(\int_0^\infty \int_{\mathbb{R}} H(t,x)\nu^v(dt,dx)\Big)dv \\
&= \mathrm{E}\Big(\int_0^u \Big(\int_0^\infty \int_{\mathbb{R}} H(t,x)\nu^v(dt,dx)\Big)dv\Big) \\
&= \mathrm{E}\Big(\int_0^\infty \int_{\mathbb{R}} H(t,x)\bar{\nu}^u(dt,dx)\Big),
\end{aligned}$$

and the result is proved. □

PROPOSITION 4.4. *Suppose that for almost every $v \leq u$, M_t^v is a martingale and that for all t, $\mathbb{E}\left[\int_0^u |M_t^v|dv\right] < +\infty$. Then $\bar{M}_t^u = \int_0^u M_t^v dv$ is a martingale and $\Delta\bar{M}_t^u = \int_0^u \Delta M_t^v dv$.*

Proof of Proposition 4.4. A simple use of Fubini's theorem shows that for $s < t$, $\mathbb{E}[\bar{M}_t^u|\mathcal{F}_s] = \mathbb{E}\left[\int_0^u M_t^v dv|\mathcal{F}_s\right] = \int_0^u \mathbb{E}[M_t^v|\mathcal{F}_s]dv = \int_0^u M_s^v dv = \bar{M}_s^u$. In other words, $\bar{M}_t^u$ is a martingale.

Now recall that for any martingale M_t, $M_{t-} = \mathbb{E}[M_t|\mathcal{F}_{t-}]$. Thus $\bar{M}_{t-}^u = \mathbb{E}[\bar{M}_t^u|\mathcal{F}_{t-}] = \mathbb{E}\left[\int_0^u M_t^v dv|\mathcal{F}_{t-}\right] = \int_0^u \mathbb{E}[M_t^v|\mathcal{F}_{t-}]dv = \int_0^u M_{t-}^v dv$ and the result follows. □

The next proposition requires the following lemma. It is known to hold for continuous local martingales (see Revuz and Yor, p. 120).

LEMMA 4.1. *Let M and N be two local martingales, denote by $\prec M,N \succ$ the total variation process of $[M,N]$. Then*

$$\prec M,N \succ_t \leq 2\sqrt{[M,M]_t\,[N,N]_t}.$$

Proof of Lemma 4.1. It follows from $[M,N]_t = \langle M^c, N^c\rangle_t + \sum_{s\le t} \Delta M_s \Delta N_s$, that

$$\begin{aligned} \prec M,N \succ_t &\le \prec M^c, N^c \succ_t + \sum_{s\le t} |\Delta M_s||\Delta N_s| \\ &\le \sqrt{\langle M^c, M^c\rangle_t \langle N^c, N^c\rangle_t} + \sqrt{\sum_{s\le t}(\Delta M_s)^2 \sum_{s\le t}(\Delta N_s)^2} \\ &\le 2\sqrt{[M,M]_t\,[N,N]_t}. \end{aligned}$$

□

PROPOSITION 4.5. *Suppose that for almost every $v \le u$, M_t^v is a (square integrable) martingale such that* $\mathbb{E}\left[\int_0^u \langle M^v, M^v\rangle_\infty \, dv\right] < +\infty$. *Then* $\bar{M}_t^u = \int_0^u M_t^v dv$ *is a square integrable martingale and*

$$\left\langle \bar{M}^u, \bar{M}^u \right\rangle_t = \int_0^u \int_0^u \langle M^v, M^w\rangle_t \, dvdw. \tag{4.1}$$

Proof of Proposition 4.5. First note that for any t, both $\mathbb{E}\left[(\bar{M}_t^u)^2\right]$ and $\mathbb{E}\left[\int_0^u |M_t^v| dv\right]^2$ are less than or equal to

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^u |M_t^v| dv\right)^2\right] &= \mathbb{E}\left[\int_0^u\int_0^u |M_t^v M_t^w| dvdw\right] = \int_0^u\int_0^u \mathbb{E}[|M_t^v M_t^w|]dvdw \\ &\le \int_0^u\int_0^u \mathbb{E}\left[(M_t^v)^2\right]^{1/2} \mathbb{E}\left[(M_t^w)^2\right]^{1/2} dvdw = \left(\int_0^u \mathbb{E}\left[(M_t^v)^2\right]^{1/2} dv\right)^2 \\ &\le u\int_0^u \mathbb{E}\left[(M_t^v)^2\right] dv = u\int_0^u \mathbb{E}\left[\langle M^v, M^v\rangle_t\right] dv \\ &\le u\mathbb{E}\left[\int_0^u \langle M^v, M^v\rangle_\infty \, dv\right], \end{aligned}$$

which is finite. This and Proposition 4.4 establish that $\bar{M}_t^u$ is a square-integrable martingale. Its sharp bracket is obtained as follows. First, Lemma 4.1 shows that the total variation of the process $[M^v, M^w]_t$ is less than or equal to $2\sqrt{[M^v, M^v]_t\,[M^w, M^w]_t}$. Since

$$\mathbb{E}\left[\int_0^u [M^v, M^v]_\infty \, dv\right] = \mathbb{E}\left[\int_0^u \langle M^v, M^v\rangle_\infty \, dv\right] < +\infty$$

and

$$\int_0^u \int_0^u \sqrt{[M^v, M^v]_t \, [M^w, M^w]_t} dvdw \;=\; \left(\int_0^u \sqrt{[M^v, M^v]_t} dv \right)^2 \le \; u \int_0^u [M^v, M^v]_t \, dv,$$

it follows from Proposition 4.2 that $\int_0^u \int_0^u [M^v, M^w]_t \, dvdw$ is an integrable process of finite variation, the compensator of which is $\int_0^u \int_0^u \langle M^v, M^w \rangle$. Finally, let τ be a finite stopping time. Then

$$\begin{aligned} \mathbb{E}\left[(\bar{M}^u_\tau)^2 \right] &= \mathbb{E}\left[\int_0^u \int_0^u M^v_\tau M^w_\tau dvdw \right] = \int_0^u \int_0^u \mathbb{E}[M^v_\tau M^w_\tau] dvdw \\ &= \int_0^u \int_0^u \mathbb{E}\left[[M^v, M^w]_\tau \right] dvdw \\ &= \mathbb{E}\left[\int_0^u \int_0^u [M^v, M^w]_\tau \, dvdw \right] \\ &= \mathbb{E}\left[\int_0^u \int_0^u \langle M^v, M^w \rangle_\tau \, dvdw \right], \end{aligned}$$

which shows (4.1) and completes the proof. □

PROPOSITION 4.6. *Suppose that for almost every $v \le u$, M^v_t is continuous and that for all t, $\int_0^u \langle M^v, M^v \rangle_t \, dv < +\infty$. Then $\bar{M}^u_t = \int_0^u M^v_t dv$ is a locally square-integrable martingale and (4.1) holds.*

Proof of Proposition 4.6. Let $T_n = \inf\{t : \int_0^u \langle M^v, M^v \rangle_t \, dv > n\}$. Then, by monotone convergence

$$\int_0^u \langle M^v, M^v \rangle_{T_n} dv = \lim_{\varepsilon \downarrow 0} \int_0^u \langle M^v, M^v \rangle_{T_n - \varepsilon} dv \le n,$$

and the local martingales M^v stopped at T_n are square-integrable martingales ($\mathbb{E}\left[\langle M^v, M^v \rangle_{T_n} \right] < +\infty$ $(a.e.)$) which satisfy the integrability condition of Proposition 4.5. The result immediately follows from the application of proposition 4.5.

□

PROPOSITION 4.7. *Suppose that for almost every $v \le u$, M^v_t is a purely discontinuous local martingale such that the increasing process $\int_0^u [M^v, M^v]_t \, dv$ is locally* *integrable. Then* $\bar{M}_t^u = \int_0^u M_t^v dv$ *is a locally square-integrable purely discontinuous martingale such that* $\Delta \bar{M}_t^u = \int_0^u \Delta M_t^v dv$*, and (4.1) holds.*

Proof of Proposition 4.6. Let T_n be such that $\mathbb{E}\left[\int_0^u [M^v, M^v]_{T_n} dv\right] < +\infty$. Then (for almost every $v \leq u$) M^v is a locally square-integrable martingale and $\mathbb{E}\left\langle \int_0^u [M^v, M^v\rangle_{T_n} dv\right] < +\infty$. Therefore we can apply Proposition 4.5 and Proposition 4.4 to the local martingales M^v stopped at T_n. We now establish the purely discontinuous feature of $\bar{M}^u$. Let L_t be a continuous square-integrable (or even bounded) martingale and τ be a finite stopping time. Then $\mathbb{E}\left[\bar{M}_\tau^u L_\tau\right] = \mathbb{E}\left[\int_0^u M_\tau^v dv L_\tau\right] = \int_0^u \mathbb{E}\left[M_\tau^v L_\tau\right] dv = 0$ since M^v is purely discontinuous and L is continuous. □

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