Newton's method in the complex plane – Basins of attraction and the Julia set

Newton Method

The Newton method is an iterative method that can approximate the roots of a particular function, which works for both real and complex numbers. The formula used is

\[ z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)} \]

where \( z_n \) is the value of your complex number at the \( n \)th iteration, \( p(z_n) \) is the function of \( z \) you are looking at, and \( p'(z_n) \) is its derivative, both of which are calculated for your value of \( z_n \). This method requires you to choose a starting value, \( z_0 \), which will return a value \( z_1 \). This value is then put back into the formula to give a value \( z_2 \), and this process is repeated until you have an accurate enough approximation for your route (in theory...).

Complex Numbers/Plane

Complex numbers have a real and imaginary part (\( \sqrt{-1} = i \), the imaginary number). Complex numbers exist on the "complex plane", where you plot the real number on the \( x \)-axis, and the imaginary number on the \( y \) axis, as shown below.

There are also a number of ways to write complex numbers: \( z = u + iv = Re^{i\theta} = R(\cos \theta + i \sin \theta) \), with each of the different variables shown on the complex plane above.

Roots of unity

Solving \( z^3 - 1 = 0 \) \( \Rightarrow \) \( z^3 = 1 \), obvious answer is \( z = 1 \).

What about complex numbers?

Rewrite \( z^3 = 1 \) \( \Rightarrow \) \( (Re^{i\theta})^3 = 1e^{i(0+2\pi k)} \) \( \Rightarrow \) \( R^3e^{3i\theta} = 1, k = 0,1,2 \ldots \)

The \( 0 + 2\pi k \) comes from the fact that rotating \( 2\pi \) radians/360\(^\circ\) around the origin brings you back to where you started.
From this, can see that $R^3 = 1, R = 1$, and $3\theta = 2\pi k, \theta = \frac{2\pi k}{3}$.

For $k = 0, z = 1$. For $k = 1, z = e^{\frac{2\pi i}{3}}$. For $k = 2, z = e^{\frac{4\pi i}{3}}$.

Using $Re^{i\theta} = R(\cos \theta + i \sin \theta)$, we can rewrite these as:

$$z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

**Basins of attraction in the complex plane**

When using Newton’s method on real numbers, the starting point on the real axis affects the behaviour of the successive iterations (i.e. if it converges to a root, which root, etc). When considering complex numbers, your starting location in the complex plane affects the behaviour of successive iterations. There are regions where Newton’s method will always converge to a specific root if your starting point is in this region. These are called “Basins of attraction”. The colour coded basins of attraction are shown below for $p(z) = z^3 - 1$.

![Basins of attraction image](plotted using matlab)

Green regions will always converge to the $z = 1$ root, red regions always converge to the $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ root, and blue regions always converge to the $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ root.

You can also see fractal patterns in the boundaries between the three main regions (a fractal is something that is self-similar, and looks the same even if you zoom in. So the patterns along the boundaries will still look the same, even if you keep zooming in).

What about the points that lie exactly on the boundary between two basins of attraction?...

**The Julia Set**

... These form the Julia set, which is defined as the set of all points for which very small movements away from one of these points will drastically change the outcome of an iterative process applied at/near the point. So, for example, at the origin of the above graph for $z^3 - 1$, very slight movements away from the origin in any direction will change whether you are in the red, green or blue region, and will therefore change which root you will converge to if you were then to apply Newton’s method to that point.