Regularity properties of optimal stopping values arising from stochastic volatility models

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Introduction

- Setting: consider the stochastic volatility (SV) model on \((\Omega, \mathcal{F}, \mathbb{P})\):

\[
\begin{align*}
\text{d}S_t &= \sigma(V)_t \text{d}B_t^\omega, \\
\text{d}V_t &= \theta(V)_t \text{d}B_t^\nu + \sigma'(V)_t \text{d}W_t \\
\text{S}_0 &= x, \\
\text{V}_0 &= y,
\end{align*}
\]

where \((B_t^\omega, B_t^\nu)\) is a two-dimensional Brownian motion, and \(\sigma, \theta, \sigma'\) are continuous functions.

We deal with the regularity of the optimal stopping problem

\[
\mathbb{E}_x\left[e^{-\gamma T}g(S_T)\right] = \sup_{\tau \in \mathcal{M}} \mathbb{E}_x\left[e^{-\gamma \tau}g(S_\tau)\right]
\]

\(\text{S}_0, \text{V}_0 \geq 0\) and the requirements are verified.

Path comparison of processes

Suppose that \(\xi (\text{resp. } \xi')\) is a unique strong solution of the corresponding SDE in (3) with \(\xi = -y \geq \xi'\) and assume that \(y \geq y'\).

FACT:

\[
\xi_t \leq \xi'_t, \quad \forall t \geq 0 \quad a.s.
\]

Note that we are NOT making specific assumptions on the coefficients of \(\xi\), we only require existence of a unique strong solution.

Theorem 1: Monotonicity

Suppose that \(g\) is a continuous, non-negative, and satisfies

\[
g(ax) \leq ag(x), \quad a \geq 1, \quad \forall x \geq 0.
\]

Then for every \(x \geq 0, T \in [0, \infty)\),

\[
y \mapsto v(x, y, T) \quad \text{is non-decreasing.}
\]

Idea of Proof:

- Given \(\xi, \xi'\) as above, \(\xi_t \leq \xi'_t\) for all \(t \geq 0\) a.s., we have \(\xi_t \leq \xi'_t\) for all \(t \geq 0\) a.s.

Choosing \(a = e^{r(t-t_0)}\),

\[
e^{-rT}g(S_T^\xi) \leq e^{-rT}g(S_T^{\xi'}).
\]

- The inverses \(A_t \leq A'_t\) and so \(M_T \leq M'_T\) implying \(v(x, y, T) \leq v(x, y', T)\).

Example 2: Heston model

The SV model proposed by Heston is of the form

\[
\begin{align*}
\text{d}S_t &= \sqrt{\text{V}_t} \text{d}B_t^\omega, \\
\text{d}V_t &= \rho \sqrt{\text{V}_t} \text{d}B_t^\nu + \theta - \kappa \text{V}_t \text{d}t, \\
\text{S}_0 &= x, \\
\text{V}_0 &= y,
\end{align*}
\]

where \(\rho, \kappa, \theta, \sigma\) are constants and \(\text{B}_t^\omega, \text{B}_t^\nu\) are independent Brownian motions.

Path comparison of processes

Suppose that \(\xi (\text{resp. } \xi')\) is a unique strong solution of the corresponding SDE in (3) with \(\xi = -y \geq \xi'\) and assume that \(y \geq y'\).

FACT:

\[
\xi_t \leq \xi'_t, \quad \forall t \geq 0 \quad a.s.
\]

Note that we are NOT making specific assumptions on the coefficients of \(\xi\), we only require existence of a unique strong solution.

Theorem 2: Continuity

Using the fact (8), if either \(g\) is bounded for \(x \geq 0\), or \(g\) is Lipschitz and the integrability condition

\[
E \left[ \sup_{0 \leq t \leq T} e^{-\gamma t}g(S_t) \right] < \infty
\]

holds, then for every \(x \geq 0, T \in [0, \infty)\),

\[
y \mapsto v(x, y, T) \quad \text{is continuous.}
\]

Idea of Proof:

- Let \(\{\xi_n\}\) be a sequence converging to \(y\).

- Under certain general assumptions, \(A_t \rightarrow A_t\) for all \(t \geq 0\) a.s.

- Writing \(S_t^\omega = G_t^\omega + S_t^\nu = G_t^\nu \), we also have \(S_t^\omega - S_t^\nu \rightarrow 0, \quad \forall \tau \geq 0 \quad a.s.\)

- The result follows using dominated convergence.