A MARTINGALE REPRESENTATION RESULT AND AN APPLICATION TO INCOMPLETE FINANCIAL MARKETS

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ABSTRACT We establish necessary and sufficient conditions for an $\mathbf{H}^1$-martingale to be representable with respect to a collection, $\mathcal{X}$, of local martingales. $M \in \mathbf{H}^1(P)$ is representable if and only if M is a local martingale under all p.m.s Q which are 'uniformly equivalent' to P and which make all the elements of $\mathcal{X}$ local martingales (Theorem 1). We then give necessary and sufficient conditions which are easier to verify, and only involve expectations (Theorem 2). We go on to apply these results to the problem of pricing claims in an incomplete financial market—establishing two conjectures of Harrison and Pliska (1981).

Key Words: martingale, martingale representation, local martingale, incomplete financial market, contingent claim, attainable claim, fair price.

§**1. Introduction** Harrison and Pliska (1981) showed that *every* contingent claim in a financial market is attainable (hedgeable) if and only if the collection of martingale measures is a singleton (see Harrison and Pliska (1981) for terminology and assumptions); such a market is called complete. The following question then arises very naturally—

'which claims are attainable in an incomplete financial market?'

Harrison and Pliska's proof relied on results in Jacod (1979) and a basic equivalence between questions of this type and questions about martingale representations. Essentially any dynamic hedging strategy corresponds to a (vector-valued) previsible process ϕ, and the (discounted) value of the corresponding portfolio corresponds to the stochastic integral of ϕ with respect to the vector of (discounted) security prices. The question of attainability then becomes one of representation—

' which contingent claims can be represented as a stochastic integral with respect to the vector of discounted security prices?'—

whilst the parallel question in martingale representation theory is:

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'given a collection $\mathcal{X}$ of martingales, which are those martingales which are representable as stochastic integrals with respect to $\mathcal{X}$?'.

In practise we pose and answer (at least initially) a slightly different question, but before we state it we recall a few concepts.

Recall first that if $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0))$ is a filtered measurable space and P is a probability measure on $(\Omega, \mathcal{F})$ then $\mathbf{H}^1(P)$ is the collection of martingales M (with respect to P) on $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0))$ such that

$$\mathbb{E}_P \sup_t |M_t| < \infty,$$

or equivalently, such that

$$\mathbb{E}_P [M, M]_\infty^{1/2} < \infty.$$

Then, given a collection $\mathcal{X}$ of cadlag processes, adapted to the filtered measurable space we say P is a martingale measure, and write

$$P \in \mathcal{M}(\mathcal{X}),$$

if each $X \in \mathcal{X}$ is a $(P, \mathcal{F}_t; t \geq 0)$ local martingale. Finally, if P is a martingale measure we say M is representable (with respect to $\mathcal{X}$, under P) if

$$M \in \mathcal{L}^1(\mathcal{X} \cup \{1\}),$$

where $\mathcal{L}^1(\mathcal{X} \cup \{1\})$ is the closure (in $\mathbf{H}^1(P)$) of the collection of elements of $\mathbf{H}^1(P)$ which can be written as linear combinations of stochastic integrals with respect to the elements of $\mathcal{X} \cup \{1\}$.

The initial task we have set ourselves is then (suppressing $(\mathcal{F}_t)$ and P) to characterise $\mathcal{L}^1(\mathcal{X} \cup \{1\})$.

We know (Theorem 11.2 of Jacod (1979)), that

$$\begin{gathered} P \text{ is extremal in } \mathcal{M}(\mathcal{X}) \\ \Leftrightarrow \\ \mathcal{L}^1(\mathcal{X} \cup \{1\}) = \mathbf{H}^1(P) \textit{ and } \mathcal{F}_0 \text{ is } P\text{-trivial,} \end{gathered}$$

so the case we want to address is that where P is *not* extremal in $\mathcal{M}(\mathcal{X})$.

Let's make a few definitions and then we can state the main results:

Definition 1: define
$\mathcal{P} \equiv \mathcal{M}(\mathcal{X}, P) \equiv \{Q \in \mathcal{M}(\mathcal{X}) : Q \sim P\}$,
$\hat{\mathcal{P}} \equiv \hat{\mathcal{M}}(\mathcal{X}, P) \equiv \{Q \in \mathcal{M}(\mathcal{X}) : Q \ll P\}$
and
$\mathcal{P}^0 \equiv \mathcal{M}^0(\mathcal{X}, P) \equiv \{Q \in \mathcal{M}(\mathcal{X}) : Q \sim P \text{ and } ||\frac{dQ}{dP}||_\infty \vee ||\frac{dP}{dQ}||_\infty < \infty\}$.

Thus $\mathcal{P}$ is the collection of martingale measures equivalent to P,
$\hat{\mathcal{P}}$ is the collection of martingale measures absolutely continuous with respect to P,

and

$\mathcal{P}^0$ is the collection of martingale measures 'uniformly equivalent' to P.

We shall write $P \approx Q$ if P and Q are uniformly equivalent (in other words $||\frac{dQ}{dP}||_\infty \vee ||\frac{dP}{dQ}||_\infty < \infty\}$, which is, of course equivalent to P and Q having equivalent $\mathbf{L}^1$ norms).

Notice that P is extremal in $\mathcal{M}(\mathcal{X})$ if and only if $\mathcal{P}^0$ (and then $\mathcal{P}$) is a singleton (see Chapter 11 of Jacod (1979) for details).

Definition 2: define, for any subset $\mathcal{R}$ of $\hat{P}$,
$\mathbf{H}^1(P,\mathcal{R}) = \{X \in \mathbf{H}^1(P) : X \in \mathbf{M}_{\text{loc}}(Q) \text{ for all } Q \in \mathcal{R}\}$

Thus

$\mathbf{H}^1(P,\mathcal{R})$ consists of all those X in $\mathbf{H}^1(P)$ which are also local martingales under each Q in $\mathcal{R}$.

Definition 3: define, for any subset $\mathcal{R}$ of $\hat{P}$,
$\mathcal{T}(Q) = \{ \text{ sequences of stopping times } (T_n) : T_n \uparrow \infty \quad Q \text{ a.s.}\}$,
and say that $X \in \mathbf{H}^1(P)$ is **stable over** $\mathcal{R}$ if, for any $Q \in \mathcal{R}$:
$\exists (T_n) \in \mathcal{T}(Q)$ such that $\mathbb{E}_Q X_{T_n} = \mathbb{E}_P X_0 \quad \forall n$.
Then define
$\mathcal{E}^0(P,\mathcal{R}) = \{X \in \mathbf{H}^1(P) : \quad X \text{ is stable over } \mathcal{R}\}$.

For any collection of adapted processes C, C_+ denotes

$$\{X \in C : X \text{ is non-negative}\} = C \cap A_+$$

where A_+ is the collection of non-negative adapted processes.

Then the results are as follows:

Theorem 1

$$\mathcal{L}^1(\mathcal{X} \cup \{1\}) = \mathbf{H}^1(P,\mathcal{P}^0) = \bigcap_{Q \in \mathcal{P}^0} \mathbf{H}^1(Q).$$

in other words, representable martingales in $\mathbf{H}^1(P)$ are those elements of $\mathbf{H}^1(P)$ which are also local martingales under each Q in $\mathcal{P}^0$, and then they are in fact in $\mathbf{H}^1(Q)$ for each Q in $\mathcal{P}^0$.

Theorem 2 If $\mathcal{F}_0$ is P-trivial then

$$\mathbf{H}^1(P,\mathcal{P}^0) = \mathcal{E}^0(P,\mathcal{P}^0),$$

and

$$\begin{aligned} \mathcal{E}^0(P,\mathcal{P}^0) =& \hat{\mathcal{E}}^0(P,\mathcal{P}^0) \\ &\stackrel{def}{=} \{X \in \mathbf{H}^1(P) : \text{ for each } Q \in \mathcal{P}^0 \quad \mathbb{E}_Q(X_\infty) = \mathbb{E}_P(X_0) = X_0\} \end{aligned} \tag{1}$$

Theorem 3 If $\mathcal{F}_0$ is P-trivial and $\mathcal{X}$ is finite then

(a) $\mathcal{L}^1_+(\mathcal{X} \cup \{1\}) = \mathbf{H}^1_+(P, \hat{\mathcal{P}}) = \mathbf{H}^1_+(P, \mathcal{P})$;

and

(b) $\mathbf{H}^1_+(P, \mathcal{P}) = \mathcal{E}^0_+(P, \mathcal{P})$ and $\mathbf{H}^1_+(P, \hat{\mathcal{P}}) = \mathcal{E}^0_+(P, \hat{\mathcal{P}})$

so that

$$\begin{aligned} \mathcal{L}^1_+(\mathcal{X} \cup \{1\}) &= \mathcal{E}^0_+(P, \mathcal{P}) \\ &= \mathcal{E}^0_+(P, \hat{\mathcal{P}}). \end{aligned}$$

(c) If, moreover, all the elements of $\mathcal{X}$ are non-negative, then we may remove the + signs from all of the above.

Remark: In the language of option pricing, Theorems 1 and 2 say (roughly) that a contingent claim is attainable if and only if it has the same price under all consistent price systems. We shall make this result precise in Theorem 8.

For more details on the financial background, the reader is referred to Harrison and Pliska (1981), Harrison and Pliska (1983), Harrison and Kreps (1979), and for the case of American options, to Karatzas (1988) and Karatzas (1989).

§2. Notation and proofs

We are given an underlying filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0))$ and a collection $\mathcal{X}$ of cadlag processes adapted to $(\mathcal{F}_t)$. For any p.m. Q on $(\Omega, \mathcal{F})$ we define

$$\mathbf{M}(Q) = \{ \text{ processes which are uniformly integrable martingales under } Q\},$$

$$\mathbf{H}^p(Q) = \{M : M \in \mathbf{M}(Q) \text{ and } \|\sup_t |M_t|\|_p < \infty\} \quad p \geq 1,$$

and, for any suitable class $\mathbf{C}$ of processes

$$\mathbf{C}_{\mathrm{loc}}(Q) = \{X : \text{ there exists } (T_n) \in \mathcal{T}(Q) \text{ such that } X^{T_n} \in \mathbf{C} \quad \forall n\}$$

We *assume* the existence of a probability measure P on $(\Omega, \mathcal{F})$ such that

$$\mathcal{X} \subseteq \mathbf{M}_{\mathrm{loc}}(P)$$

i.e. each $X \in \mathcal{X}$ is a P-local martingale[2]. Then $\mathbf{H}^1(P)$ is the collection of P-martingales with finite $\mathbf{H}^1$-norm, whilst $\mathcal{L}^1(\mathcal{X} \cup \{1\})$ is the stable sub-space (see Jacod (1979), (4.2)) of $\mathbf{H}^1$, generated by stochastic integrals of previsible processes H with respect to M, where $M \in \mathcal{X} \cup \{1\}$ and $\|\int H dM\|_{\mathbf{H}^1(P)} < \infty$. Note that this last condition is also written as $H \in L^1(M; P)$.

[2] Note that Harrison and Pliska (1981) assume that each $X \in \mathcal{X}$ is a martingale under P, but this distinction has no effect.

We work almost entirely with classes of equivalent measures so we trust no ambiguity will arise from failing to specify the measure involved in expressions involving $||X||_\infty$.

We have attempted to maintain consistency with Jacod's notation and the reader is referred to Jacod (1979) for any unexplained notation.

Finally, before we launch into the promised proofs, we recall the fact, which follows from Corollary 2.16 of Jacod (1979), that

$$\mathbf{M}(Q) \subseteq \mathbf{M}_{\mathrm{loc}}(Q) = \mathbf{H}^1_{\mathrm{loc}}(Q),$$

which we shall use repeatedly without, in future, recalling it.

The proof of Theorem 1 is just a fairly straightforward application of the Hahn-Banach Theorem (and a corollary), together with the characterisation of the dual of $\mathbf{H}^1(P)$.

Proof of Theorem 1 First note that

$$\mathbf{H}^1(P, \mathcal{P}^0) = \bigcap_{Q \in \mathcal{P}^0} \mathbf{H}^1(Q),$$

since

$$||\frac{dQ}{dP}||_\infty < \infty \quad \text{for all } Q \in \mathcal{P}^0.$$

We now wish to show that

$$\mathcal{L}^1(\mathcal{X} \cup \{1\}) \subseteq \mathbf{H}^1(P, \mathcal{P}^0) \tag{2}$$

Since both sets in (2) are closed subspaces of $\mathbf{H}^1(P)$ it is clearly sufficient to show that

$$\int H dX \in \mathbf{H}^1(Q)$$

for any $Q \in \mathcal{P}^0$, $X \in \mathcal{X} \cup \{1\}$, and $H \in L^1(X; P)$, but for this to be true we only need $|| \int H dX||_{\mathbf{H}^1(Q)} < \infty$ which follows immediately from the equivalence of the $\mathbf{H}^1(P)$ and $\mathbf{H}^1(Q)$ norms, establishing (2).

Now to establish equality in (2), first recall that the dual of $\mathbf{H}^1(P)$ is $\mathbf{BM0}(P)$, with duality given by

$$c(M) = \mathbb{E}_P[X, M]_\infty \text{ for all } M \in \mathbf{H}^1(P)\,,$$

where c is a continuous linear functional on $\mathbf{H}^1(P)$ and X is the corresponding martingale in $\mathbf{BM0}(P)$. Moreover

$$\mathbf{BM0}(P) \subseteq \mathbf{H}^\infty_{\mathrm{loc}}(P)\,. \tag{3}$$

Now suppose that c is a continuous linear functional on $\mathbf{H}^1(P, \mathcal{P}^0)$, which vanishes on $\mathcal{L}^1(\mathcal{X} \cup \{1\})$. By the Hahn-Banach Theorem there is an extension of c to $\mathbf{H}^1(P)$, and it then follows from (3), that there is an $X \in \mathbf{H}^{\infty}_{\text{loc}}(P)$ such that

$$c(M) = \mathbb{E}_P[X, M]_\infty \text{ for all } M \in \mathbf{H}^1(P)\,.$$

By localisation we may assume that $X \in \mathbf{H}^\infty(P)$ with $\| X \|_{H^\infty} = a$. Now define Q by

$$\frac{dQ}{dP} = \big(1 + \frac{X_\infty}{2a}\big).$$

Claim: $Q \in \mathcal{P}^0$.

Proof of claim: It is clear that $P \approx Q$, whilst, since $c(1) = 0$, $\mathbb{E}_P X_\infty = 0$ (indeed, since c disappears on $\mathcal{L}(1)$ we must have $X_0 = 0$) so Q is a probability measure.

Since

$$c(M) = 0 \text{ for all } M \in \mathcal{L}^1(\mathcal{X} \cup \{1\})$$

it follows that XM is a P-local-martingale for all $M \in \mathcal{X}$ so $\big(1 + \frac{X}{2a}\big)M$ is a P-local-martingale for all $M \in \mathcal{X}$, which implies that M is a Q-local-martingale for all $M \in \mathcal{X}$ so $Q \in \mathcal{P}^0$.

If we now take $N \in \mathbf{H}^1(P, \mathcal{P}^0)$ it follows, by reversing the argument above, that, since N is in both $\mathbf{H}^1(P)$ and $\mathbf{H}^1(Q)$, $\big(1 + \frac{X}{2a}\big)N$, and hence XN, is in $\mathbf{H}^1(P)$, thus $[X, N] \in \mathbf{H}^1(P)$ and so $c(N) = \mathbb{E}_P X_0 N_0 = 0$. It is now immediate that $\mathcal{L}^1(\mathcal{X} \cup \{1\})$ is dense in $\mathbf{H}^1(P, \mathcal{P})$ (suppose not and apply corollary 23.6 of Jameson (1974) to deduce a contradiction), but $\mathcal{L}^1(\mathcal{X} \cup \{1\})$ is closed so $\mathcal{L}^1(\mathcal{X} \cup \{1\}) = \mathbf{H}^1(P, \mathcal{P}^0)$ $\square$

Corollary 4

$$\mathcal{L}^1_{\text{loc}}(\mathcal{X} \cup \{1\}) = \bigcap_{Q \in \mathcal{P}^0} \mathbf{M}_{\text{loc}}(Q).$$

Proof : Suppose X is in $\mathcal{L}^1_{\text{loc}}(\mathcal{X} \cup \{1\})$, then there is a sequence (T_n) in $\mathcal{T}(P)$ such that

$$X^{T_n} \in \mathcal{L}^1(\mathcal{X} \cup \{1\}) = \mathbf{H}^1(P, \mathcal{P}^0) = \mathbf{H}^1(P) \cap \bigcap_{Q \in \mathcal{P}^0} \mathbf{M}_{\text{loc}}(Q),$$

by Theorem 1, so X^{T_n} is in $\mathbf{M}_{\text{loc}}(Q)$ for each Q in $\mathcal{P}^0$ and hence (since (T_n) is in $\mathcal{T}(Q)$ for each Q in $\mathcal{P}^0$) so is X. Thus

$$\mathcal{L}^1_{\text{loc}}(\mathcal{X} \cup \{1\}) \subseteq \bigcap_{Q \in \mathcal{P}^0} \mathbf{M}_{\text{loc}}(Q).$$

To establish the reverse inclusion, simply take X in $\bigcap_{Q \in \mathcal{P}^0} \mathbf{M}_{\text{loc}}(Q)$, then take $(T_n) \in \mathcal{T}(P)$ with $X^{T_n} \in \mathbf{H}^1(P)$; it follows that $X^{T_n} \in \mathbf{H}^1(P) \cap \bigcap_{Q \in \mathcal{P}^0} \mathbf{M}_{\text{loc}}(Q)$, so the reverse inclusion follows from Theorem 1.

Theorem 2 is perhaps more interesting, because this is a result whose formulation is driven by the financial theory: if there's a unique price for an option then it must be hedgeable (attainable). The proof is somewhat perverse so we'll describe it first. Notice that, by Theorem 1,

$$\mathbf{H}^1(P,\mathcal{P}^0) \quad \subseteq \quad \hat{\mathcal{E}}^0(P,\mathcal{P}^0) \quad \subseteq \quad \mathcal{E}^0(P,\mathcal{P}^0);$$

so all we need to prove is that $\mathbf{H}^1(P,\mathcal{P}^0) \supseteq \mathcal{E}^0(P,\mathcal{P}^0)$. The way we do this is as follows: we take an $M \in \mathcal{E}^0(P,\mathcal{P}^0)$; all we need to show is that M is a Q local-martingale for all $Q \in \mathcal{P}^0$. Now since $M \in \mathcal{E}^0(P,\mathcal{P})$, $M \in \mathbf{M}(P)$, so in particular M is a P-martingale. We then show that $M^{Q,n}$, given by $M^{Q,n}_t = \mathbb{E}_Q[M_{T_n}|\mathcal{F}_t]$, is a P-**martingale**: then, since $M^{Q,n}_\infty = M_{T_n}$, it follows that $M^{Q,n} \equiv M^{T_n}$ (since $M^{Q,n} - M^{T_n}$ is an $\mathbf{M}(P)-$martingale terminating at 0) so that M^{T_n} is a Q-**martingale**. Finally, since this is true for the whole localising sequence (T_n), M is a Q-local-martingale .

Proof of Theorem 2 As we said above, all we need to prove is that if $M \in \mathcal{E}^0(P,\mathcal{P}^0)$ then $M^{Q,n}$ is a P-martingale for any $Q \in \mathcal{P}^0$. To do this we shall show that

$$\mathbb{E}_P M^{Q,n}_\tau = \mathbb{E}_P M^{Q,n}_0 \text{ for any stopping time } \tau\,. \tag{4}$$

Given τ and Q define a probability measure R by

$$\frac{dR}{dP} = \frac{dQ}{dP}\Big/\mathbb{E}_P\left[\frac{dQ}{dP}|\mathcal{F}_\tau\right] \tag{5}$$

Notice that P and R agree on $\mathcal{F}_\tau$.

Claim: $R \in \mathcal{P}$.

Proof of claim: It is obvious that $P \approx R$. Now take any $X \in \mathcal{X}$; by localising we may assume that X is both a P- and a Q- uniformly integrable martingale. We show that

$$\mathbb{E}_R X_S = \mathbb{E}_R X_0 \text{ for any stopping time } S.$$

Now,

$$\begin{aligned}
&\mathbb{E}_R[X_S] = \mathbb{E}_R[X_S 1_{(\tau\geq S)}] + \mathbb{E}_R[X_S 1_{(\tau<S)}]\\
&= \mathbb{E}_P[X_S 1_{(\tau\geq S)}] + \mathbb{E}_P\left[\frac{dR}{dP}X_S 1_{(\tau<S)}\right]\\
&= \mathbb{E}_P[X_S 1_{(\tau\geq S)}] + \mathbb{E}_P\left[\frac{dQ}{dP}X_S\Big/\mathbb{E}_P\left[\frac{dQ}{dP}|\mathcal{F}_\tau\right]1_{(\tau<S)}\right]\\
&= \mathbb{E}_P[X_S 1_{(\tau\geq S)}] + \mathbb{E}_P\left[\mathbb{E}_P\left[X_S\frac{dQ}{dP}|\mathcal{F}_\tau\right]\Big/\mathbb{E}_P\left[\frac{dQ}{dP}|\mathcal{F}_\tau\right]1_{(\tau<S)}\right]\\
&= \mathbb{E}_P\big[X_S 1_{(\tau\geq S)} + \mathbb{E}_Q\big[X_S|\mathcal{F}_\tau\big]1_{(\tau<S)}\big]\\
&= \mathbb{E}_P\big[X_{\tau\wedge S}\big] = \mathbb{E}_P\big[X_0\big]\,.
\end{aligned}$$

This is true for any S (including $S \equiv 0$) so the claim is proved.

To establish the main result; since $R \in \mathcal{P}^0$ it follows that (if $M \in \mathcal{E}^0(P,\mathcal{P}^0)$)

$$E_R M_{T_n} = \mathbb{E}_P M_0\,,$$

$$\begin{aligned} \text{but } \mathbb{E}_R\big[M_{T_n}\big] &= \mathbb{E}_P\left[\frac{dQ}{dP}M_{T_n}\Big/\mathbb{E}_P\big[\frac{dQ}{dP}|\mathcal{F}_\tau\big]\right] \\ &= \mathbb{E}_P\left[\mathbb{E}_P\big[\frac{dQ}{dP}M_{T_n}|\mathcal{F}_\tau\big]\Big/\mathbb{E}_P\big[\frac{dQ}{dP}|\mathcal{F}_\tau\big]\right] \\ &= \mathbb{E}_P\big[\mathbb{E}_Q\big[M_{T_n}|\mathcal{F}_\tau\big]\big] = \mathbb{E}_P\big[M_\tau^{Q,n}\big], \end{aligned}$$

at least if $\tau \le T_n$. So $\mathbb{E}_P[M_\tau^{Q,n}]$ is constant over stopping times and hence $\mathbb{E}_P[M_\tau^{Q,n}] = \mathbb{E}_P[M_0^{Q,n}]$ for all τ and hence $M^{Q,n}$ is a P-martingale. $\square$

Remark A careful reading of the proof shows that we have also demonstrated the following equality

Corollary 6

$$\{X \in \mathbf{M}(P): \quad \mathbb{E}_Q(X_\infty) = \mathbb{E}_P(X_0) = X_0\} \text{ for each } Q \in \mathcal{P}^0$$
$$=$$
$$\bigcap_{Q\in\mathcal{P}^0} \mathbf{M}(Q).$$

We've already done a lot of the work to prove Theorem 3 in proving Theorems 1 and 2. The remaining ingredient is the following theorem which we believe is of independent interest.

Theorem 7 If $\mathcal{F}_0$ is P-trivial and $\mathcal{X}$ is finite then the collection $\mathcal{P}^0$ is dense (with respect to the $L^1(P)$ norm) in $\hat{\mathcal{P}}$, in the sense that if we define $D^0 = \{\frac{dQ}{dP} : Q \in \mathcal{P}^0\}$ and $\hat{D} = \{\frac{dQ}{dP} : Q \in \hat{\mathcal{P}}\}$ then

$$cl(D^0) \supseteq \hat{D}.$$

Proof Notice that $D^0 \subseteq \hat{D} \subseteq L^1(P)$ and both D^0 and $\hat{D}$ are convex sets. If we take a continuous linear functional c on $\hat{D}$ which disappears on D^0 we may extend it to $L^1(P)$. Now, since the dual of L^1 is L^∞, we see that

$$c(Y) = \mathbb{E}_P M_{(\infty)} Y \text{ for a suitable } M_{(\infty)} \in L^\infty(P).$$

Now define the P-martingale M by

$$M_t = \mathbb{E}_P[M_{(\infty)}|\mathcal{F}_t]$$

(so that $M_\infty = M_{(\infty)}$).

Claim M is in $\mathcal{L}^1(\mathcal{X} \cup \{1\}) \cap \mathbf{H}^\infty(P)$.

Proof of claim Since M_∞ is essentially bounded (under P and hence under Q for all $Q \in \hat{\mathcal{P}}$) it follows from Doob's L^∞ (in)equality that M is in $\mathbf{H}^\infty(P)$, but since c disappears on D^0 it follows that

$$\mathbb{E}_Q M_\infty = \mathbb{E}_P[\frac{dQ}{dP}M_\infty] = 0 \text{ for all } Q \in \mathcal{P}^0$$

so that

$$M \in \mathcal{E}^0(P, \mathcal{P}).$$

Hence, by Theorem 2,

$$M \in \mathcal{L}^1(\mathcal{X} \cup \{1\}) \cap \mathbf{H}^\infty(P)$$

Now, if we can prove that $M \in \mathbf{H}^1(Q)$ for all $Q \in \hat{\mathcal{P}}$ then we will have proved the theorem, for if this were true then we would have established that c disappears on $\hat{D}$ which would imply the result. But we know that $M \in \mathcal{L}^1(\mathcal{X} \cup \{1\})$, so, since $\mathcal{X}$ is finite it is sufficient to show that

$$\int H dX \in \mathbf{H}^\infty(P) \text{ implies } \int H dX \in \mathbf{H}^1(Q)$$

for any $Q \in \hat{\mathcal{P}}$. But this follows from the representation and the trite inequality

$$\mathbb{E}_Q M^*_\infty = \mathbb{E}_P[\frac{dQ}{dP} M^*_\infty] \leq ||M^*_\infty||_\infty,$$

where $M^*_t \stackrel{def}{=} \sup_{s \leq t} |M_s|$ $\square$

Proof of Theorem 3 (a) Notice that

$$\mathbf{H}^1(P, \hat{\mathcal{P}}) \subseteq \mathbf{H}^1(P, \mathcal{P}) \subseteq \mathbf{H}^1(P, \mathcal{P}^0),$$

so all we need to prove is that

$$\mathcal{L}^1_+(\mathcal{X} \cup \{1\}) \subseteq \mathbf{H}^1(P, \hat{\mathcal{P}}).$$

Claim Under the conditions of Theorem 3 if $M \in \mathcal{L}^1_+(\mathcal{X} \cup \{1\})$ then M is a non-negative Q-supermartingale for each $Q \in \hat{\mathcal{P}}$.

Proof of Claim Fix $Q \in \hat{\mathcal{P}}$ then by Theorem 7 we may take a sequence $(Q_n) \subseteq \hat{\mathcal{P}}$ such that $\frac{dQ_n}{dP} \stackrel{n \to \infty}{\longrightarrow} \frac{dQ}{dP}$ in $L^1(P)$.Take a subsequence (n_k) such that $\frac{dQ_{n_k}}{dP} \stackrel{a.s.}{\longrightarrow} \frac{dQ}{dP}$. Then

$$M_t = \mathbb{E}_{Q_{n_k}}[M_{t+s}|\mathcal{F}_t] = \mathbb{E}_P[\frac{dQ_{n_k}}{dP} M_{t+s}|\mathcal{F}_t] / \mathbb{E}_P[\frac{dQ_{n_k}}{dP}|\mathcal{F}_t]$$

so that, by Fatou's lemma, M is a Q-supermartingale.

By virtue of the (implicit) representation of M we may prove that $M \in \mathbf{M}_{\text{loc}}(Q)$ by exhibiting a sequence $(T_n) \in \mathcal{T}(Q)$ such that $M^*_{T_n} \in L^1(Q)$ for each n: if we set $T_n = n \wedge \inf\{t : M_t \geq M_0 + n\}$ then $M^*_{T_n -} \leq M_0 + n$ so, since M is non-negative, it follows that

$$\mathbb{E}_Q M^*_{T_n} \leq \mathbb{E}_Q \max(M^*_{T_n -}, M_{T_n}) \leq \mathbb{E}_Q(M^*_{T_n -} + M_{T_n}),$$

we are done if we can show that $\mathbb{E}_Q M_{T_n} < \infty$. But T_n is a bounded stopping time so that, by the optional sampling theorem for supermartingales,

$$\mathbb{E}_Q M_{T_n} \leq M_0.$$

(b) Since $\mathbf{H}^1(P,\mathcal{R}) \subseteq \mathcal{E}^0(P,\mathcal{R})$ for any $\mathcal{R}$, whilst $\mathcal{E}^0(P,\hat{\mathcal{P}}) \subseteq \mathcal{E}^0(P,\mathcal{P})$ it follows from (a) that we need only prove that $\mathcal{E}^0(P,\mathcal{P}) \subseteq \mathbf{H}^1(P,\mathcal{P})$, but a careful reading of the proof of Theorem 2 shows that it remains valid for this case.

(c)It follows from Theorems 1 and 2 and (a) and (b) above that it is sufficient to show that

$$span(\mathcal{L}^1_+(\mathcal{X} \cup \{1\})) = \mathcal{L}^1(\mathcal{X} \cup \{1\}).$$

This follows by a similar argument to that given in the proof of Theorem 1 on observing that if c is a continuous linear functional on $\mathbf{H}^1(P)$ which disappears on $\mathcal{L}^1_+(\mathcal{X} \cup \{1\})$ and the elements of $\mathcal{X}$ are non-negative then (localising if necessary) c disappears on $\mathcal{X}$ and hence disappears on $\mathcal{L}^1(\mathcal{X} \cup \{1\})$ $\square$

§3. An application to incomplete financial markets

We assume from now on that $\mathcal{X}$ is a finite collection of non-negative discounted security prices (represented as the vector $\mathbf{Z}$), stopped at some finite horizon T, $P \in \mathcal{M}(\mathcal{X})$ and that $\mathcal{F}_0$ is P-trivial, so that, in particular, all the conditions of Theorem 3 are satisfied.

Definition 4: Following Harrison and Pliska (1981) we say that $X\,(\in \mathcal{F})$ is a contingent claim if $X \geq 0$ P-a.s. X is to be interpreted as a claim for an amount X to be paid at time T. If β_t is the discount factor at time t then X is said to be $P-$**integrable** if $\mathbb{E}_P \beta_T X < \infty$, and is said to be $P-$**attainable** if there exists a V such that:

i) $V_t = V_0 + \int_0^t \phi_s . d\mathbf{Z}_s$, where ϕ is predictable,

ii) V is a P-martingale,

iii) $V_T = \beta_T X$.

Finally X is said to be **bounded** if $||\beta_T X||_\infty < \infty$.

Harrison and Pliska (1981) define the price under P of a claim as

$$\pi_P(X) \stackrel{def}{=} \mathbb{E}_P \beta_T X,$$

and it follows that the price under P of a $P-$attainable claim X is

$$\pi_P(X) = V_0, \text{ given by i).}$$

We are now in a position to state Theorem 8.

Theorem 8 If X is a contingent claim then the following are equivalent:

(i) X is P-attainable

(ii) X has the same price under all $Q \in \mathcal{P}^0$

(iii) the process V given by

$$V_t = \mathbb{E}_P[\beta_T X | \mathcal{F}_t] \tag{6}$$

is a $Q-$martingale for all $Q \in \mathcal{P}^0$

Proof Taking V given by (6), it follows from Corollary 4 that, if (iii) holds,

$$V \in \mathcal{L}^1_{\text{loc}}(\mathcal{X} \cup \{1\}, P)$$

and since $V \in \mathbf{M}(P)$, X is P-attainable thus (iii)$\Rightarrow$(i).

If X is P-attainable then V is a Q-local martingale for all $Q \in \mathcal{P}^0$, but V is P-uniformly integrable and hence is Q-uniformly integrable for any $Q \in \mathcal{P}^0$ and hence is a Q-martingale, thus (iii)$\Leftarrow$(i).

If $V \in \mathbf{M}(Q)$ then

$$\pi_Q(X) = \mathbb{E}_Q V_T = V_0,$$

so (iii)$\Rightarrow$(ii).

Finally, if (ii) holds then, by (6) V is in $\mathbf{M}(P)$ and hence by successively applying Corollary 6 and Corollary 4, we deduce that

$$V \in \mathcal{L}^1_{\text{loc}}(\mathcal{X} \cup \{1\}, P)$$

and since $V \in \mathbf{M}(P)$, X is P-attainable, thus (ii)$\Rightarrow$(i) $\square$

We are also in a position to prove two conjectures from Harrison and Pliska (1981).

Theorem 9 If X is a bounded claim then it is P-attainable iff it is Q-attainable for all $Q \in \mathcal{P}$.

Remark This result is Proposition 2.1 of Stricker (1984)

Theorem 10 If $Q, R \in \mathcal{P}$ and X is both Q- and R-attainable then

$$\pi_Q(X) = \pi_R(X)$$

Proof of Theorem 9 If X is bounded and $P-$attainable then V given by (6) is in $\mathbf{H}^\infty(P) \cap \bigcap_{Q \in \mathcal{P}} \mathbf{M}_{\text{loc}}(Q)$: it follows immediately from the equality of the $\mathbf{L}^\infty(P)$ and $\mathbf{L}^\infty(Q)$ norms that

$$V \in \bigcap_{Q \in \mathcal{P}} \mathbf{H}^\infty(Q) \subseteq \mathbf{H}^1(R) \cap \bigcap_{Q \in \mathcal{P}} \mathbf{M}_{\text{loc}}(Q)$$

for all $R \in \mathcal{P}$, but from Theorem 3,

$$\mathbf{H}^1(R) \cap \bigcap_{Q \in \mathcal{P}} \mathbf{M}_{\text{loc}}(Q) = \mathcal{L}^1(\mathcal{X} \cup \{1\}, R)$$

hence V is $R-$representable $\square$

Corollary 11 If X is a bounded claim then it is P-attainable iff it is Q-attainable for all $Q \in \hat{\mathcal{P}}$.

Proof Just observe that in the proof of Theorem 5 we may conclude (using Theorem 3) that $V \in \mathbf{H}^\infty(R)$ for all $R \in \hat{\mathcal{P}}$ and the result follows $\square$

Theorem10 is a corollary of the following theorem.

Theorem 12 X is P-attainable if and only if

$$\pi_P(X) \geq \pi_Q(X) \text{ for all } Q \in \mathcal{P} \qquad (7).$$

Proof It follows from Theorem 3 that, if X is P-attainable then, for any Q in $\mathcal{P}$, V is a positive local martingale under Q and hence is a positive Q-supermartingale, thus

$$\pi_Q(X) = \mathbb{E}_Q \beta_T X = \mathbb{E}_Q V_T \leq V_0 = \mathbb{E}_P V_T = \mathbb{E}_P \beta_T X = \pi_P(X),$$

establishing the forward implication.

Conversely, suppose that (7) holds, then in particular the inequality holds for any $Q \in \mathcal{P}^0$. Suppose $Q \in \mathcal{P}^0$ with $||\frac{dQ}{dP}||_\infty = a$. Notice that $a \geq 1$, and that if we define

$$R = (1 + {}^1\!/_a)P - {}^1\!/_a Q,$$

then $R \in \mathcal{P}^0$. If we now apply (7) to R, then we see that

$$\pi_Q(X) \geq \pi_P(X),$$

so that equality must hold in (7) whenever $Q \in \mathcal{P}^0$, and it then follows immediately from the equivalence in Theorem 8 that X is $P-$attainable.

$\square$

Proof of Theorem 10 This follows immediately from Theorem 12 and the fact that $\mathcal{P}$ is an equivalence class under the obvious equivalence relation $\square$

References:

Harrison, J.M. and Kreps, D. M. (1979). "Martingales and arbitrage in multiperiod securities markets". J. Econ. Theory 20, 381-408.

Harrison, J.M. and Pliska, S. (1981). "Martingales and stochastic integrals in the theory of continuous trading". Stoch. Proc. Appl. 11, 215-260.

Harrison, J.M. and Pliska, S. (1983). "A stochastic calculus model of continuous trading: complete markets". Stoch. Proc. Appl. 15, 313-316.

Jacod, J. (1979). "Calcul Stochastique et Problémes de Martingales". Lecture Notes in Maths 714, Springer, Berlin-Heidelberg-New York.

Karatzas, I (1988). "On the pricing of American options". Appl. Math. Opt. 17, 37-60.

Karatzas, I (1989). "Optimization problems in the theory of continuous trading". SIAM. J. Cont. Opt. 27, 1221-1259.

Jameson, G.J.O. (1974). "Topology and Normed Spaces". Chapman and Hall, London.

Stricker, C. (1984). "Integral representation in the theory of continuous trading". Stochastics 13(**4**), 249–257.