Incomplete Language as an Incentive Device

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Abstract

We analyze a game of strategic information transmission where the sender needs to exert costly, unobservable effort to acquire information. If information is sufficiently costly to acquire, full information transmission cannot occur in equilibrium even in the absence of a bias in decision-making. Acquiring information and communicating a limited amount of information can occur. Under natural conditions on the distribution of types, less revealing communication provides stronger incentives for information acquisition, so the most valuable information tends to be communicated through simple language.

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1 Introduction

Investment advisors and rating agencies commonly resort to categorical ranking systems - such as \{buy, hold, sell\} - to communicate their information to customers. Previous research (Morgan and Stocken (2003)) has emphasized conflicts of interests with respect to the use of information, e.g., an interest to inflate stock prices due to the advisor’s own asset holdings in order to rationalize this behavior. In their model an advisee is uncertain whether or not his advisor has an interest to inflate his advice. Categorical communication, much in the spirit of Crawford and Sobel (1982) arises in equilibrium. However, their analysis rests on the assumption that the advisor is unable to eliminate the conflict of interest that induces him to inflate his reports. Their leading example is securities firms that typically engage in both investment banking activities and in brokerage activities. With tight Chinese walls in place separating investment banking from brokerage services, advisees could take financial advice at face value because advisors’ interests are in predicting stock prices accurately. However, if investor’s are uncertain about whether advisors benefit in addition also from increased profits of brokerage services then advisees should be more cautious and take the advisor’s interest to inflate stock prices into account and as Morgan and Stocken (2003) shows, communication cannot be perfect but may instead take the form of categorical rankings.

What if the securities firm could credibly commit to separate its investment banking and brokerage services? Would it not be in the interest of the securities firm to do so? And if that is the case should we not observe a very rich language used for investment advice? We show that categorical ranking systems can arise endogenously even when rating agencies’ interests with respect to the use of information are perfectly aligned with those of their advisees - that is, when Chinese walls are in place and tight. Categorical ranking systems allow only a limited amount of communication, and these limits to communication effectively prevent shirking with respect to the costly acquisition of information.
Our formal model is a cheap talk game in the spirit of Crawford and Sobel (1982) with an ex ante stage of covert information acquisition. To focus on the ex ante dimension of conflict arising from the costs of acquiring information, we assume that the sender and the receiver have perfectly aligned interests ex post, and the only conflict arises from the cost of information acquisition, borne by the sender. With perfect communication, acquisition of information is incentive compatible only when the cost of information is smaller than the benefit of reduced uncertainty (the variance in our model). For higher costs, information acquisition followed by perfect communication cannot be an equilibrium. However, a coarsening of communication can increase the incentives for information acquisition. Although the sender’s on equilibrium path expected utility decreases with a coarsening of communication, his off equilibrium path expected utility - arising from deviating to not acquiring information and then lying strategically - decreases as well. Moreover, the difference between on and off equilibrium path utilities increases. Hence, categorical ranking systems and other forms of incomplete language arise endogenously as an incentive device.

Our result is closely related to Szalay (2005) which studies a sender receiver game where the receiver is able to commit to the actions he will take after listening to the sender’s advice. Similar to the present investigation, the receiver hires the sender to acquire information at a private cost and then communicate it to the receiver who then takes a decision affecting both parties’ payoffs. Although there is no conflict of interest with respect to the decision, there is a conflict of interest arising from the fact that the sender bears the private cost of acquiring information, but the sender benefits from higher quality of information. Szalay (2005) shows that the receiver may wish to commit not to take compromising choices; left with the extreme options, the sender has higher incentives to acquire information. Moreover, the joint ex ante expected utility of sender and receiver can increase when only the extreme options are available. The main difference to the current investigation is that the receiver is unable to commit to actions he takes after he has
heard the sender; that is, in this paper we analyze perfect Bayesian equilibria where the receiver’s choice of decision is sequentially rational given the sender’s advice. Compromising choices are again not taken in equilibrium even with the new equilibrium concept.

Our results provide an explanation for the incompleteness of language in other applications, ranging from refereeing for scientific journals to rating agencies and political communication, as well. We show that even if the speaker’s and listener’s state-contingent interests regarding the listener’s actions are perfectly aligned, the listener may have an incentive to only understand simple statements from the speaker. The listener’s inability to understand nuanced messages induces the speaker to invest more in learning about the true state, which may benefit the listener more than coarse communication hurts her. In the context of refereeing for scientific journals our results suggest that offering referees a wide variety of suggestions, such as \{accept, strongly revise and resubmit, revise and resubmit, reject, definitely reject\} provides referees with less incentives than the arguably simpler choice set \{accept, reject\}. Indeed some economics journals - the Journal of the European Economic Association and the Review of Economic Studies - have now adopted policies where referees in the second round of review are asked to suggest only whether a paper should be accepted or rejected, but not to make more detailed recommendations anymore at that stage. In the context of Rating Agencies, our results suggest that rating systems that have a greater number of categories are not necessarily more informative than those with fewer categories. Indeed according to our model, under some conditions, information that is very hard to acquire - and thus potentially the most valuable information in a competitive setup - should be communicated through simple language with a small number of categories.

The plan of our paper is as follows. We first introduce the model and the equilibrium concept we use. We then analyze our game by backward induction, and begin with the analysis of equilibria in the continuation game once information has been acquired. With a small extension dealing with
the receiver’s beliefs, this analysis provides a building block of our overall equilibrium construction. We then give conditions for the existence of equilibria that feature information acquisition and meaningful communication. Our central result is that if information acquisition followed by perfectly revealing communication is not an equilibrium, there may be equilibria that feature information acquisition followed by less than fully revealing communication. We provide precise conditions for the existence of such equilibria. We then specialize our model to allow for clear comparative statics results and investigate how the incentive to acquire information under meaningful communication (that is with more than one message) changes with the number of messages. Our main analysis is carried through assuming no bias and symmetric distributions. Add the end of the paper we extend our analysis to cases where these assumptions are dropped.

1.1 Literature review

tbd.

2 Model and notation

We analyze a game of information transmission (cheap talk à la Crawford and Sobel (1982)) where the sender needs to exert costly, unobservable effort to acquire information before communication.

There are two players: a Sender (he) and a Receiver (she). Before the game starts, the payoff-relevant state of the world is determined by the realization of a random variable, $\omega$. The distribution of $\omega$, denoted by $F$ and commonly known by the players, is continuous with full support on $[0, 1]$.

There are two periods. In the first period, the Sender decides whether to learn the realization of $\omega$. The Sender’s decision is unobservable and costs him $c > 0$ in utility terms. The Receiver cannot observe $\omega$ directly. In the second period, the Sender picks a message $m$ from a set $M$ and sends it to
the Receiver. Upon receiving the Sender’s message, the Receiver picks an
action \( y \in \mathbb{R} \) and the game ends.

Much of our analysis will focus on the type of “language” used by the
Sender and the Receiver in equilibrium in the communication stage. Since the
Sender’s information is two dimensional - he knows whether he has acquired
information and if he has acquired information he knows the state of the
world - it takes a message space such as \( M = [0, 1] \cup \emptyset \) to communicate all
information; \( m = \emptyset \) represents the message claiming that the Sender did not
acquire information, while \( m \in [0, 1] \) stands for the claim that he did and
the state of nature is \( \omega = m \). There are two senses in which language can
be limited; in its meaning and in its possibilities. The meaning of language
is limited if some types pool in equilibrium, so effectively it takes a smaller
message space to communicate the content of equilibrium messages. The
possibilities of language are limited if the message space is restricted to be
smaller than required to communicate the Sender’s information. The reader
is free to adopt any interpretation; in the first interpretation we will analyze
equilibrium selection, in the second one we will analyze the richness of the
message space as an incentive device to which the Receiver can commit.

The players’ ex-post payoffs are strictly concave and single-peaked in the
Receiver’s action conditional on the state of nature and independent of the
Sender’s message. Both players are expected-utility maximizers. Denote
the Sender’s ideal point by \( y^S(\omega) \) and the Receiver’s by \( y^R(\omega) \). Then, the
Sender’s ex-post payoff is \( U^S(|y - y^S(\omega)|) - ec \), while the Receiver’s ex post
payoff is \( U^R(|y - y^R(\omega)|) \). \( e \in \{0, 1\} \) denotes the Sender’s decision whether
or not he learns \( \omega \). Note that the functions \( U^S \) and \( U^R \) depend only on the
distance between the action taken and the preferred action. Moreover, we
assume that the functions are strictly concave. For expositional simplicity
we set \( U^S \) and \( U^R \) equal to the negative of a quadratic function.

In most of the paper we shall assume that \( y^S(\omega) = y^R(\omega) = \omega \). We call
this the “no bias” or “aligned preference” case. Obviously, if it were costless
for the Sender to learn \( \omega \) then it would be an equilibrium for him to learn
and report $\omega$ honestly to the Receiver, and for the Receiver to carry out $y = \omega$ for a “first-best” outcome. Our main result is that the same may not be true (and an incomplete language may emerge in equilibrium), when information acquisition is costly, even in the absence of conflict of interest between the two players. In Section 7 we extend the analysis to the case where the Sender’s and Receiver’s ideal points differ from each other.

We use the term “equilibrium” in reference for a perfect Bayesian equilibrium. We do not study any ad hoc refinements of this concept for reasons that may be familiar to the reader (e.g., it is well known that Farrell’s (1994) neologism-proofness eliminates all equilibria in standard cheap talk games).

In a perfect Bayesian equilibrium, in the second period, if the Sender is informed about the state then he reports the message for which the Receiver’s response is closest to the state of nature. If he is not informed, then he reports the message for which the response is closest to the prior mean. The Receiver, upon receiving the Sender’s message, forms beliefs over the states of nature that are consistent according to Bayes rule with the prior and the Sender’s equilibrium strategy. (Beliefs are unrestricted following out-of-equilibrium messages.) In the first period the Sender chooses whether or not to acquire information about $\omega$ depending on which action yields a greater expected payoff to him.

3 Informative Equilibria under No Bias

Throughout this section we assume that the Sender’s and the Receiver’s ideal points coincide with the state of the world, that is, $y^S(\omega) = y^R(\omega) = \omega$. Furthermore, we assume that the distribution of $\omega$ is symmetric around $1/2$, that is, $F(\omega) = 1 - F(1-\omega)$, where $F$ is a continuous cumulative distribution function. We do not restrict the distribution in any other way.

First, we characterize continuation play in equilibria where the Sender learns the value of $\omega$. We describe partitional languages where the unit interval is partitioned into a finite number of disjoint intervals corresponding
to different messages. As the number of partition elements increases, the
language approximates perfect communication (the reporting of ω).

3.1 Partitional languages and continuation play in informative equilibria

Building on Crawford and Sobel (1982), we know that any equilibrium in the
continuation play following information acquisition is equivalent to a parti-
tional equilibrium. A partitional equilibrium is described by a partition of
[0, 1] into N non-degenerate intervals, defined by the thresholds \((a_0^N, \ldots, a_N^N)\)
such that \(a_0^N = 0 < a_1^N < \ldots < a_N^N = 1\). There are \(N\) different messages
needed to sustain the equilibrium, \(m_i\) for \(i = 1, ..., N\). Let \(P_i^N = [a_{i-1}^N, a_i^N]\)
denote the \(i^{th}\) element of a partition with \(N\) elements. Types \(ω ∈ P_i^N\) send
message \(m_i\) and the receiver’s best response to message \(m_i\) is to pick the
action \(y_i = ω_i^N \equiv E[ω | ω ∈ P_i^N]\). This is an equilibrium if indeed all types
\(ω ∈ [a_{i-1}^N, a_i^N]\) weakly prefer to send message \(m_i\) rather than any other mes-
sage \(m_j\). Given the assumed preferences, the most tempting deviations are to
mimick types in adjacent partition elements. The indifference condition for
type \(a_i^N\) reporting either \(ω ∈ P_i^N\) (that is, sending message \(m_i\)) or \(ω ∈ P_{i+1}^N\)
(message \(m_{i+1}\)) is
\[
a_i^N - ω_i^N = ω_{i+1}^N - a_i^N. \tag{1}
\]
(1) forms a system of \(N − 1\) equations; initial and final condition are \(a_0^N = 0\)
and \(a_N^N = 1\), respectively. For convenience of the reader we now state
results on the existence and uniqueness of the solutions to the system. To
get uniqueness of equilibrium we impose the following

**Assumption 1:** \(\frac{∂}{∂z} E[ω | z ≤ ω ≤ τ], \frac{∂}{∂z} E[ω | z ≤ ω ≤ τ] < 1\) for all \(0 ≤ z < τ ≤ 1\).

Continuity of the distribution gives us \(\frac{∂}{∂z} E[ω | z ≤ ω ≤ τ], \frac{∂}{∂z} E[ω | z ≤ ω ≤ τ] > 0\). Assumption 1 essentially rules out that too much mass is concentrated
around some points, which could change truncated means dramatically if the
bounds of truncation are changed.
Lemma 1 Assume that the distribution of \( \omega \) is continuous and satisfies Assumption 1. Then, for any \( N \), there exists a unique solution to this equation system in \( (a_1^N, \ldots, a_{N-1}^N) \); moreover, the solution is symmetric around \( \frac{1}{2} \).

Our method of proof parallels the one given in Crawford and Sobel (1982). However, due to our more specific payoff functions and the absence of a bias we are able to get uniqueness under less restrictive assumptions on the distribution of types. Essentially, what is required is that the density of the type distribution does not vary too much. E.g., for the uniform we have 
\[
\frac{\partial}{\partial \bar{x}} E [\omega|\bar{x} \leq \omega \in \bar{x}] = \frac{\partial}{\partial \bar{x}} E [\omega|\bar{x} \leq \omega \in \bar{x}] = \frac{1}{2}.
\]
So, our result holds for a wide range of distributions around the uniform.

Notice that our result gives us uniqueness for any, arbitrary \( N \). Hence, there is still multiplicity of equilibria in the sense that \( N \) can be any positive integer.

3.2 Conditions for informative equilibrium

We can now characterize the conditions for the existence of an equilibrium with information acquisition and communication. Consider a candidate equilibrium with information acquisition and communication inducing a partition of the unit interval into \( N \) elements. On the equilibrium path the expected utility of the Sender is
\[
-c - \sum_{i=1}^{N} \Pr (\omega \in P_i^N) Var (\omega|\omega \in P_i^N)
\]
Suppose the Sender deviates to not acquiring information. In that case he wishes to induce his most preferred action among those that are feasible, that is the action that comes closest to the prior mean. Let \( \langle \frac{N}{2} \rangle \) denote the smallest integer greater than \( \frac{N}{2} \). Given the symmetry of the distribution, the best action among those that can be induced in the candidate equilibrium is \( y_{\langle \frac{N}{2} \rangle} \) and the resulting expected utility is
\[
-Var (\omega) - \left( E (\omega|\omega \in P_{\langle \frac{N}{2} \rangle}) - \frac{1}{2} \right)^2.
\]
So, information acquisition is an equilibrium with $2n$ messages if, and only if,

$$c \leq Var(\omega) - \sum_{i=1}^{N} \Pr(\omega \in P_i^N) Var(\omega|\omega \in P_i^N) + \left( E\left(\omega|\omega \in P_i^N\right) - \frac{1}{2} \right)^2. \tag{2}$$

Condition (2) is necessary and sufficient for the existence of an informative equilibrium with partitional language inducing $N$ distinct choices.

Clearly, given that we assume no bias, the focal equilibrium would be the limiting case where $N$ goes to infinity. However, to make things interesting, we focus on the case where such an equilibrium cannot exist, by imposing

**Assumption 2:** $c > Var(\omega)$.

Under Assumption 1, information acquisition followed by fully informative communication cannot be an equilibrium, because it violates condition (2). The interesting question is whether there is any other form of communication that is informative.

**Lemma 2** Given Assumption 2, perfectly aligned ex post interests and a symmetric distribution of $\omega$, an equilibrium is informative only if $N$ is even.

The proof of the lemma is obvious: given assumption 2, an equilibrium can be informative only if

$$\left( E\left(\omega|\omega \in P_i^N\right) - \frac{1}{2} \right)^2 > \sum_{i=1}^{N} \Pr(\omega \in P_i^N) Var(\omega|\omega \in P_i^N).$$

For odd numbered partitions, the value on the left-hand side is zero, while the value on the right-hand side is strictly positive for non-degenerate distributions. Hence, we need an even number of partitions to induce $E\left(\omega|\omega \in P_i^N\right) \neq \frac{1}{2}$. This result hinges on the conditions listed in the lemma; in the extensions below we show how informative communication can be an equilibrium with an odd number of partitions when either the distribution is asymmetric or there is a conflict of interest ex post between sender and receiver.
Let \( n \equiv \frac{N}{2} \). Given symmetry, it is enough to consider only one half of the support, say the upper half. Informative communication is an equilibrium under assumption 1 with communication inducing \( 2n \) different actions iff

\[
\left( E(\omega|\omega \in P_{n+i}^{2n}) - \frac{1}{2} \right)^2 - 2 \sum_{i=1}^{n} \Pr(\omega \in P_{n+i}^{2n}) \text{Var}(\omega|\omega \in P_{n+i}^{2n}) > 0 \quad (3)
\]

We term the difference on the left-hand side of this condition as the added incentive to acquire information due to the coarseness of language. We now investigate under what conditions this additional incentive is indeed positive. We do so by first analyzing simple languages with few different messages; later on we generalize the findings for arbitrary message spaces.

4 Binary messages

4.1 Incentives for Information Acquisition

We now characterize the conditions (on \( c \)) that are necessary and sufficient for the existence of an equilibrium with two messages where the Sender learns \( \omega \). In such an equilibrium, in the communication continuation game the Partitions \( P_1^2 \) and \( P_2^2 \) have a natural interpretation. We call message \( m_1 \) as \( L \) for low and \( m_2 = H \) for high. The focal equilibrium behavior is for the sender to say "H" iff \( \omega \geq \frac{1}{2} \) and "L" otherwise (this is the focal continuation equilibrium as it minimizes the ex-ante expected loss of either party). The Receiver responds by either \( y_H = \omega_H \equiv E[\omega|\omega > \frac{1}{2}] \) or \( y_L = \omega_L \equiv E[\omega|\omega < \frac{1}{2}] \) depending on the message; note that \( \omega_H - \frac{1}{2} = \frac{1}{2} - \omega_L \) by symmetry, so that \( \omega_H + \omega_L = 1 \). The Sender’s ex ante expected payoff in equilibrium is \(-c - \text{Var} [\omega|\omega > \frac{1}{2}] \). Clearly a deviation after learning \( \omega \) is not profitable. If the Sender does not learn \( \omega \) then his expected payoff is \(-\text{Var} [\omega] - (\omega_H - \frac{1}{2})^2 \). Therefore the condition for the existence of a binary-message equilibrium where the Sender learns \( \omega \) is

\[
-c - \text{Var} [\omega|\omega > \frac{1}{2}] \geq -\text{Var} [\omega] - \left( \omega_H - \frac{1}{2} \right)^2.
\]

(4)
and the added incentive to acquire information due to the coarseness of communication is positive iff

\[
\left( \omega_H - \frac{1}{2} \right) > \sigma_H,
\]

where \( \sigma_H \equiv \sqrt{\text{Var} \left[ \omega \mid \omega > \frac{1}{2} \right]} \) is the standard deviation of the truncated distribution.

**Proposition 1** Suppose the distribution of \( \omega \) is symmetric with a continuous cdf on \([0, 1]\). Then, binary communication provides the Sender with higher incentives to acquire information than fully revealing communication for any distribution
i) such that \( \omega_H \geq \frac{3}{4} \);
ii) such that \( \frac{1}{2} < \omega_H < \frac{3}{4} \), and \( \sigma_H < \omega_H - \frac{1}{2} \).

The formal proof of the theorem is in the appendix. The idea is very simple. Given quadratic payoff functions the payoff relevant characteristics of the distribution of \( \omega \) are the first two moments, truncated to the half intervals. The moments of these distributions are determined together; in particular, there is an upper bound on the standard deviation of the truncated distribution that depends on the location of the mean. We show in the appendix that the upper bound on the standard deviation is in fact described by a half circle with center .75. Thus, the set of distributions that are both feasible and provide the Sender with higher incentives to acquire information when restricted to binary communication, are those with moments in the half circle and below the 45-degree line in the following plot:
Let $y \equiv \sigma$ and $x \equiv \omega_H$. $y = \frac{1}{2} \sqrt{2 \sqrt{-2x^2 + 3x - 1}}$ is the formula for the half-circle in the positive orthant with radius $\frac{1}{4}$ and center $\frac{3}{4}$.

Since there is no measure on the feasible distributions, it is impossible to say whether binary communication provides better incentives for information acquisition than full communication most of the time. However, it is still interesting to note that the set of distributions described in the theorem covers roughly 80% of the area of feasible distributions$^1$. In that sense, it is probably fair to say that coarse communication is good for incentives for information acquisition under fairly general conditions.

### 4.2 A non-incentive example

To understand what kind of distributions are ruled our by the conditions in Proposition 1, let us construct an example that violates these conditions.

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$^1$The intersections of the 45°-line and the half-circle are $\omega_H = \frac{3}{4}$ and $\omega_H = \frac{1}{2}$. The total area below the half-circle is $\frac{1}{32}\pi$. The area between the 45° line and the half-circle is $\frac{1}{64}\pi - \frac{1}{32}$, so the ratio the two areas is $\frac{\frac{1}{64}\pi - \frac{1}{32}}{\frac{1}{32}\pi} \simeq 0.20674$. 

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Consider the truncated distribution with density

\[ \hat{f}(\omega) = \begin{cases} \frac{1-\varepsilon}{\varepsilon} & \text{for } \omega \in \left[ \frac{1}{3}, \frac{1}{2} + \varepsilon \right) \\ \frac{1}{2-\varepsilon} & \text{for } \omega \in \left[ \frac{1}{2} + \varepsilon, 1 \right] \end{cases} \]

for \( \varepsilon > 0 \).\(^2\) One easily verifies that

\[ \omega_H = \frac{3}{4}\varepsilon + \frac{1}{2} \]

and

\[ Var\left( \omega \mid \omega > \frac{1}{2} \right) = \left( \frac{1}{2}\varepsilon^2 + \frac{5}{6}\varepsilon + \frac{1}{4} - \left( \frac{3}{4}\varepsilon + \frac{1}{2} \right)^2 \right) \]

So, from condition (5) (squared again), the added incentive to acquire information is positive iff

\[ \frac{9}{16}\varepsilon^2 - \left( \frac{1}{2}\varepsilon^2 + \frac{5}{6}\varepsilon + \frac{1}{4} - \left( \frac{3}{4}\varepsilon + \frac{1}{2} \right)^2 \right) > 0. \]

The following graph shows the behavior of the added incentive as a function of \( \varepsilon \).

\(^2\)For \( \varepsilon \in (0, 0.5) \), this is indeed a density, as

\[ \int_{\frac{1}{2}}^{1} \hat{f}(\omega) \, d\omega = \int_{\frac{1}{2}}^{\frac{1}{2}+\varepsilon} \frac{1-\varepsilon}{\varepsilon} \, d\omega + \int_{\frac{1}{2}+\varepsilon}^{1} \frac{1}{2-\varepsilon} \, d\omega = 1 - \varepsilon + \varepsilon = 1, \text{ and } \hat{f}(\omega) > 0 \]
For $\varepsilon$ positive but close to zero, fully revealing communication provides better incentives for information acquisition than binary communication does. For $\varepsilon$ close to zero, the distribution has almost all its mass concentrated around the prior mean. Hence, the bias induced off equilibrium path goes to zero as $\varepsilon$ goes to zero. On the other hand, the variance in the upper half of the support goes to zero as well. Moreover, the bias term approaches zero faster than the variance term does.

Observe that $\varepsilon = \frac{1}{3}$ is the uniform distribution, for which condition (5) is satisfied. For $\varepsilon$ positive, but not too large, condition (5) is violated.

### 4.3 Joint Surplus

We now investigate under which conditions the joint utility of Sender and Receiver is higher with coarse communication and information acquisition than with full communication and no information acquisition. This is a correct measure of utility if utility is transferable, that is, if both of them have quasilinear utility and can exchange transfers ex ante. In that case they would agree on a language that maximizes their ex ante expected utility. They jointly prefer information acquisition followed by communication through binary messages relative to meaningless communication whenever

$$-c - 2\text{Var} \left[ \omega \middle| \omega > \frac{1}{2} \right] > -2\text{Var} [\omega].$$

On the other hand, information acquisition followed by communication using binary messages is an equilibrium at all when

$$-c - \text{Var} \left[ \omega \middle| \omega > \frac{1}{2} \right] \geq -\text{Var} [\omega] - \left( \omega_H - \frac{1}{2} \right)^2.$$

**Proposition 2** Whenever information acquisition followed by communication inducing two different actions is an equilibrium, then the Sender and the Receiver jointly prefer this equilibrium to the babbling equilibrium.
The proof is trivial. Using $\text{Var}(\omega) \equiv (\omega_H - \frac{1}{2})^2 + \text{Var}(\omega|\omega > \frac{1}{2})$, we can reformulate the first condition to

$$-c - 2\text{Var}\left[\omega|\omega > \frac{1}{2}\right] > -\text{Var}[\omega] - \left(\omega_H - \frac{1}{2}\right)^2 - \text{Var}\left(\omega|\omega > \frac{1}{2}\right)$$

which simplifies exactly to the second condition.

We now allow for richer message spaces.

5 Towards an optimal language

Since the Sender and the Receiver have perfectly aligned interests for given information, the Sender and Receiver should somehow agree on the “best” language that ensures that communication is meaningful. To characterize what exactly the best language is, we show that Sender and Receiver jointly prefer that information is acquired whenever it is incentive compatible for the Sender to indeed acquire information. Moreover, we show that the expected utility on the equilibrium path is increasing in the number of distinct actions that the Sender can induce. It follows that the best language, or message space, is the one that induces the highest number of distinct actions while still inducing the Sender to acquire information. These results are stated more formally in the following Proposition:

**Proposition 3** Whenever information acquisition followed by communication inducing a finite number of distinct actions is an equilibrium, then this equilibrium is preferred to uninformative communication. Moreover, the expected utility of Sender and Receiver is increasing in the number of induced actions.

Information acquisition is an equilibrium if

$$c \leq \text{Var}(\omega) - 2 \sum_{i=1}^{n} \text{Pr}(\omega \in P_{n+i}^{2n}) \text{Var}(\omega|\omega \in P_{n+i}^{2n}) + \left(E\left(\omega|\omega \in P_{n+1}^{2n}\right) - \frac{1}{2}\right)^2.$$
On the other hand, Sender and Receiver are better off with information acquisition if

\[-c - 4 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \text{Var} (\omega | \omega \in P_{n+i}^{2n}) > -2 \text{Var} (\omega).\]

The idea to show the first part of the Proposition is again to show that the incentive condition implies the welfare ranking. To prove the second statement, note that players prefer to have more communication if

\[-2 \sum_{i=1}^{n+1} \int_{a_{n+i}^{2(n+1)}}^{a_{n+1+i}^{2(n+1)}} \left( \omega - \omega_{n+1+i}^{2(n+1)} \right)^2 f(\omega) d\omega > -2 \sum_{i=1}^{n} \int_{a_{n+i}^{2n}}^{a_{n+i-1}^{2n}} \left( \omega - \omega_{n+i}^{2n} \right)^2 f(\omega) d\omega\]

Following the proof of Theorem 3 in Crawford and Sobel (1982) we continuously deform the coarser partition so as to end up with the finer partition and show this way that the equilibrium utility of the players increases along the way.

### 5.1 Comparative Statics: Comparing Languages

Intuitively, one might think that binary communication provides the most extreme incentives for information acquisition. Faced with only two options, the sender is forced to make a sharp prediction. However, this reasoning neglects the fact that the receiver takes into account that there are only two messages available. Hence the response to a sharp statement needs not be a sharp reaction. It depends on the details of the distribution of types whether a richer language is better for incentives (and hence better from a welfare perspective as well) than a binary language.

We now look formally into this question. We address the question from two angles. First, we increase the richness of language in the smallest possible way - from binary language to a language with four messages. Second, we compare binary language with languages of arbitrary richness.
5.1.1 When does binary language provide the strongest incentives for information acquisition?

The extra incentive to acquire information due to binary communication is stronger than the extra incentive with any other (even) number of messages iff

$$-2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \text{Var} (\omega | \omega \in P_{n+i}^{2n}) + \left(\frac{\omega_{n+1}^{2n} - 1}{2}\right)^2$$

$$< -2 \sum_{i=1}^{2} \Pr (\omega \in P_i^2) \text{Var} (\omega | \omega \in P_i^2) + \left(\omega_H - \frac{1}{2}\right)^2.$$  (6)

Let us begin with the case where $n = 2$. With four messages it is again natural to think of the messages as saying very high, high, low, and very low. To distinguish these messages clearly from the ones we had before, we term the messages in the upper half of the support as “HH” and “HL”. Recall that $a_i^N$ refers to the threshold value between partitions. Condition (6) specializes to

$$\left(\omega_H - \frac{1}{2}\right)^2 - \text{Var} (\omega | \omega \in P_2^2) > \left(\omega_{HL} - \frac{1}{2}\right)^2 - 2 \Pr (\omega \in P_3^4) \text{Var} (\omega | \omega \in P_3^4)$$

$$- 2 \Pr (\omega \in P_4^4) \text{Var} (\omega | \omega \in P_4^4)$$

Completing the squares in the variance terms on the right-hand side appropriately, we can write

$$\Pr (\omega \in P_3^4) \text{Var} (\omega | \omega \in P_3^4) + \Pr (\omega \in P_4^4) \text{Var} (\omega | \omega \in P_4^4)$$

$$= \frac{1}{2} \text{Var} (\omega | \omega \in P_2^2) - \Pr (\omega \in P_3^4) (\omega_H - \omega_{HL})^2 - \Pr (\omega \in P_4^4) (\omega_H - \omega_{HH})^2.$$

Substituting back, simplifying, and rearranging, we obtain

$$\left(\omega_H - \frac{1}{2}\right)^2 - \left(\omega_{HL} - \frac{1}{2}\right)^2 > 2 \Pr (\omega \in P_3^4) (\omega_H - \omega_{HL})^2 + 2 \Pr (\omega \in P_4^4) (\omega_H - \omega_{HH})^2$$

The term on the left-hand side is the difference in off-equilibrium path incentives. Having only two rather than four words to communicate implies that
the best feasible choice to induce after shirking on information acquisition moves further away from the prior mean. The term on the right-hand side is the difference in on equilibrium path incentives. If two rather than four messages are available, then the value of information on path decreases, that is the residual variance increases. So, there are always countervailing effects at work and we now show under what conditions one dominates the other.

**Proposition 4**  Two messages provide the Sender with stronger incentives to acquire information than four for all distributions of types such that \( \omega_{HL} \geq \frac{1+\omega_{HH}}{3} \).

For all distributions of types such that \( \omega_{HL} < \frac{1+\omega_{HH}}{3} \), there is \( \alpha' \) such that two messages provide the Sender with more (less) incentives for information acquisition than four if \( \Pr (\omega \in P^4_3) < \alpha' \) (\( \Pr (\omega \in P^4_3) > \alpha' \)).

Intuitively, the distribution of types satisfies \( \omega_{HL} \geq \frac{1+\omega_{HH}}{3} \) if the conditional means \( \omega_{HL} \) and \( \omega_{HH} \) are relatively close together. However, if that is the case, then the additional value (on equilibrium path) to communicate through four rather than only two messages is small. Essentially, most of the information is already contained in the words high and low; it does not make that much of a difference to say very high or just so high. Under the reverse condition it may still not be that valuable to make more precise statements as to the location of the state of the world if the distribution has relatively little mass concentrated around the mean. If that is the case then binary language provide excellent incentives because the off path threat \( \omega_H \) is relatively far away from the prior mean.

In fact, under the conditions given in the last proposition, binary language provides indeed stronger incentives for information acquisition than any other language:

**Proposition 5**  For all \( n \geq 1 \) and all distributions such that \( 2 \Pr (\omega \in P^4_3) \leq 0.5 \), the incentive to acquire information is stronger under binary communication rather than under communication inducing \( 2n \) distinct choices.
The intuition is again pretty simple; if the distribution of types has a lot of mass concentrated around the prior mean, then two messages are not able to convey a lot of information, that is the posterior mean as a function of the message does not vary much. Hence, the incentive effect off equilibrium path is not very strong. Vice versa, if the distribution has relatively little mass concentrated around the prior mean, then two messages do convey quite some information, and hence the punishment for the Sender off equilibrium path is relatively pronounced.

5.1.2 On the monotonicity of incentives in the coarseness of language

We now investigate how the incentive to acquire information depends on the coarseness of language. The incentive to acquire information is monotonic in the coarseness of language if

\[-2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \text{Var} (\omega | \omega \in P_{n+i}^{2n}) + \left( \omega_{n+1}^{2n} - \frac{1}{2} \right)^2 \]

\[\geq -2 \sum_{i=1}^{n+1} \Pr (\omega \in P_{n+1+i}^{2(n+1)}) \text{Var} (\omega | \omega \in P_{n+1+i}^{2(n+1)}) + \left( \omega_{n+2}^{2(n+1)} - \frac{1}{2} \right)^2 .\]

We can build on the characterization results on the value of information on the equilibrium path to reformulate this comparison. In particular, recall the sequence \( \{ a_n^{2n}, x, a_{n+1}^{2n} (x), \ldots, a_{2n-1}^{2n} (x), a_{2n}^{2n} \} \) that we used to continuously deform the equilibrium with \( 2n \) partition elements into the equilibrium with \( 2(n+1) \) partition elements. More precisely, for \( x = a_n^{2n} \), the remaining \( n \) non-degenerate partition elements in the upper half interval (and their counterpart in the lower half interval) correspond to the communication equilibrium with \( 2n \) messages; for \( x = a_{n+2}^{2(n+1)} \) the resulting \( n+1 \) partition elements (together with their counterparts in the lower half interval) correspond to the communication equilibrium with \( 2(n+1) \) messages. We have shown above
that
\[-2 \sum_{i=1}^{n+1} \Pr(\omega \in P_{n+1+i}^{2(n+1)}) \text{Var}(\omega|\omega \in P_{n+1+i}^{2(n+1)}) - \left(-2 \sum_{i=1}^{n} \Pr(\omega \in P_{n+i}^{2n}) \text{Var}(\omega|\omega \in P_{n+i}^{2n})\right)\]
\[= - \int_{a_{n}^{2n}}^{a_{n+1}^{2(n+1)}} \left((x - \bar{\omega}(x))^2 - (x - \omega_{n+1}^{2n}(x))^2\right) f(x) \, dx\]

Hence, the incentive to acquire information is increasing in the coarseness of language if

\[\left(\omega_{n+1}^{2n} - \frac{1}{2}\right)^2 - \left(\omega_{n+2}^{2(n+1)} - \frac{1}{2}\right)^2 \geq \int_{a_{n}^{2n}}^{a_{n+1}^{2(n+1)}} \left((x - \omega_{n+1}^{2n}(x))^2 - (x - \bar{\omega}(x))^2\right) f(x) \, dx,\]

where \(\bar{\omega}(x)\) is the conditionally expected value of \(\omega\) conditional on \(\omega\) being truncated between \(a_{n}^{2n}\) and \(x\) and \(\omega_{n+1}^{2n}(x)\) is the conditionally expected value of \(\omega\) conditional on \(\omega\) being truncated between \(x\) and \(a_{n+1}^{2n}\). Recall that, in order to get a unique equilibrium, we have imposed Assumption 1. We have the following proposition:

**Proposition 6** Suppose that
\[
\frac{\partial}{\partial x} \omega_{n+1}^{2n}(x) \leq 1 \tag{8}
\]
and in addition
\[
\left(1 - \Pr(\omega \in P_{n+2}^{2(n+1)})\right) \left(\omega_{n+1}^{2n} - \frac{1}{2}\right)^2 \geq \left(\omega_{n+2}^{2(n+1)} - \frac{1}{2}\right)^2, \tag{9}
\]
where both (8) and (9) are satisfied, e.g. by the uniform distribution. Then, the incentive to acquire information is decreasing in the number of induced distinct actions.

**Proof.** Differentiating the expression in brackets of the integrand in (7), we obtain
\[
2 \left(x - \omega_{n+1}^{2n}(x)\right) \left(1 - \frac{\partial}{\partial x} \omega_{n+1}^{2n}(x)\right) - 2 \left(x - \bar{\omega}(x)\right) \left(1 - \frac{\partial}{\partial x} \bar{\omega}(x)\right).\]
By the conditions stated in the proposition, and the facts that \( x - \omega_{n+1}^2(x) < 0 \) and \( x - \bar{\omega}(x) \geq 0 \), the derivative is negative. Hence, we can bound the right-hand side of (7) from above:

\[
\int_{a_n^2}^{a_{n+2}^{2(n+1)}} \left( (x - \omega_{n+1}^2(x))^2 - (x - \bar{\omega}(x))^2 \right) f(x) \, dx \leq \Pr(\omega \in P_{n+2}^{2(n+1)}) \left( \omega_{n+1}^2 - \frac{1}{2} \right)^2.
\]

Hence, if

\[
1 - \Pr(\omega \in P_{n+2}^{2(n+1)}) \left( \frac{1}{2} \right)^2 \geq \left( \omega_{n+2}^{2(n+1)} - \frac{1}{2} \right)^2,
\]

then the result follows, as claimed.

We now verify the conditions for the uniform distribution. If \( \omega \) follows a uniform distribution and we partition the interval \([x, 1]\) into \( n \) equal intervals, then \( \omega_{n+1}^2(x) = \frac{x + x + \frac{1-x}{n}}{2} \), so \( \frac{\partial}{\partial x} \omega_{n+1}^2(x) = 1 - \frac{1}{2n} < 1 \), so (8) is satisfied. Condition (9) amounts for the uniform to

\[
\left( 1 - \frac{1}{2(n+1)} \right) \frac{1}{16n^2} \geq \frac{1}{16(n+1)^2}.
\]

Simplifying, we find this is equivalent to

\[
(2n + 1)(n + 1) \geq 2n^2,
\]

and hence satisfied for all \( n \). 

Notice that for the uniform distribution, we can show directly the incentives are monotonic in the number of messages. With \( N = 2n \), the relevant comparison becomes

\[
\left( \frac{1}{2N} \right)^2 - \frac{1}{12N^2} > \left( \frac{1}{2(N+2)} \right)^2 - \frac{1}{12(N+2)^2}
\]

which is equivalent to

\[
\left( \frac{1}{4} - \frac{1}{12} \right) \frac{1}{N^2} > \left( \frac{1}{4} - \frac{1}{12} \right) \frac{1}{(N+2)^2}
\]

and satisfied for all \( N \).
5.1.3 Optimal Language

It is now straightforward to characterize the optimal language that Sender and Receiver should agree to. Under the conditions stated in the previous propositions, they should use the richest language that still induces information acquisition. Under the regularity conditions that induce monotonicity of incentives, it will be true that language needs to become coarser as information becomes more costly to acquire. We state this result formally in the following Proposition:

**Proposition 7** Suppose that conditions (8) and (9) are satisfied. Then, the higher is $c$ the coarser is equilibrium language.

6 Extensions

6.1 odd numbered partitions and incentives

With symmetric distributions and no bias between Sender and Receiver, we have shown that only communication inducing an even number of distinct choices provides better incentives for information acquisition compared to fully revealing communication. This result is arguably strong and in contrast with a real world language \{buy, hold, sell\} that is sometimes used. However, it is easy to eliminate this apparent discrepancy when we allow for asymmetric distributions of types. Indeed, suppose the density is given by

$$f(\omega) = \begin{cases} 
\frac{1}{2} & \text{for } 0 \leq \omega < \frac{1}{3} \\
1 & \text{for } \frac{1}{3} \leq \omega < \frac{2}{3} \\
\frac{3}{2} & \text{for } \frac{2}{3} \leq \omega \leq 1 
\end{cases}$$

and zero otherwise.

With this density, the communication continuation game has the obvious three partition equilibrium. Three distinct choices are induced, $y_1 = \omega_1 = \frac{1}{6}$, $y_2 = \omega_2 = \frac{1}{2}$ and $y_3 = \omega_3 = \frac{5}{6}$.
The unconditional expectation is $E[\omega] = \frac{11}{18}$ and the unconditional variance is $Var[\omega] = \frac{23}{324}$. The crucial point, why limited communication provides the sender with more incentives for information acquisition is that the unconditional expectation $E[\omega]$ is biased away from $\omega_2 = \frac{1}{2}$. If the receiver expects the sender to be informed, then the uninformed Sender’s most preferred act is out of reach, and the Sender suffers from a loss due to the biased choice of $(\frac{11}{18} - \frac{1}{2})^2 = \frac{1}{81}$ off equilibrium path. On equilibrium path, we have $E\left[Var\left[\omega|\omega \in \tilde{F}_i^3\right]\right] = \frac{1}{9} \frac{1}{12}$. For $c > \frac{23}{324}$ there is no equilibrium with information acquisition followed by fully revealing communication. However, for $c \leq \frac{23}{324} - \frac{1}{9} \frac{1}{12} + \frac{1}{81} = \frac{24}{324}$ there is an equilibrium with information acquisition followed by communication inducing three distinct choices.

6.2 Refinements

Most refinements do not have any bite in cheap talk games. Those that usually have bite are Neologism Proofness (Farrell (1994)) and NITS (Chen et. al (2008)).

Neologism Proofness requires that there should not be any unused messages, that when used would be self-signalling. In our context: less than fully revealing communication in the continuation game following information acquisition is never neologism proof. When fully revealing communication does not induce information acquisition, the only equilibrium that is neologism proof is the babbling equilibrium. Hence, the refinement selects a bad equilibrium.

Consider next the No Incentive to Separate (NITS) criterion: type $\omega = 0$ should have no incentive to separate if he somehow could. For the case $b = 0$, only the fully revealing communication survives NITS in the continuation game following information acquisition. When information acquisition followed by fully revealing communication is not an equilibrium, no informative equilibrium satisfies NITS. The only equilibrium that satisfies NITS is no information acquisition followed by babbling and the Receiver taking the prior optimal decision. Hence, also NITS selects again the babbling equilib-
7 Talking to a biased sender

7.1 The uniform quadratic case

We now extend our analysis to the case of a biased sender. Without further assumptions on the distribution of types, the analysis becomes extremely hard. Thus, to keep the model tractable, we assume from now on that \( \omega \) follows a uniform distribution. Moreover, the sender’s vNM utility function is now given by

\[
- (y - (b + \omega))^2 - ec,
\]

so that his preferred choice is \( y^S(\omega) = \omega + b \).

We can apply Crawford and Sobel’s results (1982) for the continuation game after information acquisition. For convenience of the reader we reproduce the main results here. In particular, Crawford and Sobel (1982) show equilibrium communication takes the following form. Since \( \omega_i^N = \frac{a_{i-1}^N + a_i^N}{2} \) for \( i = 1, \ldots, N \), the indifference conditions of threshold types \( a_i^N, i = 1, \ldots, N - 1 \) are given by

\[
- \left( \frac{a_{i-1}^N + a_i^N}{2} - (b + a_i^N) \right)^2 = - \left( \frac{a_{i}^N + a_{i+1}^N}{2} - (b + a_i^N) \right)^2
\]

which can hold only if the length of the partitions increases at rate \( 4b \). The solutions of this system of indifference conditions, together with the boundary conditions \( a_0^N = 0 \) and \( a_N^N = 1 \) takes the form

\[
a_i^N = \frac{i}{N} + 2b(i - N) \quad \text{for} \quad i = 0, \ldots, N
\]

where \( N \) is any integer smaller or equal to \( N(b) \), which in turn is defined as the largest integer \( N \) for which \( 2bN(N - 1) < 1 \). It is easy to compute the induced actions \( y_i^N = \omega_i^N = \frac{a_{i-1}^N + a_i^N}{2} \). We have

\[
\omega_i^N = \frac{i - 1}{N} + \frac{1}{2N} + 2bi(i - 1 - N) + b(N + 1) \quad \text{for} \quad i = 1, \ldots, N.
\]
Moreover, the expected utility of the receiver is given by

\[ -\sum_{i=1}^{N} a_i^N \int_{a_{i-1}^N}^{a_i^N} (\omega - \omega_i^N)^2 = -\frac{1}{12N^2} - b^2 N^2 - \frac{1}{3} \]

and the expected utility of the Sender is given by

\[ -\frac{1}{12N^2} - b^2 N^2 - \frac{1}{3} - b^2. \]  

(10)

Building on these results, we can now analyze incentives for information acquisition.

### 7.2 Optimal Deviations

To analyze incentives for information acquisition, we need to compute the marginal value of information to the Sender, that is, the difference between expected utility on equilibrium path and off equilibrium path. We begin with a discussion of optimal behavior off equilibrium path.

Suppose the Receiver expects the Sender to acquire information and to communicate his information inducing \( N \) distinct actions. To compute the marginal value of information to the Sender, we first need to derive the Sender’s best deviation after shirking on information acquisition. Let \( y_i^N \) denote the best action that an uninformed Sender can induce. The following Lemma characterizes this choice as a function of \( N \) and \( b \).

**Lemma 3** For a given bias \( b \), suppose the Receiver expects the Sender to acquire information and to communicate information inducing \( N \) distinct choices \( y_i^N \) for \( i = 1, \ldots, N \), for \( N \leq N(b) \). Then, keeping \( N \) fixed and decreasing \( b \) towards zero, the optimal action to induce after shirking on information acquisition converges to the mean of partition \( i^* = \frac{N}{2} + 1 \) for \( N \) even and \( i^* = \frac{N+1}{2} \) for \( N \) odd, respectively. More precisely, these actions become optimal for \( b < \frac{2}{N(N^2-2)} \) (\( N \) even) and \( b < \frac{1}{N(N^2+2)} \), respectively.
Proof. An uninformed Sender’s ideal action is \( y^* = \frac{1}{2} + b \). If the Sender does not acquire information, but the Receiver’s beliefs are consistent with the Sender acquiring information and communicating \( N \) distinct partition elements, then \( i^* \) is a solution to the problem

\[
\min_i \left| \frac{i - 1}{N} + \frac{1}{2N} + 2bi (i - 1 - N) + b(N + 1) - \left( \frac{1}{2} + b \right) \right|
\]

It is obvious that the statement in the Lemma is true for \( b = 0 \); so assume that \( b \) is positive but small.

Consider first the case where \( N \) is odd. We have

\[
\omega_{\frac{N+1}{2}} = \frac{1}{2} - b(N + 1) \left( \frac{N - 1}{2} \right)
\]

and

\[
\omega_{\frac{N+1}{2}+1} = \frac{1}{2} + \frac{1}{N} - \frac{b}{2} (N + 1) (N - 1) + 2b.
\]

Clearly, for \( b = 0 \), \( \frac{\partial}{\partial b} \omega_{\frac{N+1}{2}} = -(N+1) \left( \frac{N-1}{2} \right) < 0 \).

Likewise, for \( b = 0 \), \( \omega_{\frac{N+1}{2}+1} = \frac{1}{2} + \frac{1}{N} > \frac{1}{2} \). For \( b \) sufficiently large we eventually have \( \omega_{\frac{N+1}{2}+1} < \frac{1}{2} + b \).

Consider now the distance between \( \omega_{\frac{N+1}{2}} \) and \( y^* \), and the difference between \( \omega_{\frac{N+1}{2}+1} \) and \( y^* \), respectively. We have

\[
y^* - \omega_{\frac{N+1}{2}} = \frac{1}{2} + b - \frac{1}{2} + \frac{b}{2} (N^2 - 1) = \frac{b}{2} (N^2 + 1)
\]

and

\[
\omega_{\frac{N+1}{2}+1} - y^* = \frac{1}{N} - \frac{b}{2} (N^2 - 3).
\]

Moreover, \( y^* - \omega_{\frac{N+1}{2}} < \omega_{\frac{N+1}{2}+1} - y^* \) for

\[
b < \frac{1}{N(N^2 + 2)}.
\]

Hence, for \( b \) sufficiently small and \( N \) odd, \( i^* = \frac{N+1}{2} \) as claimed.

Consider now the case where \( N \) is even. We have

\[
\omega_{\frac{N}{2}+1} = \frac{1}{2} + \frac{1}{2N} - \frac{b}{2} N^2 + b
\]


and
\[ \omega^N_{\frac{N}{2}+2} = \frac{1}{2} + \frac{3}{2N} + 2b \left( 2 - \frac{N^2}{4} \right) + b \]

Clearly, for \( b = 0 \) \( \omega^N_{\frac{N}{2}+1} \) is closest to \( y^* \). For \( b < \frac{1}{N} \) we have \( \omega^N_{\frac{N}{2}+1} > y^* \). For \( b \geq \frac{1}{N^2} \), we have

\[ y^* - \omega^N_{\frac{N}{2}+1} = \frac{b}{2} N^2 - \frac{1}{2N} < \frac{3}{2N} + 2b \left( 2 - \frac{N^2}{4} \right) = \omega^N_{\frac{N}{2}+2} - y^* \]

for
\[ b < \frac{2}{N (N^2 - 2)}. \]

Thus for \( b \) positive but small it is easy to identify the Sender’s optimal deviation conditional on shirking (unexpectedly from the Receiver’s perspective) on information acquisition. In principle, the following analysis could also be carried through for arbitrary biases. However, since for larger biases the optimal deviation would become a function of the bias, the analysis would become cumbersome and not more illuminating.

Notice that \( N \) is taken as exogenous in the Lemma. In other words, the lemma says that, if we keep the number of distinct induced actions fixed and let \( b \) go to zero, then for \( b \) sufficiently small, we can identify the optimal deviations of equilibrium path as the ones given in the lemma.

The optimal deviations become more cumbersome to analyze if we wish to look into communication inducing the maximum possible distinct choices for given \( b, N (b) \). In that case, both the parameter \( b \) and the bounds given in the lemma change if we change \( b \). In general this means that the optimal deviation does not coincide with the ones in the lemma for all value of \( b \). We do not attempt to give a complete characterization of the optimal deviation, but will illustrate the ideas only.

\( N (b) \) is the largest integer satisfying \( 2bN (N - 1) < 1 \). It follows that \( 2b (N (b) + 1) N (b) > 1 \). Since \( \frac{1}{N(N^2+2)} < \frac{1}{N(N^2-2)} < \frac{1}{2N(N-1)} \) for any positive integer, the upper bound on \( b \) does not impose any additional restrictions;
however, the lower bound does. If the most informative communication equilibrium is played then $b > \frac{1}{2(N(b)+1)}$. So the condition in the lemma for $N$ even can be satisfied for some values of $b$ only if

$$\frac{1}{2(N(b)+1)} N(b) \leq \frac{2}{N(b)(N(b)^2 - 2)},$$

that is if $N(b) \leq 4$. Likewise, the condition in the lemma for $N$ odd can be satisfied only if

$$\frac{1}{2(N(b)+1)} N(b) \leq \frac{2}{N(b)(N(b)^2 + 2)},$$

that is for $N(b) \leq 3$.

Tedious but straightforward calculations show the following for specific examples: for $b \in \left[\frac{1}{12}, \frac{1}{4}\right)$ the most informative equilibrium has $N(b) = 2$ and the Sender induces the action $y_2^b$ off path for all $b \in \left[\frac{1}{12}, \frac{1}{4}\right)$; likewise, for $b \in \left[\frac{1}{24}, \frac{1}{12}\right)$ the most informative equilibrium induces three distinct actions, $N(b) = 3$ and the Sender induces action $y_3^b$ off path for $b \in \left[\frac{1}{24}, \frac{2}{33}\right)$; for $b \in \left(\frac{2}{33}, \frac{1}{12}\right)$ the Sender induces action $y_3^b$. Finally, for $b \in \left[\frac{1}{30}, \frac{1}{24}\right)$, the most informative equilibrium induces four distinct actions and the Sender induces action $y_4^b$ for $b \in \left[\frac{1}{30}, \frac{1}{28}\right)$; for $b \in \left[\frac{1}{28}, \frac{1}{24}\right)$, the Sender induces action $y_4^b$ off equilibrium path. More generally, one can show that the optimal action to induce after shirking on information acquisition and under communication inducing $N(b)$ distinct actions varies substantially with $b$. Hence, it is probably impossible to characterize the optimal deviations when both the bias and the informativeness of the on-path communication is allowed to vary.

### 7.3 Expected Utility off equilibrium path

We can now compute the Sender’s expected utility off equilibrium path, that is when he did not acquire information but the Receiver expects him to do so. Suppose that the candidate equilibrium of the communication game induces $N$ distinct actions. As we did in the previous Lemma, consider for some given bias $b$, any feasible communication, that is communication inducing
\( N \leq N(b) \) distinct actions. Keep now \( N \) fixed and let \( b \) tend towards zero. If \( b \) becomes sufficiently small, then Lemma (3) identifies the optimal action to induce after deviating to no information acquisition; hence, we can use Lemma (3) to compute expected utilities off the equilibrium path.

Suppose first that \( N \) is odd. The expected utility off path is equal to minus the variance of the uniform distribution minus the square of the difference between the induced choice and the Sender’s ideal choice. It is easy to see that \( y^* - \omega_N^{N+1} = \frac{b}{2} (N^2 + 1) \), so the Sender’s expected utility off path is

\[
-\frac{1}{12} - \frac{b^2}{4} (N^2 + 1)^2. \tag{11}
\]

Likewise, if \( N \) is even, the difference between the Sender’s ideal choice and the induced choice is \( y^* - \omega_N^{N+1} = \frac{b}{2} N^2 - \frac{1}{2N} \), and so the expected utility of the Sender is

\[
-\frac{1}{12} - \left( \frac{b}{2} N^2 - \frac{1}{2N} \right)^2. \tag{12}
\]

### 7.4 Incentives for Information Acquisition

We continue our analysis under the assumptions spelled out in Lemma (3). For \( N \) odd, acquiring information and inducing \( N \) distinct actions is an equilibrium if

\[
c < \left( \frac{1}{12} + \frac{b^2}{4} (N^2 + 1)^2 \right) - \left( \frac{1}{12N^2} + \frac{b^2}{3} \frac{N^2 - 1}{3} + b^2 \right), \tag{13}
\]

that is if the cost of information acquisition is smaller or equal to the difference in disutility off and on equilibrium path. Likewise, for \( N \) even, acquiring information is an equilibrium if

\[
c < \left( \frac{1}{12} + \left( \frac{b}{2} N^2 - \frac{1}{2N} \right)^2 \right) - \left( \frac{1}{12N^2} + \frac{b^2}{3} \frac{N^2 - 1}{3} + b^2 \right) \tag{14}
\]

We can now investigate how the disutility difference on the right-hand side of these expressions depend on \( N \) and \( b \).
Lemma 4 In a continuation equilibrium inducing an odd number of $N \geq 3$ distinct actions, the incentive to acquire information is increasing in $N$ and increasing in $b$.

In a continuation equilibrium inducing an even number of $N \geq 2$ distinct actions, the incentive to acquire information is decreasing in $N$ for $b$ sufficiently small and decreasing in $b$ for $b$ sufficiently small.

Proof. Consider first the case of $N$ odd. Take some feasible communication and suppose $\hat{N}=N+2$ is also feasible. Define $\hat{N}(z) \equiv N+2z$ for all $z \in [0,1]$. Define,

$$V^{\text{odd}}(z,b) \equiv \left(\frac{1}{12} + \frac{b^2}{4} (\hat{N}(z)^2 + 1)^2\right) - \left(\frac{1}{12\hat{N}(z)^2} + b^2 \frac{\hat{N}(z)^2 - 1}{3} + b^2\right),$$

The incentives to acquire information are greater with more informative communication if $V^{\text{odd}}(1,b)-V^{\text{odd}}(0,b) > 0$. This is equivalent to $\int_0^1 V^{\text{odd}}_z(z,b) \, dz > 0$. Differentiating with respect to $z$ we find that indeed

$$V^{\text{odd}}_z(z,b) = 2b^2 \left(\hat{N}(z)^2 + 1\right) \hat{N}(z) + \frac{1}{3\hat{N}(z)^3} - b^2 \frac{4\hat{N}(z)}{3} > 0.$$

Hence, $V^{\text{odd}}(1,b)-V^{\text{odd}}(0,b) > 0$ for $N$ odd.

Differentiating with respect to $b^2$ we obtain

$$V^{\text{odd}}_{b^2}(z,b) \equiv \left(\frac{1}{4} \hat{N}(z)^4 + \frac{1}{2} \hat{N}(z)^2 + \frac{1}{4}\right) - \frac{\hat{N}(z)^2 - 1}{3} - 1 > 0$$

where the conclusion follows from the fact that $\frac{1}{4} \hat{N}(z)^4 > \frac{5}{12}$ for meaningful communication.

Consider now the case of $N$ even. Define

$$V^{\text{even}}(z,b) \equiv \left(\frac{1}{12} + \left(\frac{b}{2} \hat{N}(z) - \frac{1}{2\hat{N}(z)}\right)^2\right) - \left(\frac{1}{12\hat{N}(z)^2} + b^2 \frac{\hat{N}(z)^2 - 1}{3} + b^2\right).$$
Differentiating with respect to $z$ we find
\[ V_{z \text{ even}}^2 (z, b) \equiv 2 \left( \frac{b}{2} \hat{N}(z)^2 - \frac{1}{2 \hat{N}(z)} \right) \left( 2b\hat{N}(z) + \frac{1}{\hat{N}(z)^2} \right) + \frac{1}{3\hat{N}(z)^3} - b^2 \frac{4 \hat{N}(z)}{3}. \]

For $b = 0$ we have
\[ V_{z \text{ even}}^2 (z, 0) \equiv \left( -\frac{1}{\hat{N}(z)} \right) \left( \frac{1}{\hat{N}(z)^2} \right) + \frac{1}{3\hat{N}(z)^3} = -\frac{2}{3\hat{N}(z)^3} < 0. \]

Differentiating $V_{z \text{ even}}^2 (z, b)$ with respect to $b$, we find
\[ V_{z \text{ even}}^{zb} (z, b) \equiv \hat{N}(z)^2 \left( 2b\hat{N}(z) + \frac{1}{\hat{N}(z)^2} \right) + 2 \left( \frac{b}{2} \hat{N}(z)^2 - \frac{1}{2 \hat{N}(z)} \right) 2\hat{N}(z) - 2b \frac{4 \hat{N}(z)}{3} \]
\[ = (2b\hat{N}(z)^3 + 1) + (2b\hat{N}(z)^3 - 2) - b \frac{8 \hat{N}(z)}{3} \]

Clearly, $V_{z \text{ even}}^{zb} (z, 0) < 0$. Differentiating once more, we find that $V_{z \text{ even}}^{zbb} (z, b) = 4\hat{N}(z)^3 - \frac{8 \hat{N}(z)}{3} > 0$. It follows that $V_{z \text{ even}}^2 (z, b) < 0$ for $b$ sufficiently small.

Differentiating with respect to $b$ we find
\[ V_{b \text{ even}}^2 (z, b) \equiv 2 \left( \frac{b}{2} \hat{N}(z)^2 - \frac{1}{2 \hat{N}(z)} \right) \left( \frac{\hat{N}(z)^2}{2} \right) - 2b \hat{N}(z)^2 - \frac{1}{3} - 2b. \]

Evaluating at $b = 0$, we have
\[ V_{b \text{ even}}^2 (z, 0) \equiv \left( -\frac{1}{\hat{N}(z)} \right) \left( \frac{\hat{N}(z)^2}{2} \right) < 0. \]

Differentiating once more we have
\[ V_{b \text{ even}}^{bb} (z, b) \equiv \hat{N}(z)^2 \left( \frac{\hat{N}(z)^2}{2} \right) - 2 \hat{N}(z)^2 - \frac{1}{3} - 2. \]

It is easy to verify that $V_{b \text{ even}}^{bb} (z, b) > 0$ for $N \geq 2$. So, for $b$ sufficiently large we have $V_{b \text{ even}}^2 (z, b) > 0$. Conversely, for $b$ sufficiently small, $V_{b \text{ even}}^2 (z, b) < 0$. More specifically, one finds that
\[ V_{b \text{ even}}^2 (z, b) = 2 \left( \frac{b}{2} \hat{N}(z)^2 - \frac{1}{2 \hat{N}(z)} \right) \left( \frac{\hat{N}(z)^2}{2} \right) - 2b \hat{N}(z)^2 - \frac{1}{3} - 2b = 0 \]
for
\[ \hat{b} = \frac{\hat{N}(z)}{\hat{N}(z)^4 - \frac{4}{3}\hat{N}(z)^2 - \frac{8}{3}}. \]

Moreover, it is straightforward to show that \( \hat{b} < \frac{2}{N(N^2-2)} \). \( \blacksquare \)

The second part of the lemma shows that our analysis of the model without any bias extends nicely into the uniform quadratic case with positive bias.

### 7.5 Bias and Language as an Incentive Scheme

Suppose the Receiver can choose the bias \( b \) and the language within which to communicate with the Sender. The bias could for instance result from a mix between explicit and implicit incentives, as in Morgan and Stockent (2003). Lemma 4 shows that there is a trade-off between these instruments, when it comes to inducing incentives for information acquisition. Since the ex-ante expected utility of both the Sender and the Receiver is decreasing in \( b \); purely from the perspective of inducing meaningful communication it would certainly be optimal to set \( b \) equal to zero. However, there is a loss in terms of the ex ante ability to provide the Sender with good incentives to acquire information. A priori it is unclear whether the coarseness of language can substitute in as an added incentive to acquire information; moreover, it is not clear whether this would be welfare improving. We now investigate these questions under the same assumptions as above, that is assuming that \( b \) is sufficiently small in the sense of Lemma 3. To make things interesting, we continue to assume that \( c > \frac{1}{12} \).

**Proposition 8** For a given bias \( b' \) and a number \( N' \leq N(b') \) of induced actions, consider small biases in the sense of \( b' < \frac{N'}{N'^4 - \frac{2}{3}N'^2 - \frac{2}{3}} \). Then, relative to this communication game with parameters \((b', N')\), both the Sender and the Receiver are better off in a game where \( b = 0 \) and \( N \) is the largest even integer such that (14) holds with \( b = 0 \).
Proof. The proof is trivial if $N_0$ is even; in that case we know that the right-hand side of (14) is decreasing in $b$. Hence, we can increase the incentive to acquire information by decreasing $b$. For a lower $b$ and still the same number of induced actions, both the Sender’s and the Receiver’s ex ante expected utility increases. We can further increase their expected utility if $N$ can be increased; the maximum expected utility is reached when $N = \max_{N'_0}$ such that (14) holds for $b = 0$.

Consider now the case where $N_0$ is odd. Suppose that $N'_0 + 1$ would be a feasible number of distinct actions to induce if the bias were $b'' < b'$. Let $V^{odd}(b, N)$ denote the difference on the right-hand side of (13) and let $V^{even}(b, N)$ denote the difference on the right-hand side of (14). Since $V^{even}(b, N)$ is decreasing in $b$ for $b < \frac{N'_0 - \frac{N'_0}{2}}{3}$, we know that $V^{even}(b'', N'_0 + 1) > V^{even}(b', N'_0 + 1)$. Hence, the Sender has a higher incentive to acquire information if $b$ is reduced to $b''$ and $N$ increased to $N'_0 + 1$ if $V^{even}(b', N'_0 + 1) \geq V^{odd}(b', N'_0)$, that is

$$\left(\frac{1}{12} + \left(\frac{b'}{2} \left(N' + 1\right)^2 - \frac{1}{2 (N' + 1)}\right)^2\right) - \left(\frac{1}{12 (N' + 1)^2} + \frac{b'^2 (N' + 1)^2 - 1}{3} + b'^2\right)$$

$$> \left(\frac{1}{12} + \frac{b'^2}{4} \left(N'^2 + 1\right)^2\right) - \left(\frac{1}{12N'^2} + \frac{b'^2 N'^2 - 1}{3} + b'^2\right).$$

Simplifying, we obtain

$$\left(\left(\frac{b'}{2} \left(N' + 1\right)^2 - \frac{1}{2 (N' + 1)}\right)^2\right) - \left(\frac{1}{12 (N' + 1)^2} + \frac{b'^2 (N' + 1)^2 - 1}{3}\right)$$

$$> \left(\frac{b'^2}{4} \left(N'^2 + 1\right)^2\right) - \left(\frac{1}{12N'^2} + \frac{b'^2 N'^2 - 1}{3}\right).$$

For $b = 0$, this inequality is verified, since the left-hand side of the inequality takes value $\frac{1}{6(N'+1)^2} > 0$ while the right-hand side takes value $-\frac{1}{12N'^2} < 0$.}

These results are reassuring in the sense that they show a certain robustness of the results derived in the case of no bias. However, ideally we would also like to know how incentives to acquire information and expected welfare depend on the underlying bias when the induced number of distinct actions
changes with the underlying bias, as it does when communication is always maximally informative. These questions are the subject of ongoing research.

8 Conclusion

We analyze a game of cheap talk with endogenous information. Our main result is that limited communication may, and quite generally does provide better incentives for information acquisition than perfectly revealing communication does. Moreover, under additional conditions on the underlying distribution of types, less revealing communication provides stronger incentives for information acquisition. Since we assume in the main part of our analysis that there is no conflict of interest ex post between the Sender and the Receiver, the best equilibrium is the one that induces the greatest number of distinct actions while still providing incentives for information acquisition. Hence, the more valuable information is - in the sense of the cost of information acquisition - the coarser the language necessary to induce its acquisition.

9 Appendix

Proof of Lemma 1. The proof is divided into two parts, existence and uniqueness. Symmetry is a corollary.

To prove existence we show that for or any $N \in \{2, 3, \ldots\}$ and any $a_N^N \in (0, 1]$, there exist $a_1^N, \ldots, a_{N-1}^N$ with $a_0^N \equiv 0 < a_1^N < \ldots < a_{N-1}^N < a_N^N$ that simultaneously satisfy equations (1) for $i = 1, \ldots, N - 1$. We prove the claim by induction on $N$.

In order to prove the claim for $N = 2$, for any given $a_2^2 \in (0, 1]$, we need to find $a_1 \in (0, a_2^2)$ such that

$$a_1 - E [\omega | 0 \leq \omega \leq a_1^2] = E [\omega | a_1 \leq \omega \leq a_2^2] - a_1.$$ 

Note that at $a_1 = 0$, the left-hand side becomes zero while the right-hand side is $E [\omega | 0 \leq \omega \leq a_2^2] > 0$, while at $a_1 = a_2^2$ the left-hand side is $a_2^2 -$
\[ E[\omega | 0 \leq \omega \leq a_i^2] > 0 \] while the right-hand side is zero. Therefore, by the continuity of the distribution, there indeed exists \( a_1 \in (0, a_i^2) \) that satisfies the equality.

Assume that the claim is true for \( N = k \). Fix \( a_{k+1}^{k+1} \in (0, 1] \) and for all \( a_k \in (0, a_{k+1}^{k+1}) \) let \( (a_1(a_k), \ldots, a_{k-1}(a_k)) \) be such that \( a_0(a_k) \equiv 0 < a_1(a_k) < \ldots < a_{k-1}(a_k) < a_k \) and

\[
a_i(a_k) - E[\omega | a_{i-1}(a_k) \leq \omega \leq a_i(a_k)] = E[\omega | a_i(a_k) \leq \omega \leq a_{i+1}(a_k)] - a_i(a_k) \quad \text{for} \quad i = 1, \ldots, k - 1. \quad (15)
\]

By the inductive assumption, such \( (a_1(a_k), \ldots, a_{k-1}(a_k)) \) exist for all \( a_k \in (0, a_{k+1}^{k+1}) \).

Note that \( \lim_{a_k \to 0} a_i(a_k) = 0 \) for all \( i = 1, \ldots, k - 1 \), hence

\[
\lim_{a_k \to 0} E[\omega | a_{k-1}(a_k) \leq \omega \leq a_k] = 0 < \lim_{a_k \to 0} E[\omega | a_k \leq \omega \leq a_{k+1}^{k+1}] .
\]

Therefore, for \( a_k \) sufficiently close to zero,

\[
a_k - E[\omega | a_{k-1}(a_k) \leq \omega \leq a_k] < E[\omega | a_k \leq \omega \leq a_{k+1}^{k+1}] - a_k.
\]

However, as \( a_k \) approaches \( a_{k+1}^{k+1} \), the difference \( E[\omega | a_k \leq \omega \leq a_{k+1}^{k+1}] - a_k \) goes to zero, while \( a_k - E[\omega | a_{k-1}(a_k) \leq \omega \leq a_k] \) converges to a strictly positive number. Therefore, for \( a_k \) sufficiently close to \( a_{k+1}^{k+1} \), the above inequality reverses. By the continuity of all the functions involved and the Intermediate Value Theorem, there exists \( a_{k+1}^{k+1} \in (0, a_{k+1}^{k+1}) \) such that

\[
a_k^{k+1} - E[\omega | a_{k-1}(a_k^{k+1}) \leq \omega \leq a_k^{k+1}] = E[\omega | a_k^{k+1} \leq \omega \leq a_{k+1}^{k+1}] - a_k^{k+1} . \quad (16)
\]

Equations (15) and (16) imply that the claim of the lemma holds for \( N = k+1 \) with \( a_0^{k+1} = 0, a_i^{k+1} = a_i(a_k^{k+1}) \) for \( i = 1, \ldots, k - 1 \). For \( a_N^{N} = 1 \), Lemma 1 implies there exists a partition of \([0, 1]\) into \( N \) non-degenerate intervals such that any type separating two adjacent intervals is indifferent to reporting either of the two partition-elements.
Consider now uniqueness. Consider, for given initial value $x$ and given end point $a_N^N$, the sequence $\{a_1^N(x), \ldots, a_{N-1}^N(x)\}$ as determined by the system of equations

\begin{align*}
a_1^N(x) &= E[\omega|\omega \in [x, a_1^N(x)]] = E[\omega|\omega \in [a_1^N(x), a_2^N(x)]] - a_1^N(x), \quad (17) \\
a_i^N(x) &= E[\omega|\omega \in [a_{i-1}^N(x), a_i^N(x)]] = E[\omega|\omega \in [a_i^N(x), a_{i+1}^N(x)]] - a_i^N(x) \quad (18)
\end{align*}

for $i = 2, \ldots, N-2$, and

\[ a_{N-1}^N(x) - E[\omega|\omega \in [a_{N-2}^N(x), a_{N-1}^N(x)]] = E[\omega|\omega \in [a_{N-1}^N(x), a_N^N]] - a_{N-1}^N(x). \]

(19)

We now show that each element in $\{a_1^N(x), \ldots, a_{N-1}^N(x)\}$ is increasing in $x$. This implies that there is exactly one way to divide the unit interval into $N+1$ equilibrium partitions. Since $N$ is arbitrary, this argument shows that for each $N$, there is a unique equilibrium.

Consider first (19), which determines $a_{N-2}^N$ as a function of $a_{N-1}^N$. Rearranging (19), we have

\[ E[\omega|\omega \in [a_{N-2}^N, a_{N-1}^N]] = 2a_{N-1}^N - E[\omega|\omega \in [a_{N-1}^N, a_N^N]]. \]  

(20)

For $a_{N-2}^N = a_{N-1}^N$, $E[\omega|\omega \in [a_{N-2}^N, a_{N-1}^N]] = a_{N-1}^N$. However,

\[ a_{N-1}^N > 2a_{N-1}^N - E[\omega|\omega \in [a_{N-1}^N, a_N^N]], \]

since $a_{N-1}^N - E[\omega|\omega \in [a_{N-1}^N, a_N^N]] < 0$. Hence, to establish equality, the left-hand side needs to be reduced. Since $E[\omega|\omega \in [a_{N-2}^N, a_{N-1}^N]]$ is strictly increasing in $a_{N-2}^N$, this is achieved by reducing $a_{N-2}^N$ below $a_{N-1}^N$. Hence, by monotonicity of $E[\omega|\omega \in [a_{N-2}^N, a_{N-1}^N]]$, there is a unique solution $a_{N-2}^N = h(a_{N-1}^N)$.

We now discuss whether and when $h(a_{N-1}^N)$ is an increasing function. Differentiating (20) totally with respect to $a_{N-2}^N$ and $a_{N-1}^N$ we obtain

\[ \frac{da_{N-1}^N}{da_{N-2}^N} = \frac{\partial}{\partial a_{N-2}^N} E[\omega|\omega \in [a_{N-2}^N, a_{N-1}^N]] - \frac{\partial}{\partial a_{N-1}^N} E[\omega|\omega \in [a_{N-1}^N, a_{N-2}^N]]. \]  

(21)
\( h(a_{N-1}^N) \) is an increasing function if and only if the numerator of this expression is positive. A sufficient condition is if both \( \frac{\partial}{\partial a_{N-1}^N} E[\omega|\omega \in [a_{N-1}^N, a_{N}^N]] \leq 1 \) and \( \frac{\partial}{\partial a_{N-1}^N} E[\omega|\omega \in [a_{N-2}^N, a_{N-1}^N]] \leq 1 \).

Likewise, for given \( a_i^N \) and \( a_{i+1}^N \), for \( i = 2, \ldots, n-2 \), (18) determines \( a_{i-1}^N \). Rearranging (18), we obtain

\[
E[\omega|\omega \in [a_{i-1}^N, a_i^N]] = 2a_i^N - E[\omega|\omega \in [a_i^N, a_{i+1}^N]].
\]

Clearly, (22) has exactly the same structure as (20) has, so exactly the same arguments apply. Totally differentiating (22), we have

\[
\frac{\partial}{\partial a_i^N} E[\omega|\omega \in [a_{i-1}^N, a_i^N]] \frac{da_i^N}{da_{i+1}^N} + \frac{\partial}{\partial a_{i+1}^N} E[\omega|\omega \in [a_i^N, a_{i+1}^N]] = \left( 2 - \frac{\partial}{\partial a_i^N} E[\omega|\omega \in [a_i^N, a_{i+1}^N]] - \frac{\partial}{\partial a_i^N} E[\omega|\omega \in [a_{i-1}^N, a_i^N]] \right) \frac{da_i^N}{da_{i+1}^N}.
\]

Since

\[
\frac{da_{i-1}^N}{da_i^N} = \frac{da_i^N}{da_{i+1}^N} \frac{da_i^N}{da_{i-1}^N},
\]

we can write

\[
\frac{da_i^N}{da_{i-1}^N} = \frac{\partial}{\partial a_{i-1}^N} E[\omega|\omega \in [a_{i-1}^N, a_i^N]] \left( \begin{array}{c}
2 - \frac{\partial}{\partial a_i^N} E[\omega|\omega \in [a_i^N, a_{i+1}^N]] \\
-\frac{\partial}{\partial a_i^N} E[\omega|\omega \in [a_i^N, a_{i+1}^N]] \\
-\frac{\partial}{\partial a_i^N} E[\omega|\omega \in [a_{i-1}^N, a_i^N]]
\end{array} \right)^{-1} \frac{da_i^N}{da_{i+1}^N}.
\]

We now compute a lower bound on \( \frac{da_i^N}{da_{i-1}^N} \).

Suppose \( \frac{\partial}{\partial \omega} E[\omega|\tilde{\omega} \leq \omega \leq \tilde{\omega}] \), \( \frac{\partial}{\partial \omega} E[\omega|\tilde{\omega} \leq \omega \leq \tilde{\omega}] < h \) for some positive \( h \leq 1 \). Then, we know that there is a lower bound \( u_{N-1} \) on \( \frac{da_{N-1}^N}{da_{N-2}^N} \); more specifically, we have

\[
u_{N-1} = \frac{1 - h}{2(1 - h) - (1 - h)} = \frac{1 - h}{2h}.
\]

If there is a lower bound \( u_i \), then

\[
\frac{da_i^N}{da_{i-1}^N} \geq \frac{1 - h}{2h - (1 - h) u_i};
\]
and so there is a lower bound on \( u_{i-1} \), which is given by

\[
u_{i-1} = \frac{1 - h}{2h - (1 - h) u_i}.
\]  

(24)

We now show that (24) defines a nondecreasing sequence (in the sense that \( u_{i-1} \leq u_i \)) with each element bounded below by a positive number. Notice that the initial value \( u_{N-1} = \frac{1 - h}{2h} \) satisfies \( 2h - (1 - h) u_{N-1} > 0 \) since \( 4h^2 > (1 - h)^2 \). Moreover, by (24), \( u_{i-1} \leq u_i \) if and only if

\[
\frac{1 - h}{2h - (1 - h) u_i} \leq u_i,
\]

which in turn is equivalent to the quadratic expression

\[
1 - 2\frac{h}{1 - h} u_i - u_i^2 \leq 0.
\]

This condition is satisfied for any \( u_i \geq \sqrt{\left(\frac{h}{1 - h}\right)^2 + 1 - \frac{h}{1 - h}} \). Since the initial value satisfies

\[
\frac{1 - h}{2h} > \sqrt{\left(\frac{h}{1 - h}\right)^2 + 1 - \frac{h}{1 - h}}
\]

for any positive \( h < 1 \), the sequence increases indeed; the lowest element is

\[
u \equiv \sqrt{\left(\frac{h}{1 - h}\right)^2 + 1 - \frac{h}{1 - h}}.
\]

So we have shown that

\[
\frac{d a_i^N}{d a_{i-1}^N} \geq \sqrt{\left(\frac{h}{1 - h}\right)^2 + 1 - \frac{h}{1 - h}} > 0.
\]

Finally, exactly the same argument applies to (17). In particular, from a total differentiation of (17), we obtain

\[
\frac{d a_i^N}{d x} = \frac{\partial}{\partial x} E [\omega | \omega \in [x, a_i^N (x)]] \left( 2 - \frac{\partial}{\partial a_i^N} E [\omega | \omega \in [a_i^N (x), a_2^N (x)]] \right)^{-1}
\]

\[
\left( -\frac{\partial}{\partial a_1^N} E [\omega | \omega \in [x, a_i^N (x)]] - \frac{\partial^2}{\partial a_1^N \partial a_2^N} E [\omega | \omega \in [a_1^N (x), a_2^N (x)]] \frac{d a_i^N}{d a_{i-1}^N} \right).
\]

(25)
Finally, symmetry is obvious. Given the symmetric distribution and the absence of a bias, one can construct the equilibrium partition on the lower half interval as the mirror image of the equilibrium partition on the upper half interval. Since there is a unique equilibrium, the symmetric equilibrium is the only equilibrium. ■

Proof of Proposition 1. We characterize the set of feasible standard deviations for any given \( \omega_H \). The smallest standard deviation, \( \sigma_H (\omega_H) \), is clearly zero; the Dirac distribution with point mass 1 on \( \omega_H \) generates a conditional expectation equal to \( \omega_H \), with a zero standard deviation. The largest standard deviation given \( \omega_H \) is achieved by the distribution that solves

\[
\begin{align*}
\max_{d\hat{F}} & \int_{\frac{1}{2}}^{1} (\omega - \omega_H)^2 d\hat{F}(\omega) \\
\int_{\frac{1}{2}}^{1} (\omega - \omega_H) d\hat{F}(\omega) &= 0 \\
0 &\leq d\hat{F} \leq 1; \quad \hat{F}(\frac{1}{2}) = 0, \quad \hat{F}(1) = 1.
\end{align*}
\]

The solution to this problem is a two point distribution with weights \( a \) on \( \frac{1}{2} \) and \( 1-a \) on 1 such that \( a \cdot \frac{1}{2} + 1-a = \omega_H \), which implies that \( a = 2(1 - \omega_H) \) and \( 1-a = 2\omega_H - 1 \). Any distribution different from this one would allow us to shift more mass outwards without changing the mean of the distribution, that is the distribution could be transformed by a sequence of mean preserving spreads in the sense of Rothschild and Stiglitz (1970) into one with larger variance. It follows that the largest standard deviation, \( \sigma_H (\omega_H) \) is given by

\[
\sigma_H^{\text{max}} (\omega_H) = \sqrt{(2 - 2\omega_H) \left( \frac{1}{2} - \omega_H \right)^2 + (2\omega_H - 1)(1-\omega_H)^2} = \frac{1}{2} \sqrt{2 \sqrt{-2\omega_H^2 + 3\omega_H - 1}}.
\]

■
Proof of Proposition 3. Information acquisition is an equilibrium if
\[
c \leq \text{Var} (\omega) - 2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \text{Var} (\omega \mid \omega \in P_{n+i}^{2n}) + \left( E (\omega \mid \omega \in P_{n+1}^{2n}) - \frac{1}{2} \right)^2.
\]

On the other hand, Sender and Receiver are better off with information acquisition if
\[
-c - 4 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \text{Var} (\omega \mid \omega \in P_{n+i}^{2n}) > -2 \text{Var} (\omega).
\]

Using
\[
\text{Var} (\omega) = 2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \text{Var} (\omega \mid \omega \in P_{n+i}^{2n})
+ 2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \left( E (\omega \mid \omega \in P_{n+i}^{2n}) - \frac{1}{2} \right)^2
\]
can rewrite the latter condition as
\[
c \leq 4 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \left( E (\omega \mid \omega \in P_{n+i}^{2n}) - \frac{1}{2} \right)^2,
\]

while the first is
\[
c \leq 2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \left( E (\omega \mid \omega \in P_{n+i}^{2n}) - \frac{1}{2} \right)^2 + \left( E (\omega \mid \omega \in P_{n+1}^{2n}) - \frac{1}{2} \right)^2.
\]

For any probability distribution
\[
2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \left( E (\omega \mid \omega \in P_{n+i}^{2n}) - \frac{1}{2} \right)^2 \geq \left( E (\omega \mid \omega \in P_{n+1}^{2n}) - \frac{1}{2} \right)^2.
\]

The inequality becomes strict if there is some positive mass outside $P_{n+1}^{2n}$. Hence, the incentive constraint implies the welfare ranking.

Consider now the second statement. Take the partition defined by the sequence of cutpoints $\{a_i^{2n}\}_{i=n}^{2n}$, solving (1) with $a_n^{2n} = \frac{1}{2}$ and $a_{n+1}^{2n} = 1$. Now, we introduce an additional cutpoint, $x$, in between $a_n^{2n}$ and $a_{n+1}^{2n}$. Define
a new sequence \( \{a_{2n}^n, x, a_{2n+1}^n(x), \ldots, a_{2n-1}^n(x), a_{2n}^n\} \), where the subsequence \( \{x, a_{n+1}^n(x), \ldots, a_{2n-1}^n(x), a_{2n}^n\} \) solves (1) as an \( n \)-partition with \( a_{n+1}^n \equiv x \) and \( a_{2n}^n = 1 \). Clearly, for \( x = \frac{1}{2} \), the sequence \( \{a_{n}^n, a_{n+1}^n(x), \ldots, a_{2n-1}^n(x), a_{2n}^n\} \) describes the equilibrium with \( n \) partition elements; for \( x = a_{n+2}^{2(n+1)} \), the sequence describes the equilibrium with \( n + 1 \) partition elements. Moreover, even though the sequence does not correspond to an equilibrium of the communication game, the sequence is well defined for all \( x \in \left[ \frac{1}{2}, a_{n+2}^{2(n+1)} \right] \).

Define \( \tilde{\omega}(x) \) as follows. For \( x = a_{n}^n \), \( \tilde{\omega}(x) = a_{n}^n \); for for any \( x > a_{n}^n \),

\[
\tilde{\omega}(x) = \frac{a_{n}^n}{x} \int_{a_{n}^n}^{x} \omega f(\omega) d\omega.
\]

Moreover, define

\[
\omega_{n+i}^{2n}(x) \equiv \int_{\omega_{n+i}^{2n}(x)}^{a_{n+i}^n(x)} \omega \Pr[\omega \in [a_{n+i}^{2n}(x), a_{n+i+1}^n(x)]] d\omega.
\]

Finally, define the expected payoff \( W(x) \) as

\[
W(x) = \int_{a_{n}^n}^{x} (\omega - \tilde{\omega}(x))^2 f(\omega) d\omega + \int_{x}^{a_{n+1}^n(x)} (\omega - \omega_{n+1}^{2n}(x))^2 f(\omega) d\omega
\]

\[
+ \sum_{i=2}^{n-1} \int_{a_{n+i}^{2n}(x)}^{a_{n+1}^n(x)} (\omega - \omega_{n+i}^{2n}(x))^2 f(\omega) d\omega + \int_{a_{2n}^{n+1}(x)}^{a_{n+i}^n(x)} (\omega - \omega_{2n}^{2n}(x))^2 f(\omega) d\omega.
\]
The derivative of $W(x)$ with respect to $x$ is

$$W'(x) = (x - \tilde{\omega}(x))^2 f(x) - 2 \int_{a_n^{2n}}^{x} (\omega - \tilde{\omega}(x)) f(\omega) d\omega \frac{\partial \tilde{\omega}(x)}{\partial x}$$

$$- (x - \omega_{n+1}^{2n}(x))^2 f(x) + (a_{n+1}^{2n}(x) - \omega_{n+1}^{2n}(x))^2 f(a_{n+1}^{2n}(x)) \frac{\partial a_{n+1}^{2n}(x)}{\partial x}$$

$$- 2 \int_{x}^{a_{n+1}^{2n}(x)} (\omega - \omega_{n+1}^{2n}(x)) f(\omega) d\omega \frac{\partial \omega_{n+1}^{2n}(x)}{\partial x}$$

$$+ \sum_{i=2}^{n-1} (a_{n+i}^{2n}(x) - \omega_{n+i}^{2n}(x))^2 f(a_{n+i}^{2n}(x)) \frac{\partial a_{n+i}^{2n}(x)}{\partial x}$$

$$- \sum_{i=2}^{n-1} (a_{n+i-1}^{2n}(x) - \omega_{n+i-1}^{2n}(x))^2 f(a_{n+i-1}^{2n}(x)) \frac{\partial a_{n+i-1}^{2n}(x)}{\partial x}$$

$$- 2 \sum_{i=2}^{n-1} \int_{a_{n+i-1}^{2n}(x)}^{a_{n+i}^{2n}(x)} (\omega - \omega_{n+i}^{2n}(x)) f(\omega) d\omega \frac{\partial \omega_{n+i}^{2n}(x)}{\partial x}$$

$$- (a_{2n-1}^{2n}(x) - \omega_{2n}^{2n}(x))^2 f(a_{2n-1}^{2n}(x)) \frac{\partial a_{2n-1}^{2n}(x)}{\partial x}$$

$$- 2 \int_{a_{2n-1}^{2n}(x)}^{a_{2n}^{2n}(x)} (\omega - \omega_{2n}^{2n}(x)) f(\omega) d\omega \frac{\partial \omega_{2n}^{2n}(x)}{\partial x}.$$ 

By definition of $\tilde{\omega}(x)$ and $\{\omega_{n+i}^{2n}(x)\}^{n}_{i=1}$ the effects through changes on the values of these functions themselves are exactly zero. So, the expression
reduces to

\[ W'(x) = (x - \bar{\omega}(x))^2 f(x) - (x - \omega_{n+1}^2(x))^2 f(x) \]

\[ + (a_{n+1}^{2n}(x) - \omega_{n+1}^2(x))^2 f(a_{n+1}^{2n}(x)) \frac{\partial a_{n+1}^{2n}(x)}{\partial x} \]

\[ + \sum_{i=2}^{n-1} (a_{n+i}^{2n}(x) - \omega_{n+i}^2(x))^2 f(a_{n+i}^{2n}(x)) \frac{\partial a_{n+i}^{2n}(x)}{\partial x} \]

\[ - \sum_{i=2}^{n-1} (a_{n+i}^{2n}(x) - \omega_{n+i}^2(x))^2 f(a_{n+i}^{2n}(x)) \frac{\partial a_{n+i}^{2n}(x)}{\partial x} \]

\[ - (a_{2n-1}^{2n}(x) - \omega_{2n}^2(x))^2 f(a_{2n-1}^{2n}(x)) \frac{\partial a_{2n-1}^{2n}(x)}{\partial x}, \]

Moreover, by the definition of the equilibrium partition ...

\( (a_{n+1}^{2n}(x) - \omega_{n+1}^2(x))^2 = (a_{n+i-1}^{2n}(x) - \omega_{n+i}^2(x))^2 \) for \( i = 2 \)

\( (a_{n+i}^{2n}(x) - \omega_{n+i}^2(x))^2 = (a_{n+i-1}^{2n}(x) - \omega_{n+i}^2(x))^2 \) for \( i = 3, \ldots, n-1 \) and \( j = i-1 \)

and finally

\( (a_{n+i}^{2n}(x) - \omega_{n+i}^2(x))^2 = (a_{2n-1}^{2n}(x) - \omega_{2n}^2(x))^2 \) for \( i = n-1 \).

Hence, we have

\[ W'(x) = (x - \bar{\omega}(x))^2 f(x) - (x - \omega_{n+1}^2(x))^2 f(x). \]

For all \( x < a_{n+2}^{2n+1} \), we have

\[ (x - \bar{\omega}(x))^2 < (x - \omega_{n+1}^2(x))^2 \]

and thus \( W'(x) < 0 \). Expected utility under the coarse partition is equal to \(-W(a_{n+1}^{2n+1})\). Expected utility under the finer partition is equal to \(-W(a_{n+2}^{2(n+1)})\).

\[-W(a_{n+2}^{2(n+1)}) - (-W(a_{n}^{2n})) = - \int_{a_{n+2}^{2n}}^{a_{n+1}^{2(n+1)}} W'(x) \, dx \]

\[ = - \int_{a_{n}^{2n}}^{a_{n+2}^{2(n+1)}} \left( (x - \bar{\omega}(x))^2 - (x - \omega_{n+1}^2(x))^2 \right) f(x) \, dx \]

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Since the integrand is pointwise negative we have

\[-W\left(a_{n+2}^{2(n+1)}\right) > -W\left(a_n^{2n}\right).\]

\[\] \[\]

\textbf{Proof of Proposition 4.} Let \(\alpha \equiv \Pr (\omega \in P_4^4)\). We have

\[\omega_H = 2\alpha \omega_{HL} + 2 \left(\frac{1}{2} - \alpha\right) \omega_{HH}\]

Letting \(\hat{\alpha} \equiv 2\alpha\), we can write the difference in incentives as

\[\Delta (\hat{\alpha}) \equiv \left(\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \frac{1}{2}\right)^2 - \left(\omega_{HL} - \frac{1}{2}\right)^2 - \hat{\alpha} \left(\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \omega_{HL}\right)^2 - (1 - \hat{\alpha}) \left(\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \omega_{HH}\right)^2.\]

\(\hat{\alpha}\) is a probability measure. So, we evaluate the function for all \(\hat{\alpha}\). We have

\[\Delta (0) = \left(\omega_{HH} - \frac{1}{2}\right)^2 - \left(\omega_{HL} - \frac{1}{2}\right)^2 > 0\]

and

\[\Delta (1) \equiv \left(\omega_{HL} - \frac{1}{2}\right)^2 - \left(\omega_{HL} - \frac{1}{2}\right)^2 - (\omega_{HL} - \omega_{HH})^2 = 0\]

Taking derivatives, we obtain

\[\Delta_{\hat{\alpha}} (\hat{\alpha}) = 2 \left(\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \frac{1}{2}\right) (\omega_{HL} - \omega_{HH}) - (\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \omega_{HL})^2 + (\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \omega_{HH})^2 - \hat{\alpha} 2 (\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \omega_{HL}) (\omega_{HL} - \omega_{HH}) - (1 - \hat{\alpha}) 2 (\hat{\alpha} \omega_{HL} + (1 - \hat{\alpha}) \omega_{HH} - \omega_{HH}) (\omega_{HL} - \omega_{HH})\]

Evaluating the derivatives, we obtain

\[\Delta_{\hat{\alpha}} (0) = 2 \left(\omega_{HH} - \frac{1}{2}\right) (\omega_{HL} - \omega_{HH}) - (\omega_{HH} - \omega_{HL})^2 < 0\]
\[ \Delta_\alpha (1) = 2 \left( \omega_{HL} - \frac{1}{2} \right) (\omega_{HL} - \omega_{HH}) + (\omega_{HL} - \omega_{HH})^2. \]

We have
\[ \Delta_\alpha (1) \leq 0 \]
whenever
\[ 2 \left( \omega_{HL} - \frac{1}{2} \right) \geq (\omega_{HH} - \omega_{HL}), \]
that is
\[ 3\omega_{HL} \geq 1 + \omega_{HH}. \]

Differentiating another time, we find
\[ \Delta_{\alpha\alpha} (\hat{\alpha}) = 4 (\omega_{HL} - \omega_{HH})^2 > 0, \]
so the function is convex in \( \hat{\alpha} \).

Hence, if the function is decreasing in \( \hat{\alpha} \) for \( \hat{\alpha} = 0 \) and non-increasing in \( \hat{\alpha} \) for \( \hat{\alpha} = 1 \), then it is decreasing everywhere. In that case \( \Delta (\hat{\alpha}) > 0 \) for all \( \hat{\alpha} < 1 \). On the other hand, if the function is increasing at \( \hat{\alpha} = 1 \), then there is \( \alpha' \) such that \( \Delta (\hat{\alpha}) > 0 \) for all \( \hat{\alpha} < \alpha' \) and \( \Delta (\hat{\alpha}) < 0 \) for all \( \hat{\alpha} \in (\alpha', 1) \).

**Proof of Proposition 5.** Consider the added incentive to acquire information comparing binary communication to any form of communication:
\[ \Delta = \left( \omega_H - \frac{1}{2} \right)^2 - \left( \omega_{n+1}^{2n} - \frac{1}{2} \right)^2 - 2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \left( \omega_{n+i}^{2n} - \omega_H \right)^2, \]
where \( \omega_{n+i}^{2n} = E \left[ \omega \mid \omega \in P_{n+i}^{2n} \right] \). For any sequence of moments \( \omega_{n+i}^{2n} \) satisfying
\[ \omega_H = 2 \sum_{i=1}^{n} \Pr (\omega \in P_{n+i}^{2n}) \omega_{n+i}^{2n}, \]
\( \Delta \) is minimized for a probability distribution that puts all the mass to the most extreme partitions. For such probability distribution, we can write
\[ \Delta (\hat{\alpha}) \equiv \left( \hat{\alpha} \omega_{n+1}^{2n} + (1 - \hat{\alpha}) \omega_{2n}^{2n} - \frac{1}{2} \right)^2 - \left( \omega_{n+1}^{2n} - \frac{1}{2} \right)^2 \]
\[ - \hat{\alpha} \left( \hat{\alpha} \omega_{n+1}^{2n} + (1 - \hat{\alpha}) \omega_{2n}^{2n} - \omega_{n+1}^{2n} \right)^2 - (1 - \hat{\alpha}) \left( \hat{\alpha} \omega_{n+1}^{2n} + (1 - \hat{\alpha}) \omega_{2n}^{2n} - \omega_{2n}^{2n} \right)^2. \]
The proof of the previous proposition has shown that

\[ \Delta(0) = \left( \omega_{2n}^2 - \frac{1}{2} \right)^2 - \left( \omega_{n+1}^2 - \frac{1}{2} \right)^2 > 0 \]

and

\[ \Delta(1) = 0. \]

Moreover, \( \Delta(\hat{\alpha}) \) is convex. Finally, for \( \omega_{n+1}^2 \leq \frac{1+\omega_{2n}}{3} \) (which eventually holds for \( \omega_{n+1}^2 \) small enough and \( \omega_{2n}^2 \) large enough), \( \Delta_{\hat{\alpha}}(\hat{\alpha}) > 0 \) at \( \hat{\alpha} = 1 \). We can simplify

\[
\Delta(\hat{\alpha}, \omega_{n+1}^2, \omega_{2n}^2) = - \left( \hat{\alpha} \omega_{n+1}^2 + (1 - \hat{\alpha}) \omega_{2n}^2 \right) - \left( \omega_{n+1}^2 \right)^2 + \omega_{n+1}^2
\]

\[
+ \hat{\alpha} \left( 2 \left( \hat{\alpha} \omega_{n+1}^2 + (1 - \hat{\alpha}) \omega_{2n}^2 \right) \omega_{n+1}^2 - \left( \omega_{n+1}^2 \right)^2 \right)
\]

\[
+ (1 - \hat{\alpha}) \left( 2 \left( \hat{\alpha} \omega_{n+1}^2 + (1 - \hat{\alpha}) \omega_{2n}^2 \right) \omega_{2n}^2 - \left( \omega_{2n}^2 \right)^2 \right)
\]

The equation

\[ \Delta(\hat{\alpha}, \omega_{n+1}^2, \omega_{2n}^2) = 0 \]

has for any given \( \frac{1}{2} \leq \omega_{n+1}^2 \leq \omega_{2n}^2 \leq 1 \) a solution

\[ \hat{\alpha}(\omega_{n+1}^2, \omega_{2n}^2) = \frac{1}{2} \frac{(\omega_{n+1}^2 + \omega_{2n}^2 - 1)}{\omega_{2n}^2 - \omega_{n+1}^2}. \]

It is easy to see that \( \hat{\alpha}(\omega_{n+1}^2, \omega_{2n}^2) \) is increasing in \( \omega_{n+1}^2 \) and decreasing in \( \omega_{2n}^2 \) over the relevant range. Hence \( \hat{\alpha}(\omega_{n+1}^2, \omega_{2n}^2) \) is minimized for \( \omega_{n+1}^2 = .5 \) and \( \omega_{2n}^2 = 1 \). In this case,

\[ \hat{\alpha} \left( \frac{1}{2}, 1 \right) = \frac{1}{2}. \]

**References**


   to be completed.