The Propeller: a counterexample to a conjectured criterion for the existence of certain convex functions.

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The Propeller: a counterexample to a conjectured criterion for the existence of certain convex functions.

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Abstract: A counterexample is given to a conjecture by Emery; it is shown how to construct a domain $\mathcal{B}$ (the "propeller") of a two-dimensional Riemannian manifold such that any pair of points in $\mathcal{B}$ is connected by one and only one geodesic in $\mathcal{B}$ and yet there is no convex function $\mathcal{Q} : \mathcal{B} \times \mathcal{B} \to [0, 1]$ vanishing only on the diagonal of $\mathcal{B} \times \mathcal{B}$.

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1: Introduction.

Emery has made various bold conjectures concerning criteria for the existence of bounded convex functions defined on domains in Riemannian manifolds and required to vanish only on specified totally geodesic submanifolds [**2**–**4**]. Of these perhaps the most important is the following.

CONJECTURE: **(Emery's Conjecture)**. *A sufficient condition for the existence of a convex function $\mathcal{Q} : \mathcal{B} \times \mathcal{B} \to [0, 1]$ vanishing only on the diagonal of $\mathcal{B} \times \mathcal{B}$ is that between any two points of $\mathcal{B}$ there should exist one and only one connecting geodesic in $\mathcal{B}$.*

In this note there is constructed a counterexample to the conjecture in the form of a two-dimensional C^2 compact Riemannian manifold with piecewise-C^2 boundary, referred to below as the *propeller*.

Emery's conjecture is important in stochastic differential geometry because the convex function $\mathcal{Q}$ above is closely related to questions concerning Γ-martingales and the probabilistic approach to harmonic maps [**5**,**7**,**8**]. These relationships suggest the following terminology: if such a $\mathcal{Q}$ exists then we say $\mathcal{B}$ has *convex geometry*, while the phenomenon of existence of unique connecting geodesics is referred to as *existence and uniqueness for the geodesic Dirichlet problem for* $\mathcal{B}$. In [**7**] convex geometry and the geodesic Dirichlet problem are related to two other nonlinear Dirichlet problems involving respectively Γ-martingales and harmonic maps.

Note that the notion of "strongly convex" described in [**1**] is related to existence and uniqueness for the geodesic Dirichlet problem. Note also that Emery discusses these questions in the more general setting of manifolds with nonmetric connections, which is also of importance in stochastic differential geometry (see also [**7**,**8**]).

Emery [2] showed that convex geometry is possessed by sufficiently small geodesic balls. In [5,6] this was quantified by exhibiting direct constructions of convex geometry for compact subsets of open hemispheres and, more generally, for regular geodesic balls. However the task of dealing directly with questions of existence and non-existence of such convex functions $\mathcal{Q}$ appears in general to be hard. This prompted the formulation (in [7,8]) of a number of related but weaker conjectures, of independent interest and in combination equivalent to Emery's conjecture, and the proof of one of them (that convex geometry for a compact domain is equivalent to existence, uniqueness. and well-posedness for a Γ-martingale Dirichlet problem).

There is a further relationship, not touched on in [7,8] but referred to in [5], which relates convex geometry to uniqueness questions for Riemannian centres–of–mass (described as "Karcher means" in [5]). If a domain $\mathcal{B}$ possesses convex geometry then a simple argument due to Emery shows that finite mass measures in $\mathcal{B}$ must have unique Riemannian centres–of–mass. Therefore one possible way to obtain a counterexample for Emery's conjecture is to construct a domain whose geodesic Dirichlet problem has solutions which exist and are unique, but which supports finite mass measures with non-unique Riemannian centres–of–mass in the domain. The following note provides just such a construction. Hence Emery's conjecture does not hold.

The paper is divided into sections as follows. After this introductory section, section 2 describes the construction of the propeller manifold $\mathcal{P}_{\epsilon,\eta}$ parametrized by two positive numbers ϵ and η such that $\eta < \epsilon$. In section 3 it is shown that existence holds for the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$ when η is smaller than a specific bound. Section 4 shows that uniqueness holds when η is sufficiently small. Section 5 shows that Riemannian centres–of–mass in $\mathcal{P}_{\epsilon,\eta}$ are nonunique when ϵ is sufficiently small. Section 6 summarizes the counterexample, as indicated above, and discusses some further issues.

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2: Construction of the Propeller.

The *propeller* $\mathcal{P}_{\epsilon,\eta}$ is a two-dimensional C^2 compact Riemannian manifold with piecewise-C^2 boundary, defined as follows when $\epsilon > \eta > 0$ and $\epsilon < 3\pi/8$ and $\eta < \pi/3$. Let (r, θ) form a system of geodesic polar coordinates for the hyperbolic plane $\mathbb{H}^2$ based at $\mathbf{o} \in \mathbb{H}^2$. Consider the open ball $\mathcal{D}$ centred at $\mathbf{o}$, of radius π, and equipped with a new metric depending on ϵ and η and defined below. The propeller $\mathcal{P}_{\epsilon,\eta}$ is a compact subset of $\mathcal{D}$ specified below.

Before giving a precise specification of $\mathcal{P}_{\epsilon,\eta}$ we define the new metric. This is such that (r, θ) remains a system of geodesic polar coordinates based on $\mathbf{o}$. The metric is therefore fully defined by specifying the curvature function $\kappa = \kappa_{\epsilon,\eta}$ on $\mathcal{D}$. To define the curvature we first define three *propeller blades* $\mathcal{B}^0_{\epsilon,\eta}$, $\mathcal{B}^{2\pi/3}_{\epsilon,\eta}$, $\mathcal{B}^{-2\pi/3}_{\epsilon,\eta}$, open subsets of $\mathcal{D}$. The first propeller blade $\mathcal{B}^0_{\epsilon,\eta}$ is defined by

$$\mathcal{B}^0_{\epsilon,\eta} \quad = \quad \{(r, \theta) \in \mathcal{D} \ : \ |\theta| < \eta, \ r \in (\gamma_\epsilon(\theta), \pi)\}$$

where the function $\theta \mapsto (\gamma_\epsilon(\theta), \theta)$ defines the locus of a geodesic through $(\epsilon, 0)$ perpendicular to the $\theta = 0$ axis under the original hyperbolic metric. The blades $\mathcal{B}^{2\pi/3}_{\epsilon,\eta}$, $\mathcal{B}^{-2\pi/3}_{\epsilon,\eta}$ are obtained from $\mathcal{B}^0_{\epsilon,\eta}$ by rotation about $\mathbf{o}$ through angles of $2\pi/3$, $-2\pi/3$ respectively. The restriction $\eta < \pi/3$ ensures the propeller blades do not overlap. It is also useful to define the *spines* $\mathcal{S}^0_\epsilon$, $\mathcal{S}^{2\pi/3}_\epsilon$, $\mathcal{S}^{-2\pi/3}_\epsilon$ of the propeller blades by

$$\mathcal{S}^\theta_\epsilon \quad = \quad \{(r, \theta) : 2\epsilon \le r < \pi\} \ .$$

The curvature function $\kappa = \kappa_{\epsilon,\eta}$ is now defined for $x \in \mathcal{D}$ (using the distance $\mathrm{dist}_{\mathbb{H}}$ obtained by viewing $\mathcal{D}$ as a subset of $\mathbb{H}^2$) by

$$\kappa_{\epsilon,\eta}(x) \quad = \quad \frac{\mathrm{dist}_{\mathbb{H}}(x, \mathcal{D} \setminus \mathcal{B}^\theta_{\epsilon,\eta}) \ - \ \mathrm{dist}_{\mathbb{H}}(x, \mathcal{S}^\theta_\epsilon)}{\mathrm{dist}_{\mathbb{H}}(x, \mathcal{D} \setminus \mathcal{B}^\theta_{\epsilon,\eta}) \ + \ \mathrm{dist}_{\mathbb{H}}(x, \mathcal{S}^\theta_\epsilon)}$$

$$\text{if } x \in \mathcal{B}^\theta_{\epsilon,\eta} \text{ for } \theta = 0, \ 2\pi/3, \text{ or } -2\pi/3 \ ,$$

$$\kappa_{\epsilon,\eta}(x) \quad = \quad -1 \qquad \text{otherwise.}$$

Consequently the curvature of the new metric lies everywhere between ± 1. Hence the system (r, θ) of geodesic polar coordinates is nonsingular within $\mathcal{D}$ under the new metric (though it may degenerate at the boundary $\partial\mathcal{D}$). This follows by a geometric comparison argument, comparing to the open ball of the same radius π under the unit sphere metric of constant positive curvature 1. Under the new metric $\mathcal{D}$ admits a $\mathbb{Z}_2 \times \mathbb{Z}_3$ group of isometries generated by $(r, \theta) \mapsto (r, -\theta)$ and $(r, \theta) \mapsto (r, \theta + 2\pi/3)$.

Remark: Of course it would be possible to make the curvature a smooth function by suitable modification of this formula. The essential points are: the curvature is bounded everywhere by ± 1, the curvature is equal to -1 over most of $\mathcal{P}_{\epsilon,\eta}$ for small η, and the curvature is equal to $+1$ along the spines. We choose the C^2 case simply for ease of exposition.

The propeller $\mathcal{P}_{\epsilon,\eta}$ is the closed and compact subset of $\mathcal{D}$ given by

$$\mathcal{P}_{\epsilon,\eta} \quad = \quad \{(r, \theta) \in \mathcal{D} : r \leq R_\eta(\theta)\}$$

where the continuous function R_η is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetry described above and

$$\begin{aligned} R_\eta(0) &= 3\pi/4\,, \\ \tan R_\eta(\theta) &= -\sec\theta \qquad \text{for } -\eta \leq \theta \leq \eta\,, \\ R_\eta(\theta) &= R_\eta(\eta)\cos(\pi/3 - \eta)\sec(\pi/3 - \theta) \\ &\qquad \text{for } \eta \leq \theta \leq 2\pi/3 - \eta\,. \end{aligned}$$

The restriction $\epsilon < 3\pi/8$ ensures that the propeller overlaps the spines of the propeller blades. The construction of $\mathcal{P}_{\epsilon,\eta}$ is illustrated in the figure.

A final definition proves convenient at a later stage of the paper:

$$\begin{aligned} \mathcal{D}_{(\eta,+)} &= \{(r, \theta) \in \mathcal{D} : \theta \in [\eta, \pi] \text{ or } \theta \in [-\pi, -\pi + \eta]\} \\ \mathcal{D}_{(\eta,-)} &= \{(r, \theta) \in \mathcal{D} : \theta \in [-\pi + \eta, \eta]\} \end{aligned}$$

Figure: The rounded triangle is the propeller manifold $\mathcal{P}_{\epsilon,\eta}$ in geodesic polar coordinates.

3: Existence for the Propeller Geodesic Dirichlet problem.

In this section it is shown that when η is sufficiently small then solutions to the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$ always exist.

The property of existence for the geodesic Dirichlet problem is implied by a local geodesic convexity property for compact domains. This kind of argument is well-known; we give it here because we are dealing with a non-complete Riemannian manifold $\mathcal{D}$, because it is convenient to work with general open domains, and for the sake of a self-contained exposition.

DEFINITION *3.1:* **Local strict convexity**. *Let U be an open set with a topological boundary ∂U in a (not necessarily complete) smooth Riemannian manifold. We say U is* **locally strictly convex** *if for any point $x \in \overline{U} = U \cup \partial U$ there is a neighbourhood W of x such that if y, $\tilde{y}$ are in $W \cap \overline{U}$ then any geodesic connecting y to $\tilde{y}$ and lying in W must lie entirely in U with the possible exception of its endpoints. We say such a neighbourhood W is a neighbourhood of strict convexity for U.*

Remark: Local strict convexity in two-dimensional cases (as in the examples below) follows when the boundary is piecewise-smooth, with corners forming interior angles and with the smooth portions having nonzero inward-pointing curvature vectors.

PROPOSITION *3.2: Let U be an open set with topological boundary ∂U in a Riemannian manifold $\mathbb{M}$. Suppose U is connected and $\overline{U} = U \cup \partial U$ is compact. If U is locally strictly convex then existence holds for the geodesic Dirichlet problem for $\overline{U}$.*

Proof: Let x, y be two points in $\overline{U}$. We have to construct a geodesic connecting x and y and lying entirely within $\overline{U}$. In fact the constructed geodesic will lie entirely within U with the possible exception of its endpoints.

Because $\overline{U}$ is compact there is $\delta > 0$ such that any ball centre z and radius less than δ is a normal convex neighbourhood for every z in $\overline{U}$. (Recall that in a normal convex neighbourhood any pair of points are connected together by a unique minimal geodesic.) Furthermore by compactness of $\overline{U}$ and local strict convexity of U we may choose δ so that there is a finite covering of $\overline{U}$ by balls of radius $\delta/2$ and centres w_1, ..., w_k, such that the balls of radius δ and centres w_1, ..., w_k are neighbourhoods of strict convexity for U as well as being normal convex neighbourhoods.

By connectedness of $\overline{U}$ there is a smooth path connecting x to y in $\overline{U}$. Hence for some integer N there are z_0, ..., z_N in $\overline{U}$ with $z_0 = x$, $z_N = y$, and $\mathrm{dist}(z_{i-1}, z_i) \leq \delta/2$. By our choice of δ the minimal geodesic between z_{i-1} and z_i must lie in U for each $i = 1, ..., N$, with the possible exception of its endpoints. For such N choose the sequence of z_i so as to minimize

$$L = \sum_{i=1}^{N} \mathrm{dist}(z_{i-1}, z_i)$$

under the constraints that $\mathrm{dist}(z_{i-1}, z_i) \leq \delta/2$ and the minimal geodesic between z_{i-1} and z_i must lie in $\overline{U}$ for each $i = 1, ..., N$.

Let $\gamma : [0, L] \to \overline{U}$ be the continuous path which is a broken unit-speed geodesic passing through each of the z_i in turn and broken only at the z_i.

Consider that portion of γ which runs through z_{i-1}, z_i, z_{i+1}. The point z_i lies in a ball of radius $\delta/2$ and centre w_j for some j. So by the triangle inequality all of z_{i-1}, z_i, z_{i+1} lie in the ball of radius δ and centre w_j, which is a neighbourhood of strict convexity for U and a normal convex neighbourhood. As a consequence the minimal geodesic from z_{i-1} to z_{i+1} must lie within U save possibly at the endpoints. Hence by minimality of the choice of the sequence of z_i it follows that $z_i \in U$ and the broken unit-speed geodesic γ is smooth at z_i.

It follows that γ is smooth everywhere and hence is a genuine geodesic as required. □

For small enough η the propeller $\mathcal{P}_{\epsilon,\eta}$ is locally strictly convex. For the boundary $\partial\mathcal{P}_{\epsilon,\eta}$ is piecewise-smooth. Calculation shows that the six corners form interior angles when η satisfies

$$\frac{\sin\eta}{1+\cos^2\eta} < \arctan\left(-\sec\eta\right)\tan\left(\pi/3-\eta\right)$$

where the branch of arctangent is chosen to take values in $[0, \pi]$. This holds for sufficiently small and positive η. On the other hand the curvature vectors of the smooth sections are inward-pointing by comparison arguments. For the section corresponding to $\theta \in (-\eta, \eta)$ is formed as part of the locus of a great circle based at $(\pi/4, 0)$ if the geodesic polar coordinates are regarded as those of a sphere of curvature $+1$, the section corresponding to $\theta \in (\eta, 2\pi/3 - \eta)$ is formed as part of the locus of a line segment if the geodesic polar coordinates are regarded as those of the Euclidean plane of curvature zero, and the remaining sections correspond to one or the other of these two under the $\mathbb{Z}_2 \times \mathbb{Z}_3$ isometries. In each case the constant curvatures used to construct the loci strictly dominate the actual curvature function of $\mathcal{P}_{\epsilon,\eta}$ in the corresponding angular wedge based on $\mathbf{o}$. It follows from the formula for second variation of arc-length (see for example [**1** chapter 1 section 6]) that the curvature vectors are nonzero and inward-pointing on each smooth segment of $\partial\mathcal{P}_{\epsilon,\eta}$.

Since $\mathcal{P}_{\epsilon,\eta}$ is compact it follows from the proposition that existence holds for the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$.

4: Uniqueness for the Propeller Geodesic Dirichlet problem.

In this section it is shown that for any fixed $\epsilon \in (0, 3\pi/8)$ the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$ has unique solutions when η is sufficiently small. This is established by means of a number of steps.

Step 1: It suffices for $\mathcal{P}_{\epsilon,\eta}$ to be free of conjugate geodesic segments.

PROPOSITION *4.1: Suppose $\mathcal{P}_{\epsilon,\eta}$ is free of conjugate geodesic segments. Then no two points in $\mathcal{P}_{\epsilon,\eta}$ are joined by more than one geodesic in $\mathcal{P}_{\epsilon,\eta}$.*

Proof: This is a well-known argument; see for example [**5** Theorem (1.7)] where it is applied to regular geodesic balls. The argument for $\mathcal{P}_{\epsilon,\eta}$ is identical to the regular geodesic ball case: if $\mathcal{P}_{\epsilon,\eta}$ is free of conjugate geodesic segments then the Riemannian exponential map Exp_p must have fibres of constant finite cardinality for each $p \in \mathcal{P}_{\epsilon,\eta}$, and the fibre cardinality must be locally constant in p. The result follows from the non-degeneracy of the system of geodesic polar coordinates based on $\mathbf{o}$. □

Step 2: There is a dichotomy between those geodesics which make small angles with polar geodesics and those which make large angles. Let $\rho_\theta : (-\pi, \pi) \to \mathcal{D}$ be the polar geodesic with $\rho_\theta(0) = \mathbf{o}$ and with tangent vector $(1, \theta)$ at that point. The dichotomy is expressed as follows.

PROPOSITION *4.2: Consider $\mathcal{P}_{\epsilon,\eta}$ and $\mathcal{D}$ with the corresponding metric, for fixed ϵ, η. For each $\xi > 0$ there is a $\phi > 0$ such that the following holds. Let γ be a unit–speed geodesic in $\mathcal{D}$ intersecting a polar geodesic ρ_θ at $\gamma(0) = \rho_\theta(u)$ in $\mathcal{P}_{\epsilon,\eta}$ and such that*

$$|\langle \gamma'(0), \operatorname{grad} r \rangle| \quad > \quad \cos\phi\,.$$

Then

$$\sup\left\{\operatorname{dist}(\gamma(t-u), \rho_\theta(t)) : t \in (-\pi, \pi) \textit{ and } \gamma(t-u), \rho_\theta(t) \in \mathcal{D}\right\} \; < \; \xi\,.$$

Proof: This is a consequence of continuous dependence on initial conditions for the system of differential equations for geodesics in $\mathcal{D}$, the compactness of $\mathcal{P}_{\epsilon,\eta}$, and the bound 2π on the lengths of all geodesics in $\mathcal{D}$ passing through the origin $\mathbf{o}$. □

Remark: Consequently a geodesic segment in $\mathcal{P}_{\epsilon,\eta}$ making a small angle with a polar geodesic can be uniformly approximated by the polar geodesic. All other geodesic segments in $\mathcal{P}_{\epsilon,\eta}$ must always cut polar geodesics at angles bounded away from zero by ϕ.

Step 3: One need consider only those conjugate geodesic segments which hit at most two of the propeller blades $\mathcal{B}^0_{\epsilon,\eta}$, $\mathcal{B}^{2\pi/3}_{\epsilon,\eta}$, $\mathcal{B}^{-2\pi/3}_{\epsilon,\eta}$. For suppose a geodesic segment hits all three propeller blades. This geodesic must make multiple hits on at least one of these geodesics forming boundaries for $\partial\mathcal{D}_{(\eta,+)}$ and its transforms under the $\mathbb{Z}_2 \times \mathbb{Z}_3$ isometries. Thus the existence of such a geodesic forces one of $\mathcal{D}_{(\eta,+)} \cap \mathcal{P}_{\epsilon,\eta}$, $\mathcal{D}_{(\eta,-)} \cap \mathcal{P}_{\epsilon,\eta}$ to have a non-unique geodesic Dirichlet problem. Arguing as in proposition 4.2, it follows that one of $\mathcal{D}_{(\eta,+)} \cap \mathcal{P}_{\epsilon,\eta}$, $\mathcal{D}_{(\eta,-)} \cap \mathcal{P}_{\epsilon,\eta}$ contains a conjugate geodesic segment, and this segment can hit at most two of the propeller blades $\mathcal{B}^0_{\epsilon,\eta}$, $\mathcal{B}^{2\pi/3}_{\epsilon,\eta}$, $\mathcal{B}^{-2\pi/3}_{\epsilon,\eta}$.

Step 4: Clearly geodesic segments of the second kind in Step 2 above exhibit a relationship between changes in angle and length.

PROPOSITION *4.3: Let $\gamma : [0, L] \to \mathcal{P}_{\epsilon,\eta}$ be a unit-speed geodesic such that*

$$|\langle \gamma', \operatorname{grad} r\rangle| \quad < \quad \cos\phi$$

everywhere. Then

$$|\theta \circ \gamma(L) - \theta \circ \gamma(0)| \quad > \quad \frac{L|\sin\phi|}{\sinh(3\pi/4)}\,.$$

Proof: The metric of $\mathcal{P}_{\epsilon,\eta}$ is given by $ds^2 = dr^2 + G_{\epsilon,\eta}(r,\theta)^2 d\theta^2$ where standard geometric comparison arguments show that $G_{\epsilon,\eta}(r,\theta) \le \sinh r \le \sinh(3\pi/4)$. By the condition on γ'

$$G_{\epsilon,\eta}(r,\theta)\frac{d\theta}{ds} \quad > \quad |\sin\phi|\,.$$

The result follows by integration. □

Step 5: A Jacobi vectorfield argument shows that geodesic segments in $\mathcal{P}_{\epsilon,\eta}$ running through only two sufficiently small intervals of curvature greater than -1 cannot be conjugate.

PROPOSITION *4.4: Consider a unit–speed geodesic segment $\gamma : [0, L] \to \mathcal{P}_{\epsilon,\eta}$ running within the region of curvature -1 save for two disjoint time-intervals $[a, b]$, $[c, d]$ such that*

$$(d-c)+(b-a) \quad < \quad \pi/2\,.$$

Then γ is a nonconjugate geodesic segment.

Proof: A standard Jacobi vectorfield argument shows that it suffices to consider the solution J to the second-order differential equation over $[0, L]$ given by

$$\begin{aligned} J'' + \tilde{\kappa} J &= 0 \\ J(0) = 0, &\quad J'(0) = 1 \end{aligned}$$

where

$$\tilde{\kappa}(t) \quad = \quad \begin{cases} +1 & \text{if } t \in [a,b] \cup [c,d] \\ -1 & \text{otherwise.} \end{cases}$$

If all such J never vanish over $[0, L]$ then comparison arguments show that γ is nonconjugate. However J can be solved for explicitly using trigonometric or hyperbolic functions according to the different regimes of curvature. We summarize the argument based on these computations. Clearly J will not vanish over $[0, a]$, and

$$(J'/J)(a) \quad = \quad \coth(a) \quad > \quad 1 = \cot(\pi/4)\,.$$

By monotonicity of cot over $(0, \pi)$ it follows J does not vanish over $[a, b]$ and indeed

$$(J'/J)(b) \quad > \quad \cot(b-a+\pi/4) \quad > \quad -1$$

(because $b - a < \pi/2$). Thus $(J'/J)(t)$ over the range $[b, c]$ will equal either $\coth(t-b+u)$ for some positive u (hence bounded below by 1) or $\tanh(t-b+u)$

for some u (hence monotonically increasing). In either case J will not vanish there, and since $\cot(b-a+\pi/4) \leq 1$ it follows

$$(J'/J)(c) \quad \geq \quad \cot(b-a+\pi/4)\,.$$

Consequently (using again the monotonicity of cot over $(0,\pi)$) it follows J does not vanish over $[c,d]$ and

$$(J'/J)(d) \quad \geq \quad \cot(d-c+b-a+\pi/4) \quad > \quad -1$$

(because $d-c+b-a<\pi/2$). Solving over the range $t \in [d,L]$ we see

$$(J'/J)|_{[d,L]} \quad > \quad -1\,.$$

So J does not vanish over $[0,L]$. □

Step 6: In this final step we put together the information from the previous steps to show that for fixed $\epsilon>0$ the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$ has unique solutions when η is sufficiently small.

PROPOSITION 4.5: *Fix $\epsilon>0$. When η is sufficiently small then for any two points in $\mathcal{P}_{\epsilon,\eta}$ there is at most one connecting geodesic in $\mathcal{P}_{\epsilon,\eta}$.*

Proof: By the proposition of step 1 and the argument of step 3 it suffices to show that there are no conjugate geodesic segments hitting at most two of the propeller blades. Suppose $\gamma:[0,L]\to\mathcal{P}_{\epsilon,\eta}$ were such a geodesic segment. We shall establish a contradiction and hence prove the result.

Determine ξ to be the largest number in $(0,\pi/4]\cap(0,\epsilon/2]$ such that no radial geodesic hits more than one of the (2ξ)-dilations of the propeller blades, where for example the β-dilation of $\mathcal{B}^0_{\epsilon,\eta}$ for $\beta>0$ is given by

$$\left\{x\in\mathcal{P}_{\epsilon,\eta} : \operatorname{dist}(x,\mathcal{B}^0_{\epsilon,\eta}) \leq \beta\right\}$$

Note that the overlap of any one radial geodesic with a ξ-dilation of a propeller blade will be contained in one of $[-\pi, -\xi]$ or $[\xi, \pi]$. (Here we use $\xi \leq \epsilon/2$.)

By step 2 there is a $\phi > 0$ such that $|\langle\gamma'(u), \operatorname{grad} r\rangle| > \cos\phi$ at any point $\gamma(u)$ implies

$$\sup\{\operatorname{dist}(\gamma(t-u), \rho_\theta(t)) : t-u \in [0, L], \rho_\theta(t) \in \mathcal{D}\} \quad < \quad \xi$$

for some radial geodesic ρ_θ. Indeed the geodesic γ may then be continued, and its time reparametrized to belong to an interval I, so that $I \subset [-3\pi/4 - \xi, 3\pi/4 + \xi]$ and

$$\sup\{\operatorname{dist}(\gamma(t), \rho_\theta(t)) : t \in I\} \quad < \quad \xi\,.$$

Thus if $|\langle\gamma', \operatorname{grad} r\rangle| > \cos\phi$ at any stage then γ hits at most one propeller blade. Furthermore by construction of ξ and under the extension and reparametrization above the collision of γ with this blade may then be contained in one of the time intervals $[-3\pi/4 - \xi, -\xi] \cap I$ or $[\xi, 3\pi/4 + \xi] \cap I$.

In this case we may employ standard Jacobi vectorfield comparison techniques to deduce, the geodesic segment γ can be conjugate only if J has a zero in $(-3\pi/4, 3\pi/4]$, where J is the solution to

$$\begin{aligned} J'' + \kappa^* J \quad &= \quad 0 \\ J(-3\pi/4) = 0, \quad &J'(-3\pi/4) = 1 \end{aligned}$$

where

$$\kappa^*(t) \quad = \quad \begin{cases} +1 & \text{if } t \in [-3\pi/4, 0] \\ -1 & \text{if } t \in (0, 3\pi/4]. \end{cases}$$

Here the range $[-3\pi/4, 0]$ arises from the above considerations of how the radial geodesic ρ_θ intersects the ξ-dilations of propeller blades. It is a matter of direct calculation to solve explicitly for J and to verify that J is nonzero over $(-3\pi/4, 3\pi/4]$.

Consequently if γ is to be a conjugate geodesic segment then it must form angles with $\operatorname{grad} r$ which are everywhere greater than ϕ in absolute value. It follows from this, from step 4, and from the requirement that γ hit at most two propeller blades, that the curvature along γ must be -1 everywhere except for two intervals each of length less than

$$\frac{2\eta\sinh(3\pi/4)}{|\sin\phi|}\,.$$

So if η is chosen with

$$\eta \quad < \quad \tfrac{1}{2}\cdot\frac{|\sin\phi|}{2\sinh(3\pi/4)}\cdot\frac{\pi}{2}$$

then by step 5 it follows γ cannot be conjugate. $\square$

It follows that for any fixed ϵ if η is chosen sufficiently small then the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$ has solutions which exist and are unique.

Remark: of course it then follows from the theory of geodesics that the geodesic Dirichlet problem for $\mathcal{P}_{\epsilon,\eta}$ is also well-posed.

5: Nonunique Riemannian Centres–of–Mass in the Propeller.

Consider the finite mass distribution on $\mathcal{P}_{\epsilon,\eta}$ given by

$$\mu \quad = \quad \delta_{(3\pi/4,0)} + \delta_{(3\pi/4,3\pi/2)} + \delta_{(3\pi/4,-3\pi/2)}\,.$$

For sufficiently small ϵ, η this fails to have a unique Riemannian centre–of–mass in $\mathcal{P}_{\epsilon,\eta}$. We establish this by showing that the second derivative or Hessian of its energy function

$$\mathcal{V}(x;\mu) \quad = \quad \tfrac{1}{2}\int \operatorname{dist}(x,y)^2\,\mu(dy)$$

is negative-definite at the origin $\mathbf{o}$ for sufficiently small ϵ, η, so that the origin cannot be a local minimum. The measure and the Riemannian manifold $\mathcal{P}_{\epsilon,\eta}$ both respect the $\mathbb{Z}_2\times\mathbb{Z}_3$ group of isometries, and $\mathbf{o}$ is the only fixed point of this symmetry group. Moreover by the local strict convexity of $\mathcal{P}_{\epsilon,\eta}$ there must be local minima of the energy function $\mathcal{V}$. It follows that for sufficiently small ϵ, η there must be at least **three** local minima of $\mathcal{V}$ (which are by definition the Riemannian centres–of–mass of μ) within $\mathcal{P}_{\epsilon,\eta}$.

First note that by symmetry $\operatorname{grad}\mathcal{V}(\mathbf{o}) = 0$. Moreover

$$\mathcal{V}(x;\mu) \quad = \quad \sum_{\theta=0,2\pi/3,-2\pi/3} \tfrac{1}{2}\operatorname{dist}(x,(3\pi/4,\theta))^2\,.$$

Suppose γ is a geodesic running through $\mathbf{o}$, with $\gamma(0) = \mathbf{o}$, $\gamma'(0) = (1,\alpha)$. Then

$$\begin{aligned}\left[\frac{d^2}{du^2}\mathcal{V}(\gamma(u);\mu)\right]_{u=0} &= \tfrac{1}{2}\sum_{\theta=0,2\pi/3,-2\pi/3}\left[\frac{d^2}{du^2}\operatorname{dist}(\gamma(u),(3\pi/4,\theta))^2\right]_{u=0}\\ &= \sum_{\theta=0,2\pi/3,-2\pi/3}(3\pi/4)\cdot\left[\frac{d^2}{du^2}\operatorname{dist}(\gamma(u),(3\pi/4,\theta))\right]_{u=0}\end{aligned}$$

and we can estimate the second derivatives on the right-hand side using Jacobi vectorfields. By symmetry it suffices to consider the case $\theta = 0$:

$$\left[\frac{d^2}{du^2}\operatorname{dist}(\gamma(u),(3\pi/4,0))\right]_{u=0} \quad = \quad [\langle J', J\rangle]_0^{3\pi/4}$$

where $J(0) = (1,\alpha)$, $J(3\pi/4) = 0$, $J'' + \kappa J = 0$.

The component J^p parallel to $(1,0)$ is linear in arclength, so

$$J^p(s) \quad = \quad \cos\alpha\left(1 - \frac{s}{3\pi/4}\right)\,.$$

The classical theory of Jacobi vectorfields shows that the component J^n normal to $(1,0)$ is dominated by $\tilde{J}$ in the sense that $[(J^n)'J^n]_0^{3\pi/4} \leq [(\tilde{J})'\tilde{J}]_0^{3\pi/4}$, where $\tilde{J}$ is the solution of

$$\begin{aligned}\tilde{J}(0) &= \sin\alpha\\ \tilde{J}(3\pi/4) &= 0\\ \tilde{J}'' - \tilde{J} &= 0 \qquad \text{over } [0,2\epsilon]\\ \tilde{J}'' + \tilde{J} &= 0 \qquad \text{over } (2\epsilon,3\pi/4]\,.\end{aligned}$$

This has the explicit solution

$$
\begin{aligned}
\tilde{J}(s) &= \sin\alpha\cosh s + \tilde{J}'(0)\sinh s \qquad \text{over } [0, 2\epsilon] \\
\tilde{J}(s) &= (\sin\alpha\cosh(2\epsilon) + \tilde{J}'(0)\sinh(2\epsilon))\cos(s-2\epsilon) + \\
&\quad + \ (\sin\alpha\sinh(2\epsilon) + \tilde{J}'(0)\cosh(2\epsilon))\sin(s-2\epsilon) \qquad \text{over } (2\epsilon, 3\pi/4]\,.
\end{aligned}
$$

Hence $\tilde{J}(3\pi/4) = 0$ implies

$$
\tilde{J}'(0) = \sin\alpha\frac{\cos(\pi/4+2\epsilon)\cosh(2\epsilon) - \sin(\pi/4+2\epsilon)\sinh(2\epsilon)}{\sin(\pi/4+2\epsilon)\cosh(2\epsilon) - \cos(\pi/4+2\epsilon)\sinh(2\epsilon)}\,.
$$

It follows,

$$
\begin{aligned}
\langle J'(0), J(0)\rangle &= \langle (J^p)'(0), J^p(0)\rangle \ + \ \langle (J^n)'(0), J^n(0) \\
\geq \quad & -\cos^2\alpha\frac{4}{3\pi} \ + \ \sin^2\alpha\frac{\cos(\pi/4+2\epsilon)\cosh(2\epsilon) - \sin(\pi/4+2\epsilon)\sinh(2\epsilon)}{\sin(\pi/4+2\epsilon)\cosh(2\epsilon) - \cos(\pi/4+2\epsilon)\sinh(2\epsilon)}\,.
\end{aligned}
$$

Summing this and the corresponding terms arising from the other two summands we find

$$
\left[\frac{d^2}{du^2}\,\mathcal{V}(\gamma(u);\mu)\right]_{u=0} \quad \leq \quad \left[\frac{4}{3\pi} - \frac{\cot(\pi/4+2\epsilon) - \tanh(2\epsilon)}{1 - \tanh(2\epsilon)\cot(\pi/4+2\epsilon)}\right]\cdot\left(\frac{9\pi}{8}\right)\,.
$$

Thus for sufficiently small $\epsilon > 0$ we find

$$
\left[\frac{d^2}{du^2}\,\mathcal{V}(\gamma(u);\mu)\right]_{u=0} \quad < \quad 0
$$

for all geodesics γ starting at the origin $\mathbf{o}$, as required.

6: Counterexample to Emery's Conjecture.

We have established that for sufficiently small ϵ, η the propeller $\mathcal{P}_{\epsilon,\eta}$ has a geodesic Dirichlet problem with solutions that exist and are unique, and supports finite mass measures with non-unique Riemannian centres–of–mass.

In the language of the present note Emery [**2**–**4**] conjectured that existence and uniqueness for the Dirichlet problem formed a sufficient condition for a compact domain $\mathcal{B}$ to support *convex geometry*, which is to say a convex function $\mathcal{Q} : \mathcal{B} \times \mathcal{B} \to [0,1]$ vanishing only on the diagonal of $\mathcal{B} \times \mathcal{B}$. But one of the consequences of convex geometry is the uniqueness of Riemannian centres–of–mass; see the argument in [**5** Theorem (7.3)], which was supplied by Emery (personal communication). It follows for sufficiently small ϵ, η that $\mathcal{P}_{\epsilon,\eta}$ as above cannot have convex geometry and therefore supplies a counterexample to Emery's conjecture.

It will be clear from the above that the construction may be modified easily to provide a counterexample which is a smooth Riemannian manifold with (smooth) boundary.

It is of course possible that further restrictions on the domain might rescue the conjecture (for example one might require that the metric be rotationally symmetric or have nonnegative curvature). Note also that in [**7**] there is expounded a sequence of conjectured relationships between various nonlinear Dirichlet problems and convex geometry which together would imply the truth of Emery's conjecture. Some of these conjectures must therefore be false – the most likely candidate is (**D**)$\Rightarrow$(**C**) in the terminology of [**7**] – and it is an interesting question which of these conjectures fail, since they each possess intrinsic interest in the field of stochastic differential geometry.

The details of the construction of the propeller lead to a couple of interesting questions in classical and in stochastic differential geometry. Firstly, the intuition behind the construction of the propeller was to build a C^2 smoothing of a piecewise C^2 Riemannian manifold with piecewise constant curvature set at $+1$ on the propeller blades and -1 elsewhere. It would be interesting to investigate to what extent the constructions of stochastic differential geometry (stochastic parallel transport, *et cetera*) could be carried out in such a context, thus facilitating explicit computations and the construction of examples. Secondly, consideration of the propeller leads one to ask the following questions. Given two Riemannian manifolds expressed in exponential charts,

(i) what functional of the difference between the two curvatures would usefully measure the extent to which the two geodesic flows deviate from each other?

(ii) and what functional would usefully measure the deviation between Brownian motions on the two manifolds expressed in terms of stochastic development of a single Euclidean Brownian motion?

We conclude by noting that in [**3**] there are formulated other conjectured criteria for similar convexity problems. It is not clear whether the construction of this note can be modified to deal with these other problems, since there appears to be a lack of concepts playing rôles analogous to that of the Riemannian centre–of–mass above.

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