A Cross-currency Markov-functional model with FX volatility skew

Jaka Gogala and Joanne Kennedy
Department of Statistics
The University of Warwick
Coventry, CV4 7AL

First version: Jan 16
This version: June 20

Abstract

In this paper we propose a two-currency Markov-functional model which can calibrate to the domestic and foreign caplet prices and foreign exchange call option prices across different maturities and strikes. The model is particularly suited for pricing power-reverse dual-currency (PRDC) swaps which are sensitive to the skew exhibited by the FX options.

1 Introduction

In the single currency setting Markov-functional models (MFMs), introduced by Hunt et al. [4], have been popular with practitioners because of their efficiency and their ability to calibrate to any arbitrage free formula for caplet and swaptions prices. Most of the literature has been focused on models driven by Gaussian processes. A recent paper by Gogala and Kennedy [3] provides new and efficient algorithms that can be used to implement a MFM driven by a not necessarily Gaussian one dimensional process.

In this paper we build on the techniques developed in Gogala and Kennedy [3] to develop a versatile three factor cross-currency Markov-functional model. This model is able to calibrate via a "Markov-functional sweep" to any arbitrage free formula for FX options across different maturities and strikes as well as offering the usual calibration to the market implied distributions of caplets in each currency that can be achieved by a single currency MFM (A swap form of the model can easily be formulated to calibrate to swaptions prices in each currency instead.)

The most widely traded cross-currency exotic in the Asian markets are Power-Reverse Dual-Currency (PRDC) swaps. PRDCs have coupons which are options on the FX rate and these options exhibit significant skew. It was shown in Piterbarg [5] that the FX skew has a significant impact on the pricing of PRDCs and cannot be ignored in the modelling process. The model described in this paper has the potential to calibrate to the foreign exchange market -matching the market prices for the relevant vanilla FX options exactly- as well as calibrate to both interest rate markets and so is well suited for this product.

In the context of MFMs a two-currency model was first introduced by [2]. A special case where the dynamics of the foreign currency are deterministic was also presented by [1]. The models introduced in the two papers use the Markov-functional approach to calibrate to the domestic and foreign interest rate markets and a parametric approach to model the foreign exchange market.

The paper is structured as follows. In Section 2 we formally define the two-currency economy and discuss how it fits in with the usual arbitrage pricing theory framework. In Section 3 we define a two-currency MFM and review the approach taken by [2]. In Section 4 we propose a new version of a two-currency MFM under the spot measure. Section 5 discusses its numerical implementation on a grid and Section 6 concludes.

2 Two-currency Economy

Let $0 = T_0 < T_1 < \ldots < T_{n+1}$ be an increasing sequence of dates. Of our interest will be an economy consisting of ZCB denominated in two different currencies – referred to as the domestic and the foreign currency – maturing on the dates $T_i, i = 1, \ldots, n+1$. More precisely, for each $i = 1, \ldots, n+1$, there exist two ZCBs maturing at time T_i, one denominated in the domestic currency and one denominated in the foreign currency. We will refer to them as the domestic (T_i-maturity) ZCB and the foreign (T_i-maturity) ZCB and denote their time $t \leq T_i$ values by D_{t,T_i} and $\tilde{D}_{t,T_i}$ respectively. Note that the prices of domestic ZCBs are denominated in the domestic currency and the prices of foreign ZCBs are denominated in the foreign currency.

We will assume that the two currencies can be exchanged at any time without frictions and will denote by $FX = (FX_t)_{t\in[0,T_{n+1}]}$ the (spot) foreign exchange rate process in the direct quotation. In particular, $FX_t, t \leq T_{n+1}$, is the price of a unit of the foreign currency denominated in the domestic currency. Furthermore, we will assume that the foreign exchange process is strictly positive and finite-valued $\mathbb{P}$-almost surely.[1]

Note that the usual arbitrage pricing theory underpinning the models presented in Gogala and Kennedy [3] assumed that the economy consists of assets denominated in the same currency and cannot be applied directly to the two-currency setting. We will overcome this issue by taking the foreign ZCBs denominated in the domestic currency, $FX_{.}\tilde{D}_{.,T_i} = (FX_t\tilde{D}_{t,T_i})_{t\in[0,T_i]}$, $i = 1, \ldots, n+1$, as the fundamental assets alongside the domestic ZCBs.

This allows us to use all the usual tools and in particular we know that if there exists a numeraire pair, such an economy is arbitrage-free and the fundamental pricing formula holds for the replicable contingent claims. However, it is beneficial to introduce more 'natural' terminology when considering the foreign denominated claims.

We will say that $\tilde{V}_T$ is an attainable foreign claim expiring at time $T \leq T_{n+1}$ if $V_T := FX_T\tilde{V}_T$ is an attainable claim in the usual sense (i.e. there exists an admissible replicating strategy). Note that for the claim V_T its time $t \leq T$ price V_t and can be obtained from the fundamental pricing formula. This allows us to define the foreign currency denominated time $t \leq T$ price $\tilde{V}_t$ of $\tilde{V}_T$ by $\tilde{V}_t := \frac{V_t}{FX_t}$. Moreover, it is easy to see that for any numeraire pair $(N, \mathbb{N})$

$$\tilde{V}_t = \frac{N_t}{FX_t}\mathbb{E}_{\mathbb{N}}\left[\frac{FX_T\tilde{V}_T}{N_T}\middle|\mathbb{F}_t^A\right]. \tag{1}$$

where $\mathbb{F}^A$ is the natural augmented filtration generated by the assets. We will refer to equation (1) as the fundamental pricing formula for foreign claims.

Now we can define the concept of foreign numeraire. We will say that a strictly positive process $\tilde{N} = (\tilde{N}_t)_{t\in[0,T_{n+1}]}$ is a foreign numeraire if $\tilde{N}_{T_{n+1}}$ is an attainable foreign claim and $\tilde{N}$ is its foreign denominated price process.

Note that if $\tilde{N}$ is a foreign numeraire then $FX\tilde{N} := (FX_t\tilde{N}_t)_{t\in[0,T_{n+1}]}$ is a numeraire i.e. it is a strictly positive process which can be replicated by a self-financing trading strategy using assets expressed in domestic currency. Moreover, if the economy is arbitrage-free there exists an equivalent martingale measure $\tilde{\mathbb{N}}$ associated with $FX\tilde{N}$. Then the fundamental pricing formula can be expressed in terms of $(FX\tilde{N}, \mathbb{N})$ as

$$\frac{V_t}{FX_t} = \tilde{N}_t\mathbb{E}_{\tilde{\mathbb{N}}}\left[\frac{V_T}{FX_T\tilde{N}_T}\middle|\mathbb{F}_t^A\right] \tag{2}$$

[1] We assume we are working on a filtered probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$.

and the fundamental pricing formula for foreign claims (1) can be expressed in terms of $(FX\tilde{N}, \mathbb{N})$ as

$$\tilde{V}_t = \tilde{N}_t \mathbb{E}_{\tilde{\mathbb{N}}}\left[\frac{\tilde{V}_T}{\tilde{N}_T}\middle|\mathbb{F}^A_t\right]. \tag{3}$$

The importance of equations (2) and (3) is in showing that by changing the measure to the one associated with a foreign numeraire process, we effectively obtain the pricing formulae that one would get if we have chosen view the two-currency economy in the single currency economy denominated in foreign currency.

Remark 1: A careful reader will notice that there is a slight difference between the two choices for the currency which is used to define the single-currency 'embedding' of the two-currency economy. In particular, the asset generated filtration $\mathbb{F}^A$ will in general depend on the choice of the domestic currency. However, this is a minor difference which is not of practical importance. In particular, in the rest of the section we will simply condition on $\mathbb{F}_t$ in the fundamental pricing formula.

For $i = 1, \dots, n$, and $t \leq T_i$ we will denote the time t value of the domestic forward/spot LIBOR by L^i_t and the value of the foreign forward/spot LIBOR by $\tilde{L}^i_t$.We have

$$L^i_t = \frac{D_{t,T_i} - D_{t,T_{i+1}}}{\alpha_i D_{t,T_{i+1}}} \tag{4}$$

$$\tilde{L}^i_t = \frac{\tilde{D}t, T_i - \tilde{D}_{t,T_{i+1}}}{\alpha_i \tilde{D}_{t,T_{i+1}}}. \tag{5}$$

Note that here we have implicitly assumed that the accrual factors $\alpha_i, i = 0, \dots, n$, are the same in both economies. This can be easily relaxed and allow the accrual factors to be different.[2]

In similar fashion we can then define domestic and foreign swaptions and caplets. As the reader will probably guess we will add a tilde to the to the notation for the domestic instrument prices to denote the prices of foreign counterparts, in particular we will denote the time $t \leq T_i, i \in \{1, \dots, n\}$ price of the digital caplet in-arrears with strike K written on $L^i_{T_i}$ and $\tilde{L}^i_{T_i}$ by $V^{\mathrm{dca},i}_t(K)$ and $\tilde{V}^{\mathrm{dca},i}_t(K)$ respectively.

For the model constructed in this paper we will work with the numeraire pair $(B, \mathbb{F}^0)$, where B, the domestic discretely compounded or rolling bank account, is the numeraire and $\mathbb{F}^0$ is the associated EMM referred to as spot measure. The value of the rolling bank account on the setting dates $T_1, \dots, T_{n+1}$ is

$$B_{T_1} = {D_{0,T_1}}^{-1} \tag{6}$$

$$B_{T_i} = {D_{0,T_1}}^{-1} \prod_{j=1}^{i-1} (1 + \alpha_j L^j_{T_j}),\ i > 1. \tag{7}$$

Markov-functional models under spot measure were first introduced by Fries and Rott [2]. Gogala and Kennedy [3] Section 4.1 reviews this construction in the single currency case and Section 4.2 shows how to efficiently implement a single currency MFM under spot measure driven by a diffusion process with continuous marginal distributions. We will build on these ideas in the multicurrency case.

3 Two-currency Markov-functional Models

Recall that in a (single-currency) MFM the prices of (domestic) ZCBs can be expressed as functions of some driving process x which is a Markov process under some equivalent mar-

[2]For example it might be the case that two economies have different day-count conventions.

tingale measure $\mathbb{N}$ corresponding to a numeraire process N. We now generalise this to a two-currency economy.

In a two-currency economy we will consider a model to be Markov-functional if there exist a triplet of processes (x, y, z) which are Markov under the measure $\mathbb{N}$ and with respect to the augmented natural filtration generated by them. Moreover, we will additionally require that

(i) for $i \in \{1, \ldots, n\}$ and $t \leq T_i$ the time t price of the domestic T_i-maturity ZCB D_{t,T_i} is a function of x_t;

(ii) for $t \leq T_{n+1}$ the time t value of the foreign exchange rate FX_t is a function of y_t;

(iii) for $i \in \{1, \ldots, n\}$ and $t \leq T_i$ the time t price of the foreign T_i-maturity ZCB $\tilde{D}_{t,T_i}$ is a function of z_t.

Of our interest will be Markov-functional models driven by one-dimensional processes x, y and z. In particular, we wish to construct a Markov-functional model by calibrating it to prices of domestic and foreign caplets and foreign exchange call options in the spirit of the MFMs described in [3].

As observed in [2] this turns out to be a difficult problem. One of the reasons for this comes from the fact that when the numeraire process N is a function of process x only (for example if we take the domestic T_{n+1}-maturity ZCB or a domestic discretely compounded rolling bank account as the numeraire) it is easy to see that processes x, y and z cannot be independent.

Fries and Rott [2] address this issue by discretising the processes x, y and z in time and assumes that they are of the form

$$x_{T_i} = x_{T_{i-1}} + \sigma_{i-1}^x (W_{T_i}^x - W_{T_{i-1}}^x), \tag{8}$$

$$y_{T_i} = y_{T_{i-1}} + \sigma_{i-1}^y (W_{T_i}^y - W_{T_{i-1}}^y) + \mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}), \tag{9}$$

$$z_{T_i} = z_{T_{i-1}} + \sigma_{i-1}^z (W_{T_i}^z - W_{T_{i-1}}^z), \tag{10}$$

where W^x, W^y and W^z are independent Brownian motions, $\sigma_{i-1}^x, \sigma_{i-1}^y, \sigma_{i-1}^z > 0$, and μ_{i-1} is a drift term that is determined during the calibration process.

Under the above assumptions, it turns out that for any foreign claim $\tilde{V}_{T_i}$ expiring at time T_i its time $T_j < T_i$ price is given by

$$\tilde{V}_{T_j} = \tilde{B}_{T_j} \mathbb{E}_{\mathbb{N}} \left[\frac{\tilde{V}_{T_i}}{\tilde{B}_{T_i}} \middle| \mathbb{F}_{T_j} \right] \tag{11}$$

as long as the numeraire process N is dependent on the process x only (see [2]). Note that in equation (11) the foreign claim is discounted using the foreign rolling bank account $\tilde{B}$ but the expectation is taken with respect to the domestic EMM $\mathbb{N}$. This turns out to be the consequence of the fact that the one time-step increments of the discretised processes x, y and z are conditionally independent.

To retain the 'symmetry' – in the sense that the domestic and foreign currency can be interchanged – in their model, Fries and Rott [2] choose to set up the model under the domestic spot measure. They calibrate the domestic and foreign ZCB markets to the domestic and foreign digital-caplets in-arrears – note that equation (11) allows this to be done independently for each of the currencies. Then they choose to calibrate the foreign exchange rate market iteratively forwards in time to prices of call options by using a parametric functional form to model the dependence of the foreign exchange rate on process y.

In the next section we will propose a construction that will allow for the Brownian motions W^x, W^y and W^z to be *dependent* and that will also use the Markov-functional sweep to calibrate the model to the prices of foreign exchange options. This allows for an exact fit to the FX volatility skew.

4 The Model

In this section we propose a new algorithm for constructing a two-currency MFM under the domestic spot measure. As done in [2] we will discretise the driving process (x, y, z) in time. In particular, we will assume that the driving process is of the form

$$
\begin{aligned}
x_{T_i} &= x_{T_{i-1}} + \sigma^x_{i-1}(W^x_{T_i} - W^x_{T_{i-1}}), && (12)\\
y_{T_i} &= \sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}), && (13)\\
z_{T_i} &= z_{T_{i-1}} + \sigma^z_{i-1}(W^z_{T_i} - W^z_{T_{i-1}}), && (14)
\end{aligned}
$$

where $x_0 = y_0 = z_0 = 0$, W^x, W^y and W^z are possibly correlated Brownian motions, $\sigma^x_{i-1}, \sigma^y_{i-1}, \sigma^z_{i-1} > 0$, and μ_{i-1} is a drift term that will be determined during the construction. Next we make the following assumptions

(i) In our model:

  (a) $L^i_{T_i}, i = 1, \ldots, n$, can be written as an increasing càdlàg function of x_{T_i};

  (b) $FX_{T_i}, i = 1, \ldots, n+1$, can be written as an increasing càdlàg function of y_{T_i};

  (c) $\tilde{L}^i_{T_i}, i = 1, \ldots, n$, can be written as an increasing càdlàg function of z_{T_i};

(ii) We are given:

  (a) the initial value of the T_1-maturity domestic ZCB D_{0,T_1} and prices of the *digital caplets in-arrears* written on $L^i_{T_i}, i = 1, \ldots, n$, for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$
V_0^{\mathrm{dca},i}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}\{L^i_{T_i} > K\}\right]; \tag{15}
$$

  (b) the prices of digital call options written on the time $T_i, i = 1, \ldots, n+1$ value of the foreign exchange rate for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$
V_0^{\mathrm{dFX,i}}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}\{FX_{T_i} > K\}\right]; \tag{16}
$$

  (c) the initial value of the T_1-maturity foreign ZCB $\tilde{D}_{0,T_1}$ and prices of the *digital caplets in-arrears* written on $\tilde{L}^i_{T_i}, i = 1, \ldots, n$, for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$
\tilde{V}_0^{\mathrm{dca},i}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}}{B_{T_i}}\mathbf{1}\{\tilde{L}^i_{T_i} > K\}\right]. \tag{17}
$$

Note that assumptions (i)(a) and (ii)(a) are exactly the same as the assumptions in the construction of a single-currency MFM under the spot measure. Assumptions (i)(a),(b) and (c) will ensure that the 'Markov-functional sweep' can be performed. On the other hand assumptions (ii)(a),(b) and (c) provide us with the market data needed to calibrate the model.

4.1 Main Idea

Ideally the one would like to perform the construction of an MFM iteratively forwards in time by performing the following steps at time $T_i, i \in \{1, \ldots, n\}$

1. Recover the functional form of $L^i_{T_i}$ from prices of domestic digital caplets in-arrears;
2. Recover the functional form of FX_{T_i} from prices of digital foreign exchange call options;[3]
3. Recover the functional form of $\tilde{L}^i_{T_i}$ from prices of foreign digital caplets in-arrears.

Unfortunately, the procedure is not so straightforward. The reason for this comes from the fact that a model of a two-currency economy is determined by the dynamics of the numeraire – in our case the discretely compounded rolling bank account B – and of the spot foreign exchange rate process FX. In particular, note that $FX_{T_{i-1}}\tilde{D}_{T_{i-1},T_i}$ is the time T_{i-1} price of the claim paying FX_{T_i} at time T_i and therefore

$$FX_{T_{i-1}}\tilde{D}_{T_{i-1},T_i} = B_{T_{i-1}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}}{B_{T_i}}\middle|\mathbb{F}_{T_{i-1}}\right]. \tag{18}$$

Observing that B_{T_i} is $\mathbb{F}_{T_{i-1}}$-measurable and that $\tilde{D}_{T_{i-1},T_i} = (1+\alpha_i\tilde{L}^{i-1}_{T_{i-1}})^{-1}$ then allows us to rewrite equation (18) as

$$FX_{T_{i-1}}\frac{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}}{1+\alpha_{i-1}\tilde{L}^{i-1}_{T_{i-1}}} = \mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}\middle|\mathbb{F}_{T_{i-1}}\right]. \tag{19}$$

In particular, observe that the functional form of $\tilde{L}^{i-1}_{T_{i-1}}$ which was determined in the previous step is also uniquely determined by the functional forms of $L^{i-1}_{T_{i-1}}$, $FX_{T_{i-1}}$ and FX_{T_i}. Or alternatively, the functional form of FX_{T_i} has to be chosen so that equation (19) holds. This demonstrates the importance of the flexibility to choose the drift during the calibration. Had we chosen the drift term μ_{i-1} in advance, calibrating to the digital swap options would in general result in a functional form for FX_{T_i} that would not satisfy equation (19).

However, the ability to freely choose the drift term does not solve problems of performing the second step in the above procedure entirely. In particular, it is not trivial to determine the drift term μ_{i-1} and the functional form of FX_{T_i}, because to determine one we need to know the other. From a theoretical perspective this is not a problem as we only need a solution – pair of functional forms (FX_{T_i},μ_{i-1}) – to exist. However, to apply the model in practice we need to be able to construct the solution, which is not a trivial task.

Here we propose to find a suitable pair (FX_{T_i},μ_{i-1}) using a predictor-corrector type of approach. In particular, we propose to use the following procedure instead of step 2:

2.1 Choose an initial functional form for μ_{i-1};

2.2 Determine the functional from of FX_{T_i} by calibrating to digital call option prices;

2.3 Adjust the drift so that equation (19) is satisfied;

2.4 Re-evaluate prices of digital call options, if the fit to the market is acceptable proceed to step 3, otherwise go to step 2.1.

Our conjecture is that for a reasonable initial choice of μ_{i-1} the above algorithm converges.

In the next three subsections we describe in more detail how the ith time step of the proposed algorithm can be implemented. In particular, we assume that prior to ith step we have already recovered the functional forms of $L^j_{T_j}$, $\tilde{L}^j_{T_j}$, FX_{T_j} and μ_{j-1} for $j\in\{1,\dots i-1\}$.

[3] Note that we need to perform this step also for time T_{n+1}.

4.2 Calibration to the Domestic Digital Caplets In-arrears

We can determine the functional form of $L^i_{T_i}$ and consequently of $B_{T_{i+1}}$ from the prices of digital caplets in-arrears as was done in Section 4 of [3] for the single-currency MFM under the spot measure.

First, we define a function $J^{x,i}$ by

$$J^{x,i}(x^*) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}\{x_{T_i} > x^*\}\right]. \tag{20}$$

Note that, we have already determined the functional form of B_{T_i} in the previous step and therefore the function $J^{x,i}$ is well defined. We can now use assumptions (i)(a) and (ii)(b) to recover the functional form of $L^i_{T_i}$ using the same argument as in [3], in particular

$$L^i_{T_i}(x^*) = \sup\big\{K \geq 0; V_0^{\mathrm{dca},i}(K) \geq J^{x,i}(x^*)\big\}. \tag{21}$$

Observe that determining the functional form of $L^i_{T_i}$ involves only integration over the joint distribution of the process x. Therefore, one can perform it for all the time steps independently of the calibration to the foreign exchange and the foreign interest rate markets.

4.3 Calibration to the Foreign Exchange Digital Call Options

To calibrate the model at time step T_i to the prices of digital call options we have proposed a predictor-corrector type scheme. Before describing its two main steps, we need to choose an initial value for of the drift term. To do so, let us return back to equation (19)

$$FX_{T_{i-1}}\frac{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}}{1+\alpha_{i-1}\tilde{L}^{i-1}_{T_{i-1}}} = \mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}\big|\mathbb{F}_{T_{i-1}}\big].$$

By assumption (i)(b) FX_{T_i} is an increasing function of

$$y_{T_i} = \sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}).$$

In particular, the Brownian increment $W^y_{T_i} - W^y_{T_{i-1}}$ is independent of the σ-algebra $\mathbb{F}_{T_{i-1}}$ while the drift term $\mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})$ is $\mathbb{F}_{T_{i-1}}$-measurable. Therefore the conditional expectation on the right-hand side of equation (19) has to be $\sigma(\mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}))$-measurable.

On the other hand, we know by observing the left-hand side of (19) the exact functional form of the conditional expectation on the right-hand side. Consequently, the drift term has to be measurable with respect to σ-algebra $\sigma\big(FX_{T_{i-1},T_i}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})\big)$ where

$$FX_{T_{i-1},T_i}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}) := FX_{T_{i-1}}(y_{T_{i-1}})\frac{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}{1+\alpha_{i-1}\tilde{L}^{i-1}_{T_{i-1}}(z_{T_{i-1}})} \tag{22}$$

is the value at time T_{i-1} of the time T_i forward foreign exchange rate and we can write

$$\mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}) = \mu_{i-1}\Big(FX_{T_{i-1},T_i}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})\Big). \tag{23}$$

Moreover, we can write equation (19) as

$$FX_{T_{i-1},T_i} = \mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}\big|\mathbb{F}_{T_{i-1}}\big]. \tag{24}$$

Remark 2: Note that equation (24) is the reason for a difference between our setup of the process y given in equation (13) and the one given in equation (9) of [2]. In particular, it allows for the drift to be dependent on the $(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})$ only through FX_{T_{i-1},T_i}.

Having, fixed the initial drift, we can now describe the predictor step, that is recovering the functional form of FX_{T_i} from foreign exchange digital call option prices. Similarly to the previous step we now define a function $J^{y,i}$ by

$$J^{y,i}(y^*) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}\{y_{T_i} > y^*\}\right]. \tag{25}$$

Note, that the functional form of the numeraire B_{T_i} is known and that we have already fixed the drift term μ_{i-1} and consequently the distribution of y_{T_i}. Therefore, the function $J^{y,i}$ is well defined. We can then use assumptions (i)(b) and (ii)(b) to recover the functional form of FX_{T_i} as

$$FX_{T_i}(y^*) = \sup\{K \geq 0; V_0^{\mathrm{dFX,i}}(K) \geq J^{y,i}(y^*)\}. \tag{26}$$

Determining the functional form as above will in general result in the foreign exchange rate $FX_{T_i}(y_{T_i})$ that no longer satisfies equation (19). Therefore, we need to perform the corrector step that adjusts the drift appropriately.

In particular, we need to determine a new functional form for the drift $\mu_{i-1}(FX_{T_{i-1},T_i})$ such that

$$\begin{aligned} FX_{T_{i-1},T_i} &= \mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}(y_{T_i})\middle|(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})\right] & (27)\\ &= \mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}\left(\sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(FX_{T_{i-1},T_i})\right)\middle|FX_{T_{i-1},T_i}\right] & (28)\\ &= \int_{-\infty}^{\infty} FX_{T_i}\left(\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}u + \mu_{i-1}(FX_{T_{i-1},T_i})\right)\varphi(u)du, & (29)\end{aligned}$$

where φ is the density function of a standard normal random variable. By assumption (i)(b) FX_{T_i} is an increasing function of y_{T_i} and therefore μ_{i-1} has to be an increasing function of FX_{T_{i-1},T_i}.

4.4 Calibration to the Foreign Digital Caplets In-arrears

Finally, we describe how to recover the functional form of the foreign LIBOR $\tilde{L}^i_{T_i}$. Unsurprisingly, we keep with the existing theme and define a function $J^{z,i}$ by

$$J^{z,i}(z^*) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}}{B_{T_i}}\mathbf{1}\{z_{T_i} > z^*\}\right]. \tag{30}$$

Note, that we already know the functional form of B_{T_i} and FX_{T_i} and therefore the function $J^{z,i}$ is well defined. Again, we can use assumptions (i)(c) and (ii)(c) to perform the Markov-functional sweep and determine the functional form of $\tilde{L}^i_{T_i}$ from

$$\tilde{L}^i_{T_i}(z^*) = \sup\{K \geq 0; \tilde{V}_0^{\mathrm{dca},i}(K) \geq J^{z,i}(z^*)\}. \tag{31}$$

Remark 3: Recall that the model presented in [2] could calibrate to the foreign digital caplets in-arrears independently of calibrating to domestic ones and the foreign exchange rate. In our model this is in general not the case as we allow for the Brownian motions W^x, W^y and W^z to be correlated. Nevertheless, if the increments of Brownian motions are independent, one can show that (11) also holds in our model and the calibration to the foreign caplets can be performed independently.

5 Numerical Implementation

Let us now outline how to implement the model presented in the previous section on a grid. For each of the time steps $T_i, i \in \{1, \ldots, n\}$, we choose grid-points

$$h^x_{i,1} < \ldots < h^x_{i,m} \tag{32}$$

$$h^y_{i,1} < \ldots < h^y_{i,m} \tag{33}$$

$$h^z_{i,1} < \ldots < h^z_{i,m} \tag{34}$$

corresponding to states of x_{T_i}, y_{T_i} and z_{T_i} respectively. Our aim is to recover the functional forms of $L^i(h^x_{i,j})$, $FX_{T_i}(h^y_{i,j})$, $\tilde{L}^i(h^z_{i,j})$ for $j = 1, \ldots, m$, and determine the functional form of the drift $\mu_i(h^x_{i,j_x}, h^y_{i,j_y}, h^z_{i,j_z}) = \mu_{i-1}(FX_{T_{i-1},T_i}(h^x_{i,j_x}, h^y_{i,j_y}, h^z_{i,j_z}))$ for $j_x, j_y, j_z = 1, \ldots, m$.

To ease the burden of notation we will adopt the following convention; by s we will denote the vector valued process defined by

$$s_i := (x_{T_i}, y_{T_i}, z_{T_i}), \quad i = 0, \ldots, n, \tag{35}$$

and by $h_{i,j}, i \in \{1, \ldots, n\}, j = (j_x, j_y, j_z) \in \{1, \ldots, m\}^3$, we will denote the grid-point

$$h_{i,j} := (h^x_{i,j_x}, h^y_{i,j_y}, h^z_{i,j_z}). \tag{36}$$

First recall that we can calibrate to the domestic digital caplet in-arrears prices and determine the functional form of L^i independently of the calibration to the foreign digital caplets in-arrears and foreign exchange digital call options. To do so, we can use the algorithm presented in Section 4 of [3]. In the rest of the section we will therefore assume we have already determined the functional forms of $L^i_{T_i}$ on the grid points and therefore also of the rolling bank account.

One of the problems we face when implementing the proposed model is that distribution of s_{T_i} is not Gaussian for $i \geq 2$ (we later discuss the case $i = 1$ separately) and we can only build it iteratively by observing that the conditional distribution $s_{T_i}|s_{T_{i-1}}$ is Gaussian. In particular, we need to build the information about the joint dynamics of process s in a way that will allow us to efficiently evaluate the functions $J^{y,i}$ and $J^{z,i}$ on the grid-points and later also allow us to price other derivatives.

Recall that we were faced with a similar, albeit only one-dimensional, problem in Gogala and Kennedy [3]. There, we introduced the piecewise polynomial basis functions and defined suitable expectations $E_{i,j}$'s that allowed us to build up the distributions of the model efficiently. Here, we will outline how to extend this idea to our setting.

Let $i \in \{1, \ldots, n\}$ we will say that functions $b_{i,j} : \mathbb{R}^3 \to \mathbb{R}, j = (j_x, j_y, j_z) \in \{1, \ldots, m\}^3$ are basis functions if they are piecewise polynomial and satisfy the following condition for all $j = (j_x, j_y, j_z), k = (k_x, k_y, k_z) \in \{1, \ldots, m\}^3$

$$b_{i,j}(h_{i,k}) = \delta_{j_x,k_x}\delta_{j_y,k_y}\delta_{j_z,k_z}. \tag{37}$$

Then we can define for any function $f : \mathbb{R}^3 \to \mathbb{R}$ a function $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{f}(x, y, z) := \sum_{\substack{j=(j_x,j_y,j_z) \\ j\in\{1,\ldots,m\}^3}} f(h_{i,j})b_{i,j}(x, y, z), \quad x, y, z \in \mathbb{R}. \tag{38}$$

In particular, note that the two functions agree on gridpoints

$$\tilde{f}(h_{i,j}) = f(h_{i,j}), \quad j \in \{1, \ldots, m\}^3. \tag{39}$$

Moreover, for a suitable choice of basis functions and any 'smooth enough'[4] function f the approximation $\tilde{f}$ is a 'good' (piecewise polynomial) approximation of f on the domain $[h_1^x, h_m^x]\times[h_1^y, h_m^y]\times[h_1^z, h_m^z]$.

In particular, at each time step $T_i, \in \{1,\dots,n\}$, we use the basis functions to define constants $E_{i,j}, j\in\{1,\dots,m\}^3$, by

$$E_{i,j} := \mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{B_{T_i}}\right]. \tag{40}$$

We now sketch how $E_{i,j}$'s and the functional forms of FX_{T_i} and $\tilde{L}^i_{T_{i-1}}$ can be recovered. First, we note that $i=1$ is a special case since at time zero the drift term $\mu_0(x_0,y_0,z_0)$ is constant and therefore the joint distribution of $(x_{T_1}, y_{T_1}, z_{T_1})$ is Gaussian. In particular, without loss of generality we can set $\mu_0 = 0$ since any deterministic drift can be absorbed into the functional form of FX_{T_1}. Moreover, observe that by fixing $\mu_0 = 0$ we have also made $E_{1,j}$'s well defined.

We can therefore evaluate $J^{y,1}$ on the grid-points directly

$$\begin{aligned}
J^{y,1}(h^y_{1,j_y}) &= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_1}}\mathbf{1}\{y_{T_1} > h^y_{1,j_y}\}\right] &(41)\\
&= (1+\alpha_0 L_0^0)^{-1}\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}\{\sigma_0^y W^y_{T_1} > h^y_{1,j_y}\}\right] &(42)\\
&= (1+\alpha_0 L_0^0)^{-1}\varphi\left(-\frac{h^y_{1,j_y}}{\sigma_0^y\sqrt{T_1}}\right), &(43)
\end{aligned}$$

where φ is the cumulative distribution of a standard normal random variable. Consequently we can recover, the functional form of FX_{T_1}.

Finally, we can determine the functional form of $\tilde{L}^1_{T_1}$ by first evaluating $J^{z,1}$ on the grid-points

$$\begin{aligned}
J^{z,1}(h^z_{1,j_z}) &= \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_1}(y_{T_1})}{B_{T_1}}\mathbf{1}\{z_{T_1} > h^z_{1,j_z}\}\right] &(44)\\
&= (1+\alpha_0 L_0^0)^{-1}\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_1}(y_{T_1})\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}\{z_{T_1} > h^z_{1,j_z}\}\middle|y_{T_1}\right]\right] &(45)\\
&= (1+\alpha_0 L_0^0)^{-1}\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_1}(y_{T_1})\varphi\left(\frac{\frac{\sigma_0^z}{\sigma_0^y}\rho^{y,z}_{T_1}y_{T_1} - h^z_{1,j_z}}{\sigma_0^z\sqrt{T_1\big(1-(\rho^{y,z}_{T_1})^2\big)}}\right)\right], &(46)
\end{aligned}$$

where $\rho^{y,z}_{T_1} = corr(W^y_{T_1}, W^z_{T_1})$. We have manipulated $J^{1,z}(h^z_{1,j_z})$ to equation (46) which only involves integrating over a one-dimensional Gaussian distribution and can be performed by many existing numerical integration techniques.

Now we show how to perform a general time step. In particular, we assume that we have recovered the functional forms of $L^{i-1}_{T_{i-1}}$, $FX_{T_{i-1}}$, $\tilde{L}^{i-1}_{T_{i-1}}$ and the values of $E_{i-1,j}$'s for some $i\in\{2,\dots,n\}$.

We first show how to recover the functional form of FX_{T_i} by evaluating $J^{y,i}$ on the grid-points. We assume that we have chosen an initial guess for the drift μ_{i-1} or we have obtained

[4]The smoothness condition depends on the choice of basis functions.

it from the corrector step.

$$J^{y,i}(h^y_{i,j_y}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}\{y_{T_i} > h^y_{i,j_y}\}\right] \tag{47}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}\{y_{T_i} > h^y_{i,j_y}\}\big|\mathbb{F}_{T_{i-1}}\right]\right] \tag{48}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}\{y_{T_i} > h^y_{i,j_y}\}\big|s_{T_{i-1}}\right]\right] \tag{49}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{(1+\alpha^{i-1}L^{i-1}_{T_{i-1}})B_{T_{i-1}}}\varphi\left(\frac{\mu_{i-1}(s_{T_{i-1}}) - h^y_{i,j_y}}{\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}}\right)\right]. \tag{50}$$

Note that $L^{i-1}_{T_{i-1}}$ is a known function of $x_{T_{i-1}}$. We can now use the approximation using the basis functions $b_{i-1,j}$ and express $J^{y,i}(h^y_{i,j_y})$ in terms of $E_{i,j}$'s which are already known.

$$J^{y,i}(h^y_{i,j_y}) \simeq \sum_k D_{T_{i-1},T_i}(h^x_{i-1,k_x})\varphi\left(\frac{\mu_{i-1}(h_{i,k}) - h^y_{i,j_y}}{\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}}\right)\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i-1,k}(s_{T_{i-1}})}{B_{T_{i-1}}}\right] \tag{51}$$

$$= \sum_k \frac{E_{i-1,k}}{1+\alpha^{i-1}L^{i-1}_{T_{i-1}}(h^x_{i-1,k_x})}\varphi\left(\frac{\mu_{i-1}(h_{i-1,k}) - h^y_{i,j_y}}{\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}}\right). \tag{52}$$

After we have recovered the functional form of FX_{T_i} we need to use the corrector step to make sure our model remains arbitrage-free. We then repeat the two steps until we achieve sufficient level of convergence at which point we can freeze the drift term μ_{i-1} and determine the $E_{i,j}$'s, by conditioning on $\mathbb{F}_{T_{i-1}}$

$$E_{i,j} = \mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{B_{T_i}}\right] \tag{53}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\bigg|\mathbb{F}_{T_{i-1}}\right]\right] \tag{54}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\bigg|s_{T_{i-1}}\right]\right] \tag{55}$$

$$\simeq \sum_k \frac{E_{i-1,k}}{1+\alpha^{i-1}L^{i-1}_{T_{i-1}}(h^x_{i-1,k_x})}\mathbb{E}_{\mathbb{F}^0}\left[b_{i,j}(s_{T_i})\big|s_{T_{i-1}} = h_{i-1,k}\right] \tag{56}$$

Finally we can recover the functional form of $\tilde{L}^i_{T_i}$ by evaluating function $J^{z,i}$ on the gridpoints

$$J^{z,i}(h^z_{i,j_z}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}(y_{T_i})}{B_{T_i}}\mathbf{1}\{z_{T_i} > h^z_{i,j_z}\}\right] \tag{57}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}(y_{T_i})\mathbf{1}\{z_{T_i} > h^z_{i,j_z}\}\big|\mathbb{F}_{T_{i-1}}\right]\right] \tag{58}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}(y_{T_i})\mathbf{1}\{z_{T_i} > h^z_{i,j_z}\}\big|s_{T_{i-1}}, y_{T_i}\right]\right] \tag{59}$$

Now observe that, the random vector $(y_{T_i}, z_{T_i})|s_{T_{i-1}}$ has a known bivariate normal distribution and therefore $z_{T_i}|y_{T_i}, s_{T_{i-1}}$ has a known normal distribution. In particular, using the tower property of conditional expectation we can show that

$$\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}(y_{T_i})\mathbf{1}\{z_{T_i} > h^z_{i,j_z}\}\big|s_{T_{i-1}}\right] = \mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}(y_{T_i})\varphi\big(g_{i,j_z}(s_{T_{i-1}}, y_{T_i})\big)\big|s_{T_{i-1}}\right], \tag{60}$$

for a known function $g_{i,j_z} : \mathbb{R}^4 \to \mathbb{R}$. This, then allows us to evaluate conditional expectation $\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_i}(y_{T_i})\mathbf{1}\{z_{T_i} > h^z_{i,j_z}\}\middle|s_{T_{i-1}}\right]$ on the grid points $h_{i-1,k}, k \in \{1,\ldots,m\}^3$, by using only one-dimensional numerical integration. Then we can finally determine the value of $J^{i,z}$ on the grid-points as a linear combination of $E_{i-1,j}$'s.

6 Conclusion

In this paper we have proposed a new two-currency Markov-functional model, which can be calibrated to the smile in the domestic and foreign caplet prices and foreign exchange options. We have used the idea of [2] and discretised the driving process first and then constructed the model. By doing so we avoided dealing with state-dependent drifts which would occur in the continuous time setting.

The resulting three factor model is very flexible and can incorporate correlation between the driving factors which could not be achieved in [2].

References

[1] Christian P. Fries and Fabian Eckstaedt, *A hybrid Markov-functional model with simultaneous calibration to the interest rate and FX smile*, Quantitative Finance **11** (2011), no. 4, 587–597.

[2] Christian P. Fries and Marius Rott, *Cross currency and hybrid Markov functional models*, 2004, http://dx.doi.org/10.2139/ssrn.532122.

[3] Jaka Gogala and Joanne E. Kennedy, *One-dimensional Markov-functional models driven by a non-Gaussian driver*, Journal of Computational Finance **23** (2019), no. 3, 1–39.

[4] P. J. Hunt, J. E. Kennedy, and A. Pelsser, *Markov-functional interest rate models*, Finance and Stochastics **4** (2000), no. 4, 391–408.

[5] V Piterbarg, *A multi-currency model with FX volatility skew*, SSRN pre-print (2005), https://ssrn.com/abstract=685084 or http://dx.doi.org/10.2139/ssrn.685084.