Size versus versus fairness in the assignment problem

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Abstract

When not all objects are acceptable to all agents, maximizing the number of objects actually assigned is an important design concern. We compute the guaranteed size index of the Probabilistic Serial mechanism, i.e., the worst ratio of the actual expected size to the maximal feasible size. It converges decreasingly to $1 - \frac{1}{e} \approx 63.2\%$ as the maximal size increases. It is the best index of any Envy-Free assignment mechanism.

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1 Introduction

1.1 The context

The assignment of indivisible objects to economic agents by means of lotteries is an important example of a “market without money”, where randomizing the allocation of objects, or, equivalently in some contexts, implementing time sharing, is the only way to achieve a fair outcome. The familiar real life examples include assigning workers to jobs, jobs to time slots, classes or dormitory rooms to students, school choice ([2], [18]), etc.. See [22] for a survey.

The three normative goals of mechanism design, efficiency, incentive compatibility and fairness, lead the discussion of the assignment problem in the economic literature. The recent literature on algorithmic mechanism design introduces the fourth goal of maximizing a simple measure of social optimality. One of the earliest instances of this approach is [21], discussing the tradeoff between Strategy-Proofness and the utilitarian minimization of aggregate cost. Another seminal example, closer to home, is in the bilateral matching problem. When preferences have ties and are incomplete (remaining single is preferred to some potential partners) not all stable matchings are of the same size (the "rural hospital theorem" does not apply), so it is natural to look for a stable matching of maximal size ([15]), or for a maximal cardinality matching with the smallest number of blocking pairs ([5]): both questions turn out to be NP-hard. The project [19], from which the present work is born, explores the tradeoffs between Strategy-Proofness on one hand, and maximizing the size
of the match on the other, in a variety of assignment and matching problems.

1.2 The problem and the punchline

In most practical instances of the assignment problem, incomplete preferences are the norm: in school choice, parents can opt out of the public system; jobs have deadlines which render certain time slots useless; students can live off campus, and so on. Even with strict preferences, efficient assignments can then have very different sizes (number of agents who receive an object), so the goal of maximizing the size of the actual assignment becomes important in its own right: filling the largest possible number of slots/seats/jobs, is a component of the system performance, to which public school administrators, the housing office on campus, etc., are paying attention. We define the size index of an assignment as the ratio of the size of the actual assignment to the maximal feasible size.

Note that the largest feasible size of an assignment only depends upon the bipartite graph of acceptability, and ignores the finer information in the profile of individual preferences. This implies that size maximization frequently conflicts with the goals of fairness and incentive compatibility, as is obvious in the following elementary example with two objects $a, b$ and two agents Ann, Bob, who both prefer $a$ to $b$. If both objects are acceptable to Bob but Ann only accepts $a$, then assigning $a$ to Ann and $b$ to Bob is the only assignment of maximal size, but it is obviously unfair to Bob, and makes it profitable for him to report that only $a$ is acceptable, if he prefers a 50% chance of getting $a$ to a 100% chance of $b$. Here, as in [8] and much of the subsequent literature, we interpret fairness as the well-known Envy-Freeness property, and incentive compatibility as Strategy-Proofness (both defined in section 4).

We define the guaranteed $m$-size index of a random assignment mechanism as its worst size index over all assignment problems such that the maximal size of a feasible assignment is $m$. We compute the largest guaranteed $m$-size index $r_m$ of any Envy-Free mechanism, and show that it is achieved by the Probabilistic Serial mechanism ([8], [7], [14]; see section 4), the only known mechanism to date combining Envy-Freeness with Ordinal Efficiency. Moreover the sequence $r_m$ converges decreasingly to $1 - \frac{1}{e}$ as $m$ grows.

Although the Probabilistic Serial mechanism is not strategy-proof, this result throws some light on the tradeoff between size maximization and Strategy-Proofness. Indeed the familiar assignment mechanism Random Priority (a.k.a. serial dictatorship, see [1]) is strategy-proof, and it was shown in [12] to have a smaller size index than Probabilistic Serial precisely for those problems where the latter achieves its worst case index $r_m$ (see section 6 for details). Therefore $r_m$ is also an upper bound for the guaranteed $m$-size index of Random Priority. On the other hand [20] show that a lower bound is $1 - (1 - \frac{1}{m+1})^m - \frac{1}{m}$, and the latter sequence converges increasingly to $1 - \frac{1}{e}$.\footnote{This result is closely related to the well known online algorithm maximizing the guaranteed size of a bilateral matching, relative to the maximal size feasible offline. The Ranking algorithm of [17] selects randomly and uniformly an ordering of the objects, then assign to the incoming agent the highest acceptable object in that ordering; its $m$-guaranteed size is no less than $1 - (1 - \frac{1}{m+1})^m$ (see also [4] for a simpler proof and [16] for a generalization to multiple objects).} This confirms the earlier results in [11] about the asymptotic equivalence of these two benchmark mechanisms. It may well be that the performance of Ran-
dom Priority cannot be improved by any other strategy-proof mechanism.

2 Random assignment with outside options

Fix \( N \) the set of agents and \( A \) of objects, with respective cardinalities \( n \) and \( q \). A preference \( R_i \) of agent \( i \in N \) is a possibly empty ordered subset of \( A \), written \( R_i = (a_1,a_2,\ldots,a_k) \) where \( a_1 \) is the best object for \( i \) and \( a_k \) her least preferred acceptable object. Abusing notation, \( a \in R_i \) means that \( a \) is an acceptable object for \( i \), and \( R_i = \emptyset \) means that no object is acceptable to \( i \). We write \( \mathcal{R}(A) \) for the set of individual preferences.

A profile of preferences \( R \in \mathcal{R}(A) \) defines a compatibility bipartite graph \( E \subseteq N \times A \): \( ia \in E(R) \iff a \in R_i \), describing which objects are acceptable to which agents. An assignment problem is a triple \( \Delta = (N,A,R) \), and its compatibility graph is written \( E(\Delta) \).

An assignment is a \( N \times A \) substochastic matrix \( P = [p_{ia}] \in \mathbb{R}^{N \times A}_+ \): \( \sum_N p_{ia} \leq 1 \) for all \( a \) and \( \sum_A p_{ia} \leq 1 \) for all \( i \). It is feasible at \( R \) if, in addition, \( p_{ia} > 0 \Rightarrow ia \in E(\Delta) \). We write \( \mathcal{P}(E(\Delta)) \), or simply \( \mathcal{P}(E) \), for the set of feasible assignments at \( \Delta \), and \( \mathcal{P}^d(E) \) for the subset of deterministic feasible assignments \( (p_{ia} = 0,1 \text{ for all } i,a) \). A well known fact (a variant of Birkhoff’s Theorem) is that the convex hull of \( \mathcal{P}^d(E) \) is \( \mathcal{P}(E) \).

The expected number of objects (or agents) assigned at \( P \) is \( s(P) = \sum_{N \times A} p_{ia} \), we call it the size of \( P \). Note that \( s(P) \leq \min\{n,q\} \).

The following nice fact refines Birkhoff’s Theorem. A random assignment is implemented by deterministic assignments of (almost) equal size: any \( P \in \mathcal{P}(E) \) is a convex combination of deterministic assignments of size \( \lfloor s(P) \rfloor \) or \( \lceil s(P) \rceil \) (lower and upper integral part).

In particular the program

\[
s^*(E) = \max_{P \in \mathcal{P}(E)} s(P) \tag{1}
\]

has at least one deterministic solution, and every solution is a convex combination of such deterministic assignments. We call \( s^*(E(\Delta)) \) the size of the problem \( \Delta \), i.e., the maximal number of objects/agents it is feasible to assign. The set of assignment problems of size \( m \) is denoted \( \mathcal{A}^m \).

An assignment mechanism \( F \) associates to every assignment problem \( \Delta \) a feasible assignment \( F(\Delta) = P \in \mathcal{P}(E(\Delta)) \). We focus in this paper on the worst possible match size that a mechanism can reach, relative to the size of the problem. Define the guaranteed \( m \)-size index of \( F \) as follows

\[
\sigma_m(F) = \min_{\Delta \in \mathcal{A}^m} \frac{1}{m} s(F(\Delta))
\]

The absolute guaranteed size index of \( F \) is

\[
\sigma_\infty(F) = \inf_{m \geq 2} \sigma_m(F).
\]

3 Efficiency and guaranteed size

Given a problem \( \Delta \) and two deterministic assignments \( P, P' \in \mathcal{P}^d(E(\Delta)) \), we say that \( P \) is Pareto superior to \( P' \) if \( P \neq P' \) and

\[
\{p_{ia} = 1 \text{ and } p'_{ib} = 1\} \Rightarrow aRib
\]

\[
\{p_{ia} = 0 \text{ for all } a\} \Rightarrow \{p'_{ia} = 0 \text{ for all } a\}
\]

An efficient (Pareto optimal) deterministic assignment is one that is not Pareto dominated.

\^This follows from the results in [10].
In any problem $\Delta \in \mathcal{A}^m$ there is at least one efficient deterministic assignment of maximal size $m$. This follows because if an assignment $P \in \mathcal{P}^d(E)$ is Pareto dominated by $P'$, then $s(P) \leq s(P')$. On the other hand it is easy to construct problems with efficient deterministic assignments of size $\frac{m}{2}$. The example in subsection 1.2 is the simplest one:

\begin{tabular}{cc}
Ann & Bob \\
$\varnothing$ & $a$ \\
$a$ & $b$
\end{tabular}

Here $m = 2$ yet $\{a \to 2, \varnothing \to 1\}$ is an efficient assignment. If $m$ is even (resp. odd), we can replicate this two-agent	wo-object pattern to get a problem in $\mathcal{A}^m$ with an efficient assignment of size $\frac{m}{2}$ (resp. $\frac{m+1}{2}$).

A useful and well known observation is that in any problem of size $m$, any efficient deterministic assignment is of size at least $\frac{m}{2}$.\footnote{If $P \in \mathcal{P}^d(E)$ is efficient and of size $m'$, and both agent $i$ and object $a$ are not matched at $P$, then $ia \notin E$, otherwise assigning $a$ to $i$ would be a Pareto improvement of $P$. It follows that any edge used by a matching feasible at $E$ has at least one endnode matched in $P$, and there are $2m'$ such nodes.} Therefore any efficient deterministic mechanism has a size index of at least $\frac{1}{2}$.

For a general (random) assignment mechanism $F$, the weakest efficiency requirement is \textbf{Ex Post Efficiency (EPE)}, requiring that the assignment $P$ be a convex combination of efficient deterministic assignments. The above observation implies that any ex post efficient assignment mechanism has a guaranteed size of at least $\frac{1}{2}$. This good news is mitigated by the fact, to which we now turn, that other normative requirements of fairness and incentive compatibility place an upper bound on the guaranteed size of the match.

\section{Three main axioms}

Given a problem $\Delta$, agent $i$ compares two feasible assignments $P, P' \in \mathcal{P}(E(\Delta))$ by means of her own allocations $p(i) = (p_{ia})_{a \in A}$ and $p'(i)$, the $i$-th rows of $P$ and $P'$ respectively. We define a familiar incomplete preference relation for agent $i$ such that $R_i = (a_1, \ldots, a_k), 1 \leq k \leq r$ (this relation is empty if $R_i = \varnothing$). We say that $p(i)$ is \textbf{sd-preferred} to $p'(i)$ (where sd stands for stochastic dominance) if

$$\sum_{1 \leq t \leq k} \sum_{1 \leq t \leq k} p_{iat} \geq \sum_{1 \leq t \leq k} p'_{iat}$$

and we write $p(i) \overset{sd}{\succeq} p'(i)$. Note that sd-indifference is just equality. We say that $p(i)$ is \textbf{strictly sd-preferred} to $p'(i)$ if $p(i) \overset{sd}{\succeq} p'(i)$ and $p'(i) \not\succeq p(i)$, so that at least one of the inequalities above is strict; then we write $p(i) \overset{sd}{>} p'(i)$. We now define the three normative properties key to the discussion of random assignment mechanisms.

The feasible assignment $P \in \mathcal{P}(E(\Delta))$ is \textbf{Ordinarily Efficient (OE)} if for all $P' \in \mathcal{P}(E(\Delta))$,

$\{p'(i) \succeq p(i) \text{ for all } i \in N\} \implies P' = P$

\textbf{Envy-Free (EF)} if $p(i) \overset{sd}{\succeq} p(j)$ for all $i, j \in N$

For a deterministic assignment, OE and EPE are the same thing, but for general random assignments OE is a strictly stronger requirement than EPE.

A deterministic mechanism (i.e., selecting $F(\Delta) = P \in \mathcal{P}^d(E(\Delta))$ for any problem) cannot be Envy-Free, so EF requires randomization. The Probabilistic Serial mechanism, explained in the next section, is Ordinarily efficient and Envy-
Free, and the only example to date of a random mechanism with these two properties.

The assignment mechanism $F$ is **Strategy-proof (SP)** if for all $\Delta$, all $i \in N$, and all $R'_i \in \mathcal{R}(A)$ we have $p(i) \succeq p'(i)$, where $F(N, A, R) = P$ and $F(N, A, (R'_i, R_{-i})) = P'$.

The simplest example of a strategyproof mechanism is the Fixed $\pi$-Priority mechanism, where $\pi$ is an arbitrary ordering $\pi = \{i_1, i_2, \cdots, i_n\}$ of the agents in $N$: agent $i_1$ gets her best acceptable object in $R_{i_1}$, next agent $i_2$ gets his best remaining acceptable object in $R_{i_2}$, if any, and so on. This mechanism is clearly Strategy-Proof and Ordinally Efficient, thus its guaranteed size is at least $\frac{m}{2}$. By replicating the $(\text{Ann}, \text{Bob}) \times \{a, b\}$ example in the previous section, we see that its guaranteed size is exactly $\frac{m}{2}$ if $m$ is even, and $\frac{m+1}{2}$ if it is odd. Moreover, the same example also shows that the guaranteed size of any deterministic strategyproof mechanism cannot be more than $\frac{1}{2}$ if $m$ is even, or $\frac{1}{2}(1 + \frac{1}{m})$ if it is odd.

There is in fact no assignment mechanism meeting OE, EF, and SP (Theorem 2 in [8]). However the two benchmark mechanisms known as **Random Priority (RP)** and **Probabilistic Serial (PS)** almost fit the bill. Here we only discuss PS, postponing until section 7 the discussion of RP.

The simplest definition of the Probabilistic Serial (PS) mechanism $PS$ is recursive. Each agent fills his allocation by eating at constant speed 1, from time $t = 0$ until at most time $t = 1$, from her best acceptable object still available. At time 0, one unit of each object is available. For brevity we only illustrate the definition by an example with 5 agents and 4 objects. Here $a$ is the best object for agents 1, 2, 3, $b$ is best for 4, 5, and $c, d$ for nobody. Then $a$ is fully eaten at time $t = \frac{1}{3}$, and 1, 2, 3 each get a $\frac{1}{3}$ share of it. Suppose agent 1 only accepts $a$, then she is done; say the next acceptable object is $b$ for agent 2 and $c$ for agent 3. Then starting from $t = \frac{1}{3}$ we have 2, 4, 5 eating the remaining $\frac{1}{3}$ unit of $b$, thus $b$ is exhausted at $t' = \frac{1}{3} + \frac{1}{9}$, and is divided in $\frac{4}{9}$ for each of 4 and 5, and $\frac{1}{9}$ for agent 2; and so on.

The PS mechanism is Ordinally Efficient, Envy-Free, but not Strategy-Proof. That is, under the premises of this axiom, the preference $p(i) \succeq p'(i)$ may not hold; however the reverse strict preference $p'(i) \succ p(i)$ does not hold either: based on her ordinal preferences only, an agent never has a compelling incentive to misreport his preferences. This latter property is called **Weak Strategy-Proofness**.

## 5 Size versus Envy-Freeness: the result

We compute first the guaranteed $m$-size index $\sigma_m(PS)$ of the PS mechanism. Then we show that this is the best feasible guaranteed size index for any Envy-Free mechanism.

The main clue comes from considering the following canonical diagonal problem of size $m$, denoted $\Delta^*_m$. This problem already played a role in three relevant earlier papers: [17], [12], and [9]. There are $m$ agents $N = \{1, \cdots, m\}$ and $m$ objects $A = \{a_1, \cdots, a_m\}$, and agent $i$’s preferences are $R_i = (a_m, a_{m-1}, \cdots, a_i)$. One interpre-

\[\text{At the profile where both Ann and Bob report that only } a \text{ is acceptable, if } a \text{ is not assigned, the size index is } 0; \text{ if } a \text{ is given to agent Bob, say, then by SP Bob still gets } a \text{ at the canonical example.}\]

\[\text{See [6] for another, more compact, though somewhat less transparent definition.}\]
tation is of a scheduling problem where objects are time slots (higher label means earlier time) and agents are jobs that are processed in exactly one time slot; each job prefers an earlier slot, and job $i$ has a deadline at time $i$ (cannot be processed later than $i$). Here is $\Delta^*_m$:

$\Delta^*_m = \begin{pmatrix}
5 & 4 & 3 & 2 & 1 \\
\emptyset & a_5 & a_5 & a_5 & a_5 \\
\emptyset & a_4 & a_4 & a_4 & a_4 \\
\emptyset & a_3 & a_3 & a_3 & a_3 \\
\emptyset & a_2 & a_2 & a_2 & a_2 \\
\emptyset & a_1 & a_1 & a_1 & a_1
\end{pmatrix}$

Problem $\Delta^*_m$ is in $A_m$ because we can assign object $a_i$ to agent $i$ for all $i$. In the PS eating algorithm, object $a_m$ is eaten first by all agents, who each get a share $\frac{1}{m}$; next object $a_{m-1}$ is eaten by agents $1, \ldots, m-1$, who each get a share $\frac{1}{m-1}$; and so on until the critical object $a_k$, such that

$$\frac{1}{k_m+1} + \frac{1}{k_m+2} + \cdots + \frac{1}{m} \leq 1 < \frac{1}{k_m+1} + \frac{1}{k_m+2} + \cdots + \frac{1}{m} + \frac{1}{k_{m+1}+1} + \frac{1}{k_{m+1}+2} + \cdots + \frac{1}{m}$$

Object $a_{k+1}$ is eaten in full, but not so object $a_k$: agents $k_m, k_m - 1, \ldots, 1$, can only eat a full unit, therefore their share of $a_k$ is $1 - (\frac{1}{k_m+1} + \frac{1}{k_{m+1}+1} + \frac{1}{k_{m+2}+1} + \cdots + \frac{1}{m})$ (which is less than $\frac{1}{k_m}$). Objects $a_{k+1}, \ldots, a_1$, are not eaten (consumed) at all.

Define for any integers $1 \leq k < m$

$$S(m, k) = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{m}$$

so that $k_m$ is defined by the inequalities $S(m, k_m) \leq 1 < S(m, k_m - 1)$. We just saw that the assignment matrix of $PS(\Delta^*_m)$ is as follows. For all $i \in N$, $a_j \in A$

$$p_{ia_j} = 0 \text{ if } i > j \text{ and/or } j < k_m$$

$$p_{ia_j} = \frac{1}{j} \text{ if } i \leq j \text{ and } j \geq k_m + 1$$

$$p_{ia_j} = 1 - S(m, k_m) \text{ if } i \leq j \text{ and } j = k_m$$

so that

$$s(PS(\Delta^*_m)) = \sum_{1 \leq i, j \leq m} p_{ia_j} = m - k_m + k_m (1 - S(m, k_m)) = m - k_m S(m, k_m)$$

$$\Rightarrow \frac{1}{m} s(PS(\Delta^*_m)) = 1 - \frac{k_m S(m, k_m)}{m} \text{ def } r_m$$

Recalling $\Delta^*_m \in A_m$, this implies $\sigma_m(PS) \leq r_m$.

**Lemma** The sequence $r_m$ is decreasing and converges to $1 - \frac{1}{e} = 0.632$ at the speed $O(\frac{1}{m})$.

For instance $r_2 = 0.750$, $r_3 = 0.722$, $r_4 = 0.708$, $r_5 = 0.687$, $r_{10} = 0.662$, $r_{20} = 0.648$.

It turns out that the canonical diagonal profile achieves the worst possible size index for the $PS$ mechanism, on all problems of $A_m$.

**Theorem**

i) $\sigma_m(PS) = r_m$

ii) The $m$-size index of any Envy-Free mechanism is at most $r_m$.

There are inefficient Envy-Free mechanisms with a worst performance than $PS$: for instance we can assign objects sequentially, uniformly among all the still unmatched agents, throwing the object away if it is not acceptable to the winner; this gives the index $\frac{m+1}{2}$ at $\Delta^*_m$.

We conjecture that the following refinement of statement ii) is true: the $m$-size index of any Ordinally Efficient and Envy-Free mechanism is $r_m$. The intuition comes from the following result about the class $D_m$ of problems such that, for a common ordering $\{a_1, \ldots, a_m\}$ of the objects, all individual preferences take the form $R^k = (a_m, a_{m-1}, \ldots, a_k)$; thus $D_m$ contains $\Delta^*_m$, as well as problems with different numbers of preferences $R^k$ for each $k$. Theorem 1 in [9] states that if $F$ is Ordinably Efficient and Envy-Free, it coincides with $PS$ on $D_m$. The conjecture is that the problems $\Delta^*_m$ are also the worst case configuration for $F$. 

6
6 The Random Priority mechanism

The \( RP \) mechanism runs the Fixed \( \pi \)-Priority mechanism after selecting \( \pi \) randomly and with uniform probability on all orderings of \( N \). It is StrategyProof and Ex Post Efficient, but not Ordinally Efficient. Moreover, \( RP \) is not Envy-Free, that is to say in the assignment \( P = RP(\Delta) \), the sd-preference \( p(i) \geq p(j) \) may fail for some \( i,j \); however \( p(j) \succ p(i) \) cannot hold either. In other words, based on her ordinal preferences \( R_i \) only, agent \( i \) never has a compelling reason to envy agent \( j \)'s allocation. This latter property is called Weak Envy-Freeness.

Our Theorem is helpful to place an upper bound on the guaranteed \( m \)-size index of \( RP \). Recall that Theorem 1 in [12] states that \( s(RP(\Delta)) \leq s(PS(\Delta)) \) for all \( m \) and all \( \Delta \in \mathcal{D}^m \) (defined in the previous paragraph). In particular \( s(RP(\Delta^*_m)) \leq s(PS(\Delta^*_m)) \), and this inequality is strict as soon as \( m \geq 4 \). Combined with our Theorem, this implies \( \sigma_m(RP) \leq r_m \).

Next [17] show that their Ranking algorithm yields the lower bound \( 1 - (1 - \frac{1}{m+1})^m \), which converges to \( 1 - \frac{1}{e} \) precisely at the canonical diagonal profile \( \Delta^*_m \). Now Ranking is the same algorithm as Random Priority when preferences are identical (but acceptable objects vary across agents). From there [20] deduces the general lower bound \( 1 - (1 - \frac{1}{m+1})^m - \frac{1}{m} \) for \( RP \).

We conclude that the performance of \( RP \) is inferior to that of \( PS \), but not asymptotically so.

7 Concluding comments

1. Other worst case indices to measure the welfare performance of \( RP \), \( PS \), and other random assignment mechanisms, are proposed in [3]. Their linear welfare factor uses Borda scores as a proxy for cardinal utilities; the performance of \( PS \) is nearly \( \frac{2}{3} \), and is superior to that of \( RP \). More work is needed to understand the connection of those results to ours.

2. Many concrete instances of assignments to jobs, schools, etc., forces participants to report only a fixed number \( q_0 \) of acceptable objects, while other objects are deemed unacceptable by the mechanism. It is therefore natural to look for the the guaranteed sizes of \( RP \) and \( PS \) in this context.

3. In many assignment instances, there are exogenous differences between the agents so that it matters more to match some agents, or some objects, than others. An example is the assignment of overdemanded slots in Dutch universities, where a student record increases her probability of admission. The design objective is now to maximize a weighted sum of the matches, as discussed in [?] for bilateral matching, and in [13] for the assignment problem. The hard question is how should we adapt \( RP \) and \( PS \) to take this new objective into account?

8 Appendix: proofs

8.1 Lemma

Step 1 \( \frac{k}{m} S(m,k) \leq \frac{1}{e} \) for all \( k, 1 \leq k \leq m - 1 \)

The Euler constant is the positive number \( C \) such that \( \lim_{m} \varepsilon_m = 0 \) where \( \varepsilon_m \overset{\text{def}}{=} ln(m) + C - (\sum_{j=1}^{m} \frac{1}{j}) \). It is easy to check that \( \varepsilon_m \) decreases to zero, as \( \varepsilon_{m+1} < \varepsilon_m \Leftrightarrow \ln(1 + \frac{1}{m}) > \frac{1}{m+1} \), which
follows from \( \ln(1 + x) > \frac{x}{x + 1} \) for \( x > 0 \). This implies

\[
S(m, k) = \ln(m) - \varepsilon_m - (\ln(k) - \varepsilon_k) \leq \ln\left(\frac{m}{k}\right)
\] (2)

Now for \( x \in [0, 1] \) we have \( |x \ln(x)| \leq \frac{1}{e} \), hence

\[
k \frac{m}{k} S(m, k) \leq k \frac{m}{k} \ln\left(\frac{m}{k}\right) \leq \frac{1}{e}
\]

as desired.

Step 2 \( k \frac{m}{k} S(m, k) \) increases strictly in \( m \).

Compare \( k_m \) and \( k_{m+1} \). We have \( S(m+1, k_m - 1) > S(m, k_m - 1) > 1 \) hence \( k_{m+1} \geq k_m \). Moreover \( S(m+1, k_m + 1) \leq S(m, k_m) \leq 1 \) implies \( k_{m+1} \leq k_m + 1 \). We distinguish two cases. If \( k_{m+1} = k_m = k \) we want to prove \( \frac{1}{m+1} S(m+1, k) > \frac{1}{m} S(m, k) \) which easily reduces to \( S(m+1, k) < 1 \), and the latter is true by definition of \( k_{m+1} \), and the fact that \( S(m, k) = 1 \) holds only for \( m = 1, k = 0 \). If \( k_{m+1} = k_m + 1 \), and we write simply \( k_m = k \), a straightforward computation gives

\[
\frac{k + 1}{m + 1} S(m + 1, k + 1) > \frac{k}{m} S(m, k)
\]

\[
\Leftrightarrow \frac{m - k}{m(m + 1)} S(m + 1, k + 1) > \frac{k}{m(k + 1)} - \frac{k + 1}{(m + 1)^2}
\]

\[
\Leftrightarrow S(m + 1, k) < 1
\]

and the latter inequality follows from the assumption \( k_{m+1} > k \).

Step 3 \( \lim_m k \frac{m}{m} S(m, k_m) = \frac{1}{e} \)

Set \( \alpha_m = k \frac{m}{m} S(m, k_m) \). By definition of \( k_m \) we have \( 1 - \frac{1}{k_m} \leq S(m, k_m) \leq 1 \), implying \( k \frac{m}{m} - \frac{1}{m} \leq \alpha_m \leq k \frac{m}{m} \). We know from Steps 1, 2 that \( \alpha_m \) converges to some \( \alpha \leq \frac{1}{e} \), so that \( \lim_m k \frac{m}{m} = \alpha \) as well. In particular \( \lim_m k_m = \infty \), therefore \( \lim_m S(m, k_m) = 1 \). From the equality in (2) we deduce \( \lim_m \ln\left(\frac{m}{k_m}\right) = 1 \), and the conclusion \( \alpha = \frac{1}{e} \) follows.

### 8.2 Theorem

#### 8.2.1 Statement i)

It remains to prove \( \sigma_m(PS) \geq m \times r_m \).

**Step 1 an auxiliary result**

In this step we consider the variant of the model where in addition to \( N, A, R \), a problem specifies a common positive capacity \( \gamma \) for each agent, and a profile of non negative capacities \( \delta = (\delta_a)_{a \in A} \) for the objects. An augmented assignment problem is now \( \tilde{\Delta} = (N, A, R, \gamma, \delta) \), and an assignment is a \( N \times A \) non negative matrix \( P = [p_{ia}] \in \mathbb{R}^{N \times A} \) such that \( \sum_N p_{ia} \leq \delta_a \) for all \( a \) and \( \sum_A p_{ia} \leq \gamma \) for all \( i \). We drop the probabilistic interpretation of \( P \), where \( p_{ia} \) was the probability that agent \( i \) is assigned to object \( a \), and think instead of the deterministic assignment of \( q \) divisible commodities, such that the initial endowment of good \( a \) is \( \delta_a \) and agent \( i \) cannot consume more than \( \gamma \) units in total. The size of \( P \) is \( s(P) = \sum_{N \times A} p_{ia} \) as before, and represents now the total capacity assigned at \( P \). Note that \( s(P) \leq \min\{n \gamma, \sum_A \delta_a\} \).

Although the \( RP \) mechanism is no longer defined, the eating algorithm runs for \( \gamma \) units of time and works as before, thus defining a feasible assignment \( PS(\tilde{\Delta}) \).

**Lemma 2** Fix \( \varepsilon > 0 \) and two augmented problems \( \tilde{\Delta} = (N, A, R, \gamma, \delta) \), \( \tilde{\Delta}' = (N, A, R, \gamma, \delta') \), such that \( \delta \leq \delta' \). Then

\[
s(PS(\tilde{\Delta})) \leq s(PS(\tilde{\Delta}')) \leq s(PS(\tilde{\Delta}))+\sum_A (\delta'_a-\delta_a)
\]

**Proof** By induction on the number of objects.

The statement is obvious if \( q = 1 \). Fix now \( q \) and assume the property holds until \( q - 1 \). Choose \( \Delta, \Delta' \), two augmented problems with \( q \) objects, that only differ in that \( \delta'_a = \delta_a + \varepsilon \) for a single object \( a \) and \( \varepsilon > 0 \). We must
prove $s(P) \leq s(P') \leq s(P) + \varepsilon$, where $P, P'$ are the corresponding assignments under $PS$. We write $D, D'$ for the two corresponding eating algorithms, and $\delta_b(z), \delta'_b(z)$ for the remaining capacity of object $b$ at time $z$ in $D, D'$.

If in $D$ object $a$ is fully consumed at time $\gamma$, then $D' = D$ and we are done. Now we assume that $a$ “dies” at some time $t, t < \gamma$. If any other object dies at $t$ in $D$, then $D$ and $D'$ coincide up to $t$, and the restriction of $D_{[t, \gamma]}$, $D'_{[t, \gamma]}$ to $[t, \gamma]$ is simply $PS$ applied to two augmented problems with at most $q - 1$ objects, capacities $(\gamma - t)$ for agents, $\delta(t)$ and $\delta'(t)$ for objects, that only differ in that $\delta'_b(t) = \varepsilon$ while $\delta_b(t) = 0$, so we can apply the inductive assumption. Thus we assume now that only object $a$ dies at $t$, and we define $t'$ to be the first time after $t$ where an object dies in $D'$, or $t' = \gamma$ if there is no such object. Note that in $D'$, $a$ is not dead at $t$, and no agent can die or switch objects during the interval $[t, t']$, because this only happens when some object dies.

We check that $\delta_b(t') \leq \delta'_b(t')$ for all $b \in A$. This is clear for $a$ because $\delta_a(t) = 0$, and also for any $b$ that nobody is eating at $t$ in $D$ (and $D'$): in $D'$ nobody switches object in $[t, t']$, thus nobody eats $b$ in that interval. Consider finally $b, b \neq a$, that the agents in the subset $N_b$ are eating at $t$ in $D$ (and $D'$): in $D'$ the agents in $N_b$ and only them continue to do so in $[t, t']$; in $D$ the agents in $N_b$ may be joined by new agents switching to $b$, and if $b$ does not die before $t'$ nobody switches in $N_b$, thus $\delta_b(t') \leq \delta'_b(t')$ as desired; this is also true if $b$ dies in $[t, t']$.

We compare now $D_{[t, \gamma]}$ and $D'_{[t, \gamma]}$: they are $PS$ applied to two augmented problems with at most $q - 1$ objects (for $b$ dying at $t'$ in $D'$, we just showed $\delta_b(t') = 0$ as well), so by the inductive assumption

$$s(D_{[t, \gamma]}) \leq s(D'_{[t, \gamma]}) \leq s(D_{[t, \gamma]}) + \sum_{b \in A} (\delta'_b(t') - \delta_b(t'))$$

$$= s(D_{[t, \gamma]}) + \delta'_a(t') + \sum_{b \in A \setminus \{a\}} (\delta'_b(t') - \delta_b(t'))$$

We also have two accounting identities

$$s(D_{[t, t']}_{[0, t]}) = \sum_{b \in A} (\delta_b(t) - \delta_b(t')) = \sum_{b \in A \setminus \{a\}} (\delta_b(t) - \delta_b(t'))$$

$$s(D'_{[t, t']}_{[0, t]}) = \sum_{b \in A} (\delta'_b(t) - \delta'_b(t'))$$

and the equalities $D_{[0, t]} = D'_{[0, t]} \ , \ \delta_b(t) = \delta'_b(t)$ for all $b \neq a$. Combining those and the two previous equalities gives

$$s(D'_{[t, t']}_{[0, t]}) - s(D_{[0, t]}) = \delta'_a(t') + \sum_{b \in A \setminus \{a\}} (\delta'_b(t') - \delta_b(t'))$$

Plugging this in the right hand inequality in (3) gives $s(D') \leq s(D) + \varepsilon$. For inequality $s(D) \leq s(D')$, recall that in $D'$, no agent still alive at $t$ dies in $[t, t']$, and the agents still alive at $t$ in $D$ are a subset of those, therefore $s(D_{[t, t']}_{[0, t]}) \leq s(D'_{[t, t']}_{[0, t]})$ completing the proof.

A useful consequence of Lemma 2 is the following monotonicity result:

**Lemma 3** Consider two (non augmented) problems $\Delta = (N, A, R), \Delta' = (N, A, R')$ where for all $i \in N$, $R'_i$ is a truncation of $R_i$: for all $i$ we have $\{ R'_i = R_i \}$ or $\{ R_i = (a_1, \ldots, a_k), k \geq 2, and R'_i = (a_1, \ldots, a_{k'}) with k' < k \}$ or $\{ R_i = (a_1) \}$ and $R'_i = \emptyset$. Then $s(PS(\Delta')) \leq s(PS(\Delta))$. 

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Proof We use the the notation of the previous proof. It is enough to assume that a single agent \( i \) truncates her preferences from \( \mathcal{R}_i = (a_1, \cdots, a_k) \), \( k \geq 2 \), to \( \mathcal{R}_i' = (a_1, \cdots, a_{k-1}) \), or from \( \mathcal{R}_i = (a_1) \) to \( \mathcal{R}_i' = \emptyset \). If in the \( \text{PS} \) algorithm \( D \) at \( R \) agent \( i \) eats no \( a_k \), then the \( \text{PS} \) algorithm \( D' \) at \( R' \) is identical. If \( i \) eats \( a_k \) units of object \( a_k \) starting at time \( t \), then it is the last object she eats. Therefore the restriction \( \widetilde{D} \) of \( D \) to \( N \setminus \{i\} \) and to interval \([t, 1]\) is \( \text{PS} \) applied to the augmented problem \( \Delta \) with capacities \( \gamma = 1 - t \) for agents, \( \delta_b(t) \) for each \( b \neq a_k \), and \( \delta_{a_k}(t) = -\alpha_k \) for object \( a_k \). On the other hand agent \( i \) dies in \( D' \) at time \( t \), and the restriction \( \widetilde{D}' \) of \( D' \) to \([t, 1]\) is \( \text{PS} \) applied to the augmented problem \( \Delta' \) on \( N \setminus \{i\} \) with capacities \( \gamma = 1 - t \), and \( \delta_b(t) \) for all \( b \). Therefore Lemma 2 implies

\[
s(D'_{[t, 1]}) = s(\widetilde{D}'_{[t, 1]}) \leq s(\widetilde{D}_{[t, 1]}) + \alpha_k = s(D_{[t, 1]})
\]

and the conclusion follows from combining this inequality with \( D'_{[0, t]} = D_{[0, t]} \). \( \blacksquare \)

Step 2 proof of statement i)

We fix now an arbitrary (non augmented) problem \( \Delta_0 = (N, A, R) \) of size \( m \), and we must prove \( s(\text{PS}(\Delta_0)) \geq m \). We construct first another problem \( \Delta \) resembling the canonical diagonal problem \( \Delta^*_m \), and such that \( s(\text{PS}(\Delta)) \leq s(\text{PS}(\Delta)) \). Pick an efficient deterministic assignment \( P \in \mathcal{P}_d(F(\Delta_0)) \) where \( m \) agents are matched to \( m \) objects. It is well known, and easy to check, that we can order these agents \( \{1, \cdots, m\} \) and these objects \( \{a_m, \cdots, a_1\} \) in such a way that \( P \) assigns object \( a_i \) to agent \( i \), so \( a_i \in R_i \), and \( a_i \) is the best object for agent \( i \) among \( \{a_1, \cdots, a_i\} \) (some of which may not be acceptable to \( i \)). By Lemma 3 if we fix \( R_i = \emptyset \) for all agents unmatched at \( P \), and for each \( i \in \{1, \cdots, m\} \) we truncate \( R_i \) at \( a_i \), thus making all objects \( \{a_1, \cdots, a_i\} \) unacceptable, then the expected size of the resulting problem \( \Delta \) is weakly smaller than at \( \Delta_0 \).

We now show \( s(\text{PS}(\Delta)) \geq m \). Let \( \{i_1, i_2, \cdots, i_H\} \) the set of agents in \( \{1, \cdots, m\} \) who do not get a full allocation in \( \text{PS}(\Delta) \) \((\sum_{A} P_{ia} < 1)\), ordered according to the time \( t_1 \leq t_2 \leq \cdots \leq t_H \) at which they die in the \( \text{PS} \) algorithm. Set \( \tau_h = t_h - t_{h-1} \), with the convention \( t_0 = 0 \). Then agent \( i_h \) eats \( \sum_{l=1}^{h} \tau_l \), therefore

\[
s(\text{PS}(\Delta)) = m - H + \sum_{h=1}^{H} (H + 1 - h) \tau_h
\]

We set \( k = m - H \) and list \( H \) inequalities that the non negative numbers \( \tau_h \) must satisfy:

\( (k + H) \tau_1 \geq 1 \), because at least object \( a_{i_1} \) is dead at \( t_1 \);

\( (k + H) \tau_1 + (k + H - 1) \tau_2 \geq 2 \), because at least objects \( a_{i_1}, a_{i_2} \) are dead at \( t_2 \), and in \([t_1, t_2]\) one agent is absent;

and for all \( h, 1 \leq h \leq H \):

\[
\sum_{l=1}^{h} (k + H + 1 - l) \tau_l \geq h
\]

because objects \( a_{i_1}, \cdots, a_{i_h} \) are dead at \( t_h \), and \( l - 1 \) agents are dead in \([t_{l-1}, t_l]\).

Define \( \Theta = \{\tau = (\tau_h) \in \mathbb{R}_+^H | \tau \) meets (4) for all \( h, 1 \leq h \leq H\} \). Then \( s(\text{PS}(\Delta)) \geq k + \min_{\tau \in \Theta} (\sum_{h=1}^{H} (H + 1 - h) \tau_h) \). We claim that the value of the latter program is \( \sum_{h=H}^{H} h \cdot \frac{\tau_h}{H} \).

To check this, we change variables to \( \lambda_h = (k + H + 1 - h) \tau_h \), so the program becomes

\[
\min \sum_{h=1}^{H} \frac{(H + 1 - h)}{h} \lambda_h
\]

such that \( \lambda \geq 0 \) and \( \sum_{l=1}^{h} \lambda_l \geq h \) for all \( h, 1 \leq h \leq H \).
Its optimal solution is $\lambda_h = 1$ for all $h$. Indeed if $\lambda_1 > 1$, a transfer from $\lambda_1$ to $\lambda_2$ lowers the objective, so $\lambda_1$ must be 1; and so on.

We just proved $s(PS(\Delta)) \geq k + \sum_{h=1}^{H} \frac{h}{k+h}$, and this sum is $k + \sum_{h=1}^{1} \frac{h}{k+h} = m - kS(m,k)$. Finally we check that the sequence $k \rightarrow kS(m,k)$ is single-peaked with its peak at $k_m$. This is because the inequality $kS(m,k) \geq (k+1)m$ is single-peaked with its peak at $m$. Feasibility w.r.t objects gives $kS(m,k) \geq (k+1)m$ (resp. $<$) is rearranged as $S(m,k) \leq 1$ (resp. $S(m,k) > 1$).

8.2.2 Statement ii)

Consider the canonical diagonal profile $\Delta^*_m$ and an Envy-Free assignment $P \in \mathcal{P}(E(\Delta^*_m))$. We check $s(P) \leq mr_m$.

Because $a_m$ is the top object for everyone, EF implies $p_{ia_m} = p_{ja_m} = x_m$ for all $i,j$. Because $a_{m-1}$ is the second best object for agents $1, \ldots, m-1$, and they all eat the same amount of $a_m$, EF implies $p_{ia_{m-1}} = p_{ja_{m-1}} = x_{m-1}$ for all $i,j \leq m-1$. Repeating the argument we see that for all $k$, $p_{ix_k} = x_k$ is independent of $i \leq k$. Feasibility w.r.t objects gives $kx_k \leq 1$, and w.r.t. agent 1 it gives $\sum_{k=1}^{m} x_k \leq 1$. Moreover $s(P) = \sum_{k=1}^{m} kx_k$. Now we claim

$$mr_m = \max_{x \in \mathbb{R}_+^m} \left\{ \sum_{k=1}^{m} kx_k \mid \sum_{k=1}^{m} x_k \leq 1; kx_k \leq 1 \text{ all } k \right\}$$

If $x$ is optimal, $x_k > 0$ and $x_{k+1} < \frac{1}{k+1}$ cannot both be true, otherwise a transfer from $x_k$ to $x_{k+1}$ improves the objective. Hence there is at most one $k^*$ such that $0 < x_{k^*} < \frac{1}{k^*}$, and then $x_k = 0$ for $k < k^*$, and $x_k = \frac{1}{k}$ for $k > k^*$. Call this case 1. Case 2 is when no such $k^*$ exists, then $x_k = 0$ up to some $k$, after which $x_k = \frac{1}{k}$.

In Case 1 we have $\sum_{k=1}^{m} x_k = S(m,k^*) + x_{k^*} \leq 1$, in particular $S(m,k^*) \leq 1$. Moreover this constraint must be tight, else we can improve the objective by raising $x_{k^*}$. Therefore $1 - S(m,k^*) = x_{k^*} < \frac{1}{k^*} \Rightarrow S(m,k^* - 1) > 1 \Rightarrow k^* = k_m$. Now $\sum_{k=1}^{m} kx_k = m - k^* + k^*x_{k^*} = m - k_mS(m,k_m)$ as desired.

In Case 2 we have $\sum_{k=1}^{m} x_k = S(m,k) \leq 1$, implying $k \geq k_m$. Moreover $\sum_{k=1}^{m} kx_k = m - \tilde{k} \Rightarrow \sum_{k=1}^{m} kx_k \leq m - k_m \leq m - k_mS(m,k_m)$.

References


