EC9A0: Pre-sessional Advanced Mathematics Course

Fixed Point Theorems

Pablo F. Beker
Department of Economics
University of Warwick

September 2023

Lecture Outline

Definition of Contraction

Definition

Let (X, d) be a metric space and $f : X \mapsto X$. f is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$, $d(f(x), f(y)) \leq \beta d(x, y)$, $\forall x, y \in X$.

Example

Let $a, b \in \mathbb{R}$ with $a < b$, $X = [a, b]$ and $d(x, y) = |x - y|$. Then f is a contraction if for some $\beta \in (0, 1)$,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \beta < 1, \text{for all } x, y \in X \text{ with } x \neq y$$

That is, f is a contraction mapping if it is a function with slope uniformly less than one in absolute value.

Blackwell's sufficient conditions

Theorem : Blackwell's sufficient conditions for a contraction

Let $X \subset \mathbb{R}^K$*, and let* $B(X)$ *be a space of bounded functions* $f : X \mapsto \mathbb{R}$ *with the sup norm. Let* $T : B(X) \mapsto B(X)$ *satisfy*

1. *(monotonicity)* $f, g \in B(X)$ *and* $f(x) \leq g(x)$*, for all* $x \in X$*, implies* $(Tf)(x) \leq (Tg)(x)$*, for all* $x \in X$*;*
2. *(discounting) there exists some* $\beta \in (0, 1)$ *such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ for all } f \in B(X), a \geq 0, x \in X$$

where $(f + a)(x)$ *is the function defined by* $(f + a)(x) = f(x) + a$*.*

Then T *is a contraction with modulus* β

Blackwell's sufficient conditions: Proof

Proof:

1. For any $f, g \in B(X)$,
$$f(x) - g(x) \leq \|f - g\|$$
2. By monotonicity:
$$Tf(x) \leq T\left(g + \|f - g\|\right)(x)$$
3. By discounting:
$$T\left(g + \|f - g\|\right)(x) \leq Tg(x) + \beta\|f - g\|$$
4. Thus,
$$Tf(x) \leq Tg(x) + \beta\|f - g\| \tag{1}$$
5. Reversing the roles of f and g we obtain
$$Tg(x) \leq Tf(x) + \beta\|f - g\|. \tag{2}$$
6. Combining (1) and (2) we get $\|Tf - Tg\| \leq \beta\|f - g\|$, as desired.

Application I: Neoclassical Growth Model

Example

In the one sector optimal growth problem, an operator T is defined by

$$(Tv)(x) = \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\}$$

- If $v(y) \leq w(y)$ for all y, then $Tw \geq Tv$ and so monotonicity holds.
- To show discounting note that:

$$\begin{aligned} T(v+a)(x) &= \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta[v(y) + a]\} \\ &= \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\} + \beta a \\ &= (Tv)(x) + \beta a \end{aligned}$$

Complete Metric Space

Definition

A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Fact: $\mathbb{R}$ with $d(x, y) = |x - y|$ is a complete metric space.

Exercise: Show that:

1. The set of integers with $d(x, y) = |x - y|$ is a complete metric space.
2. The set of continuous, strictly increasing functions on $[0, 1]$, with

$$d(x, y) = \max_{0 \leq t \leq 1} |x(t) - y(t)|. \tag{3}$$

is not a complete metric space. (Hint: show the sequence $x_n(t) = 1 + \frac{t}{n}$ converges to the constant function $x(t) = 1$)

Contraction Mapping Theorem

Theorem

If (X, d) is a complete metric space and $T : X \mapsto X$ is a contraction mapping with modulus β, then

1. *T has exactly one fixed point $x \in X$, and*
2. *for any $x_0 \in X$, $d(T^n x_0, x) \leq \beta^n d(x_0, x), n = 0, 1, 2, ...$*

- Define $\{T^n\}_{t=0}^{n}$ by $T^0 x = x$ and $T^n x = T(T^{n-1}x)$, $n = 1, 2, ...$

STEP 1: v_n converges.

- Let $v_0 \in X$, $\{v_n\}_{n=0}^{\infty}$ by $v_{n+1} = Tv_n$ so that $v_n = T^n v_0$.
- By the contraction property:

$$\begin{aligned} d(v_2, v_1) = d(Tv_1, Tv_0) &\leq \beta d(v_1, v_0) \\ d(v_{n+1}, v_n) &\leq \beta^n d(v_1, v_0), n = 1, 2, ... \\ d(v_m, v_n) &\leq d(v_m, v_{m-1}) + ... + d(v_{n+2}, v_{n+1}) + d(v_{n+1}, v_n) \\ &\leq \left[\beta^{m-1} + ... + \beta^{n+1} + \beta^n\right] d(v_1, v_0) \\ &= \beta^n \left[\beta^{m-n-1} + ... + \beta + 1\right] d(v_1, v_0) \leq \frac{\beta^n}{1-\beta} d(v_1, v_0). \end{aligned}$$

- Thus $\{v_n\}_{n=0}^{\infty}$ is Cauchy. Since X is complete, $v_n \to v \in X$.

STEP 2: Show $Tv = v$

- $\forall n$ and $\forall v_0 \in X$, $d(Tv, v) \leq d(Tv, T^n v_0) + d(T^n v_0, v) \leq \beta \underbrace{d(v, T^{n-1} v_0)}_{\to 0} + \underbrace{d(T^n v_0, v)}_{\to 0} \to 0$

Contraction Mapping Theorem

Step 3: Uniqueness

- Suppose $\exists \hat{v} \neq v$ such that $T\hat{v} = \hat{v}$. Then,

$$0 < d(\hat{v}, v) = d(T\hat{v}, Tv) \leq \beta d(\hat{v}, v) < d(\hat{v}, v).$$

- To prove (2), note that for any $n \geq 1$:

$$d(T^n v_0, v) = d(T\left(T^{n-1}v_0\right), Tv) \leq \beta d(T^{n-1}v_0, v)$$

Q.E.D.

Application II: Differential Equations

Example

Consider the differential equation and boundary condition $\frac{dx(s)}{ds} = f[x(s)]$, all $s \geq 0$, with $x(0) = c \in \mathbb{R}$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous, and for some $B > 0$ satisfies the Lipschitz condition $|f(a) - f(b)| \leq B|a - b|$, all $a, b \in \mathbb{R}$. For any $t > 0$, consider $C[0, t]$, the space of bounded continuous functions on $[0, t]$, with the sup norm.

1. Show that the operator T defined by

$$(Tv)(s) = c + \int_0^s f[v(s)]dz, 0 \leq s \leq t$$

   maps $C[0, t]$ into itself.

2. Show that for some $\tau > 0$, T is a contraction on $C[0, \tau]$.
3. Show that the unique fixed point of T on $C[0, \tau]$ is a differentiable function, and hence that it is the unique solution on $[0, \tau]$ to the given differential equation.

Definitions

- f maps the set $X \subset \mathbb{R}^K$ into itself if $f(x) \in X$ for all $x \in X$.
- We would like to find conditions ensuring that any continuous function mapping X into itself has a fixed point.
- The following example shows that some restrictions must be placed on X:
    - $f(x) = x + 1$ maps $\mathbb{R}$ into itself.
    - $f(x)$ has no fixed point since $f(x) = x$ implies $1 + x = x$, an absurd.

Brouwer's Fixed Point Theorem

Theorem L.E.J. Brouwer's fixed point theorem

Let X be a nonempty compact convex set in $\mathbb{R}^K$, and f be a continuous function mapping X into itself. Then f has a fixed point x^*.

- For $X = \mathbb{R}$, a nonempty compact convex set is a closed interval $[a, b]$.
- A continuous function $f : [a, b] \mapsto [a, b]$ must have a fixed point.
- This follows from the IVT:
    - Define $g(x) = f(x) - x$.
    - x is a fixed point of f if and only if $g(x) = 0$.
    - Since $g(a) \geq 0$ and $g(b) \leq 0$, there is $x^* \in [a, b]$ such that $g(x^*) = 0$.
- We use Brouwer's fixed point Theorem to prove existence of equilibrium in a pure exchange economy.

Kakutani's Fixed Point Theorem

- Brouwer's Theorem deals with fixed points of continuous functions.
- Kakutani's theorem generalises the theorem to correspondences.

Definition

An element $x \in X$ is a fixed point of a correspondence $F : X \mapsto X$ if $x \in F(x)$.

Theorem Kakutani's Fixed Point Theorem

Let X be a nonempty compact convex set in $\mathbb{R}^K$ and $F : X \mapsto X$ be a correspondence. Suppose that:

1. *$F(x)$ is a nonempty convex set in X for each $x \in X$*
2. *F is upper hemicontinuous.*

Then F has a fixed point x^* in X.

- We use Kakutani's Fixed Point Theorem to prove existence of a Mixed Strategy Nash Equilibrium in an N-player game with finite (pure) strategy sets.