Abstract

An emergent theme in the study of organizations is the broad differences in managerial practices and performances across organizations. In this paper we develop an explanation for these phenomena that turns on the complexity of the environments that firms operate in. We construct a model that formally captures the difficulty of the manager’s problem and show how the combination of theoretical knowledge and experience drive the choice of managerial practices. In this setting the evolution of firms is path dependent, marked by numerous failures, successes and strategy reversals. Nevertheless, patterns emerge. We show in particular how initial differences in performances persist and grow in expectation over time. We then apply the model to several long-standing questions in the study of organizations, exploring how imitation and coordination interact with the difficulty of the manager’s problem. We also apply the model to the growth and development of nations, showing how the complexity of the environment resonates with historical experience.

Keywords: learning, complementarities, complexity

JEL classifications: D21, D83, L25
1 Introduction

The corporate histories of even the most successful firms are littered with failed products, abandoned initiatives, and reorganizations that didn’t pan out (see Harford (2011) for many examples). These failures often reflect less the failings of the managers themselves than the sheer complexity of the environments that firms operate in. Managers face many alternatives to choose from yet possess only a tenuous understanding of the production function that maps these choices into outcomes. They are left with little choice but to try an action to see if it works, learning as they go.

Understanding the manager’s problem is an important step to understanding organizations and their role in the economy. Progress on this question, however, has been impeded by the absence of a model that captures formally the difficulty of the problem that managers face. The standard economic approach of smooth objective functions and convex action spaces leaves little scope for failures and missteps, describing a learning problem that is tractable yet overly simplistic. As Brynjolfsson and Milgrom (2013) observe: 

"[...] as noted by Roberts (2004), much of the standard economic treatment of firms assumes that performance is a concave function of a set of infinitely divisible design choices, and the constraint set is convex. Under these conditions, decisionmakers can experiment incrementally to gradually identify an optimal combination of practices."

An alternative modeling approach from outside of economics is the strategic management literature on “rugged landscapes” (Levinthal 1997; Rivkin 2000). This approach relaxes the assumption of concavity and generates “rugged” production functions with many peaks and troughs that can ensnare a manager’s search. The downside of this richness, however, is that strategy itself is removed from the manager’s toolkit. Managers search over the landscape via exogenously imposed local search rules—“hill climbing” algorithms—that explain why search may yield inefficient outcomes but does not address why managers would search in the manner prescribed. Indeed, in these models managers do not possess any understanding of the underlying process that generates the environment they face, so it is not clear how they would form beliefs to generate rational strategic action or even decide in which direction they should search. The trade-off at the heart of this approach is concisely critiqued by Roberts and Saloner (2013, p.819): "This newer model has pluses and minuses. On the one hand, dropping unwarranted but conventional assumptions [...] is clearly desirable. On the other hand, the approach assumes that there is no logic or theory that can be used to guide the selection. This conclusion seems to be too nihilistic."

In this paper we tread a middle ground between the rugged landscapes literature and the traditional economic approach to modeling organizations. We provide a formal model of the manager’s problem that captures the inherent difficulty of the task whilst allowing managers to search opti-
mally over the space of actions. To this end, we follow Callander (2011) in modeling the production function as the realized path of a Brownian motion. An advantage of this approach is that managers possess an understanding of the underlying process that generates the environment they face. Knowledge of the drift and variance parameters of the Brownian motion provides the manager with theoretical knowledge that they can then combine with the lessons of experience to inform their beliefs and guide their search behavior. We characterize for this environment the optimal search rule and show that it takes a simple form when agents possess standard risk aversion and maximize their utility on a period-by-period basis. We then use this result to explore several long-standing questions in organizations.

The optimal search rule is intuitive and simple to describe, yet the performance dynamics that it generates are far from smooth. The sequence of actions taken is highly path dependent, exhibiting a mixture of successes, failures, and strategy reversals. A key property is that efforts by a manager to improve performance can be counter-productive and actually lead to worse performance. We show that the fear of these missteps restrains managers and slows down learning. Whilst in moderation these missteps present little more than speed bumps that wash out over time, significant dips can have permanent effects on performance, so much so that a sufficiently poor outcome can cause a firm to abandon the search for better performance, derailing growth altogether. The ups and downs of the growth path also imply that firms can overtake—and be overtaken by—other firms. This richness captures the perplexing regularity that a market leader one day can retreat to the middle of the pack the next, and disappear altogether from the landscape the day after that.

We then turn to aggregate properties of the growth path to understand broader patterns in performance. Our first main finding is that good performance begets good performance. A well-known stylized fact in the economics of organizations is that firms exhibit Persistent Performance Differences, or PPDs (Gibbons 2010). Taking initial differences as given, we show how an initial advantage can persist and feed on itself, growing in expectation over time. In complex decision environments, therefore, performance does not converge across firms. The difficulty in finding good actions restrains the ability of under-performing firms to catch up. More surprisingly, better performance actually eases pressures on managers, enabling them to experiment more boldly. This explains how non-trivial differences in performance can emerge and persist in seemingly similar organizations.

A second puzzle in the study of organizations—one that follows naturally from the existence of PPDs—is why performance differences don’t disappear by imitation. That is, why do underperforming firms simply not imitate the actions of their better performing rivals? This failure is not only
a matter of observability as the actions of better performing firms are often well known.\textsuperscript{3} Milgrom and Roberts (1995, p. 203-4) suggest informally one potential explanation based on their work on complementarities in organizations. They argue: "An important puzzle is why Lincoln’s successes have not been copied. ... Our discussion suggests that Lincoln’s piece rates are a part of a system of mutually enhancing elements, and that one cannot simply pick out a single element, graft it onto a different system without the complementary features, and expect positive results."

We operationalize this intuition by supposing that firm performance depends on a common, industry-wide choice as well as an idiosyncratic, individual firm choice. This is consistent with Milgrom and Roberts’ intuition that some firm characteristics are inherently not imitable. We then show that, even with strategies that are perfectly observable, imitation is not profitable when the complementarity is sufficiently high. Our main contribution, however, is to show how this logic depends also on firm strategy and performance and to quantify this dependence. Holding constant the size of the complementarity between actions, we show how imitation is profitable only when competitors are ‘close’ to each other. This closeness is not measured in performance, however, as indeed the better the performance of a competitor the more attractive is imitation. Rather, the measure of closeness that matters is in the action space. It is possible that a competitor is simply ‘too far ahead’ in the action space to make imitation feasible. Restraining imitation in this case is risk, not risk in imitation \textit{per se} but risk in the match of the industry-wide and the firm-specific dimensions. This implies a connection between PPDs and imitation as it is the common factor of risk that holds back the convergence in performance in both cases.

The importance of complementarities in organizations is only magnified when decision making is decentralized. Although decentralized control is ubiquitous in modern organizations, modeling has largely focused on issues of asymmetric information (e.g., Alonso, Dessein, Matouschek 2008). So far unexplored is how decentralized decision making impacts experimentation and learning in organizations. The importance of this connection—and how it impacts managerial decision making—is a recurring them in Roberts’ (2004) book-length treatment of modern management: "Search and change must be coordinated. [...] leaving individual managers in charge of particular elements of the organization to find improvements on their own can fail miserable, as can experimentation that is limited in scope. Both can fail to find the better solution and instead leave the firm stuck at an inferior coherent point."

\textsuperscript{3} "Such firms as Lincoln Electric, Walmart, or Toyota enjoyed sustained periods of high performance. As a result, they were intensively studied by competitors, consultants, and researchers, and many of their methods were documented in great detail. Nonetheless, even when competitors aggressively sought to imitate these methods, they did not have the same degree of success as these market leaders." (Brynjolfson and Milgrom (2013, p.14))
Our model enables us to explore this question formally. We extend the model in the classic manner to a multidivisional firm with each of two divisional managers holding authority within her division. We then provide a positive theory of coordination problems, showing how they emerge endogenously as a function of firm performance and thereby explaining why some firms are plagued by coordination difficulties whereas others appear immune. The main predictive factor of coordination failures is the consistency of firm performance. A firm with constantly improving performance, loosely speaking, leaves little doubt as to the most promising route forward and is less likely to succumb to coordination problems. In fact, a firm that always beats expectations is immune from coordination failures. This result reveals a novel virtue of slow and steady growth as such a firm is less susceptible to coordination failures than a firm that has faster but more haphazard growth. Coordination problems are also important for their connection to PPDs. The emergence of coordination problems implies that otherwise identical firms can make different choices of equilibrium behavior. Complementarities and decentralized search, therefore, are consistent with the initial emergence of performance differences across firms.

Throughout the paper we cast our model in the context of firms, although the connection to other types of organizations is readily apparent. One natural application is to economic development and the growth of nations. Societies face many decisions about how to structure their economies, decisions that are fraught with uncertainty and that necessitate trial-and-error learning. For instance, the futility of communism and laissez faire as ways to organize an economy was revealed only through painful experience. Even with this experience—and agreement that better performing systems lay between these extremes—identifying the optimal arrangement remains a difficult task. In this setting the question of imitation applies a fortiori given the widening differences in wealth across countries.

The generality of our model—with its focus on decision making, imitation, and coordination—allows broad application. Applied to the problems of economic development and growth, our results on persistent performance differences, path dependence, and coordination failures, provide an explanation as to why nations do not converge in wealth over time and why the practice of poor countries imitating rich countries has such a checkered history. The tight connection between the problems facing firms and those facing nations is laid out particularly clearly in a policy paper by Matsuyama (1995) who writes: "The graph [of a one dimensional function with many local maxima and minima] represents the performance of economic systems. The rugged nature of the graph captures the inherent complementarity of activities in each system; the performance of an economy can change drastically by a small change in selection of activities. [...] There are a large
number of locally optimal systems, and each society has evolved into one of them. There is no way for society to search in a systematic way for the global optimum, or other local optima that are more efficient [...] The very diversity of the manners in which different developed economies cope with coordination also imposes a problem for underdeveloped economies if they try to learn from the experiences of more successful economies. They cannot pick and choose different parts from different systems, because of the complementarity inherent in any system. [...] Even if we could decide which system to adopt among all the systems currently known, and then replicate the system completely, it is not at all clear whether this is a desirable thing to do." (Matsuyama 1995, pp.17 & 20)

We find the similarity between this description of economic development and the problems facing firms particularly striking. The connections, we expect, will be apparent as we proceed through the text. Nevertheless, for simplicity in presentation, we relegate an explicit treatment of development and growth to the discussion section.

2 Related Literature

The use of Brownian motion to model experimentation and learning was first introduced in Callander (2011). Formally, the Brownian motion framework corresponds to a bandit model with a continuum of correlated, deterministic arms. This contrasts with the standard assumption in the experimentation literature of stochastic and independent arms (see, for instance, Bolton and Harris 1999). The correlation across arms is important as it captures learning across alternatives. With independent arms a failed experiment is just that and nothing more: the failed alternative is discarded and does not inform subsequent choice. In contrast, in the Brownian motion framework failed experiments are themselves discarded, but the information gleaned from them informs future choices. This information dictates whether to experiment further, which direction to experiment in and how bold to be. It is this path dependence that is the focus of our analysis. To be sure, this increase in realism comes with an analytic cost as we model decision makers who maximize utility on a period-by-period basis. The substantive conclusions we draw from this model do not obviously depend on this assumption, although further exploration is clearly warranted.

The current paper departs from the model of Callander (2011) in key respects. That paper imposes strong restrictions on the objective function and risk preferences, restrictions that fit some applications but are less applicable in understanding learning in firms and most other organizations. In particular, in Callander (2011) the maximum attainable outcome is known, implying that any outcome received can be evaluated relative to this known maximum. Formally, agents possess an
ideal outcome and they search for an action that delivers an outcome as close as possible to that ideal. This property plays a key role in the optimal search rule in Callander (2011). It implies that agents possess information in addition to their performance level as they know whether they are to the left or right of the optimum. This directional information, combined with knowledge of the maximum attainable, leads agents to stop experimenting when they observe an outcome that they deem “good enough” relative to the maximum. This reasoning does not exist in our model. Our agents always seek better performance and the level of performance is not bounded. These differences imply that we can not default to Callander’s (2011) assumption of quadratic utility. Instead, we identify the type of utility functions that support well-defined search in this setting and, rather than singling out a particular functional form, we characterize the class of risk preferences that satisfy this requirement.

These differences allow a broad and significant expansion of the applicability of the Brownian motion framework. Notably, it allows our application to organizational learning and enables insights into several long-standing questions of complementarities, coordination, and imitation that have not previously yielded to economic methods. Some of these questions have been covered in the rugged landscapes literature, although the intuition behind their results is distinct from ours. In a major contribution to the rugged landscapes literature, Rivkin (2000) explains difficulties in imitation through the theory of NP-completeness. He argues that when the set of actions that must be undertaken grows large, and the complementarities between actions is unknown, it is computationally infeasible for a firm to exactly imitate the strategy of a successor. Our focus, in contrast, is on the deliberate choice of managers whether to imitate or not rather than the feasibility of imitation per se.

3 The Model

There is a single manager. At the beginning of every period \( t = 1, 2, \ldots \), the manager takes an action that determines his income. After the manager has consumed his income, time moves on to the next period. Our aim is to characterize the manager’s optimal actions given the technology, preferences, and information structure that we describe next.

**Technology:** The manager’s action \( a_t \in \mathbb{R} \) determines his income level \( m_t \in \mathbb{R} \) according to the production function \( m(a_t) \), where \( m : \mathbb{R} \to \mathbb{R} \). We follow Callander (2011) and model \( m \) as the realized path of a Brownian motion with drift \( \mu > 0 \) and variance \( \sigma^2 > 0 \). For reasons that will become apparent, we interpret the variance \( \sigma^2 \) as a measure of the complexity of the production
process. Moreover, we refer to \( a_0 = 0 \) as the status quo action and denote status quo income by \( m_0 = m(a_0) \). The realized path of the Brownian motion is determined by nature before the start of the game and does not change over time. Figure 1 shows one possible realization of the Brownian motion.

**Preferences:** The manager’s utility is given by \( u(m) \), where \( m \) is his income. We assume that this function is four times continuously differentiable and satisfies \( u'(m) > 0 \) and \( u''(m) < 0 \) for all \( m \in \mathbb{R} \). The first condition implies non-satiation and the second risk aversion.

We further assume that the utility function satisfies *standard risk aversion* which requires that a loss-aggravating risk can never make an independent, undesirable risk more desirable (Kimball 1993). Formally, suppose there are two independent random variables \( x \) and \( y \) such that

\[
E[u(m + x)] - u(m) \leq 0 \quad \text{and} \quad E[u'(m + y)] - u'(m) \leq 0.
\]

The risk \( x \) is undesirable since the manager would turn it down if someone offered it to him. And the risk \( y \) is loss aggravating since it increases the manager’s expected marginal utility; it therefore makes a sure loss more undesirable and is itself more undesirable if the manager is exposed to a sure loss. The utility function then satisfies standard risk aversion if and only if

\[
E[u(m + x + y) - u(m + y)] \leq E[u(m + x) - u(m)],
\]
that is, if and only if the loss aggravating risk $y$ does not make the undesirable risk $x$ more desirable.

Most commonly used utility functions that exhibit non-satiation also exhibit standard risk aversion, including exponential, logarithmic, and power functions. Standard risk aversion, however, is stronger than decreasing absolute risk aversion: the former implies the latter but reverse does not hold (Kimball 1993). We need to assume more than decreasing absolute risk aversion since this type of risk aversion is not enough to ensure that independent risks are substitutes. As such, decreasing absolute risk aversion is less useful in settings such as ours in which an agent has to choose between multiple, risky payoffs than it is in settings in which an agent has to choose between a single, risky payoff and a safe one (see, for instance, Chapter 9 in Gollier (2001)).

We denote the coefficient of absolute risk aversion by $r(m) = -u''(m)/u'(m)$. Since standard risk aversion implies decreasing absolute risk aversion we have $r'(m) \leq 0$ for all $m \in \mathbb{R}$. We will see below that for a non-trivial solution to the manager’s problem to exist, $r(m)$ has to cross the ratio $2\mu/\sigma^2$, where $\mu$ and $\sigma^2$ are the drift and the variance of the Brownian motion. Unless we explicitly say otherwise, we therefore assume that $r(m)$ does cross $2\mu/\sigma^2$ and we denote the largest income level for which $r(m) = 2\mu/\sigma^2$ by $\hat{m}$.

**Information:** In any period, the manager knows the income generated by the status quo action and by any action he took in any previous period. We refer to these actions as “known actions” and to all other actions as “unknown actions.” In addition to the known actions, the manager knows that the production function was generated by a Brownian motion with drift $\mu$ and variance $\sigma^2$. The manager does not, however, know the realization of the Brownian motion. In any period $t$, the manager’s information set is therefore given by $I_t = \{\mu, \sigma^2, (a_0, m_0), \ldots, (a_{t-1}, m_{t-1})\}$.

**Optimal Learning Rule:** For simplicity we assume that the manager maximizes expected utility on a period-by-period basis. An optimal learning rule is therefore given by $(a_1^*, a_2^*, \ldots)$, where

$$a_t^* \in \arg \max_{a_t} \mathbb{E}[u(m_t) | I_t].$$

Our goal is to characterize the set of optimal learning rules.

4 **Beliefs and Expected Utility**

We start by examining the manager’s beliefs about the income generated by any unknown action. For this purpose, consider any period $t$ and let $l_t$ and $r_t$ denote the left-most and right-most known actions. Consider now an unknown action $a_t$ that is to the right of $r_t$. For any such action, the
manager believes that income $m(a_t)$ is drawn from a normal distribution with mean

$$E[m(a_t)] = m(r_t) + \mu(a_t - r_t)$$

and variance

$$\text{Var}(m(a_t)) = (a_t - r_t)\sigma^2,$$

where $\mu$ and $\sigma^2$ are the drift and the variance of the Brownian motion. The manager therefore expects an action to generate higher income, the further it is to the right of $r_t$. At the same time, however, the further an action is to the right of $r_t$, the more uncertain the manager is about the income generated by that action. The beliefs for actions to the left of the left-most action $l_t$ are analogous.

Notice that the manager’s beliefs about any action to the right of $r_t$ depend only on $r_t$ and that his beliefs about any action to the left of $l_t$ depend only on $l_t$. Similarly, the manager’s beliefs about any action between $l_t$ and $r_t$ depend only on the known actions closest to that action. To see this without having to introduce more notation, suppose that there are no known actions between $l_t$ and $r_t$. For any action $a_t \in [l_t, r_t]$, the manager then believes that income $m(a_t)$ is normally distributed with mean

$$E[m(a_t)] = \frac{a_t - l_t}{r_t - l_t}m(r_t) + \frac{r_t - a_t}{r_t - l_t}m(l_t)$$

and variance

$$\text{Var}(m(a_t)) = \frac{(a_t - l_t)(r_t - a_t)}{r_t - l_t}\sigma^2.$$  

The manager’s expected income is therefore a convex combination of the income generated by $l_t$ and $r_t$. Moreover, the further the action is from the closest known action, the more uncertain the manager is about the income generated by that action.

The assumption that the production function is generated by a Brownian motion therefore ensures that the manager’s beliefs take a simple form that satisfies several intuitive properties. First, beliefs are normally distributed. Second, the manager knows more about an action, the closer the action is to a known action, and the less complex is the production process. Third, the manager engages in directed search, that is, he knows where he can expect better actions and, as we will see below, he focuses his search in that direction. And finally, even if, over time, the manager learns the income generated by a very large number of actions, he can never infer the entire production function. In this sense, there is a limit to theoretical knowledge and thus a deep need to learn about by trial-and-error.
Now that we have examined the manager’s beliefs, we can specify his expected utility. Suppose that the manager believes that income is normally distributed with mean \( M \) and variance \( V \) and let \( z \) denote a random variable that is drawn from a standard normal distribution. We can then state our first lemma, which is proven in "Hilfsatz" 4.3 in Schneeweiss (1966) and Theorem 1 in Chipman (1973).

**Lemma 1 (Schneeweiss 1966 and Chipman 1973).** Suppose that \( |u(m)| \leq A \exp(-Bm^2) \) for some \( A > 0 \) and \( B > 0 \). Then the expected utility function

\[
W(M, V) = E\left[u\left(M + \sqrt{V} z\right)\right]
\]

exists for all \( M \in (-\infty, \infty) \) and \( V \in (0, 1/(2B)) \).

The restriction in the lemma ensures that expected utility is integrable, and for the remainder of the paper we assume that it holds. Notice that since we are free to choose any positive parameters \( A \) and \( B \), this restriction is mild and, for the remainder of this paper, we assume that it holds.

Next, we can show that expected utility is concave.

**Lemma 2 (Chipman 1973 and Lajeri and Nielsen 2000).** The expected utility function \( W(M, V) \) is concave.

The lemma follows from Theorem 3 in Chipman (1973) and Theorem 2 in Lajeri and Nielsen (2000). Specifically, Theorem 3 in Chipman (1973) states a condition that ensures concavity of the expected utility function. And Theorem 2 in Lajeri and Nielsen (2000) shows that this condition is equivalent to absolute prudence \( -u'''(m)/u''(m) \) being decreasing in \( m \) for all \( m \in \mathbb{R} \). The lemma then follows from the fact that in a setting such as ours, in which income is unbounded, decreasing absolute prudence is equivalent to standard risk aversion (Proposition 3 in Kimball (1993)). In the next section we will see that in the relevant range both expected income and its variance are linear in the manager’s action. Standard risk aversion therefore ensures that the manager’s problem is concave.

## 5 Managerial Learning

We first focus on the optimal action in the first period and then turn to subsequent periods.

### 5.1 The First Period

In the first period, the manager can take the status quo action, in which case he is certain to realize status quo income \( m_0 \). Or he can take some action \( a_1 \neq a_0 \), in which case he is uncertain about
what income he will realize. In the previous section, we saw that for any action \( a_1 < a_0 \), expected income is strictly less than status quo income. The manager will therefore never take an action strictly to the left of the status quo.

Suppose then that \( a_1 \geq a_0 \) and let \( \Delta_1 = a_1 - a_0 \geq 0 \) denote the size of the step the manager takes in the first period. We then know from (1) and (2) that the manager’s expected income is given by \( m_0 + \mu \Delta_1 \) and its variance is given by \( \sigma^2 \Delta_1 \). We can therefore write the manager’s problem as

\[
\max_{\Delta_1 \geq 0} W (m_0 + \mu \Delta_1, \sigma^2 \Delta_1),
\]

where \( W(\cdot) \) is the expected utility function defined in Lemma 1. As observed above, Lemma 2 ensures that this problem is concave.

Next, by differentiating \( W(\cdot) \) with respect to \( \Delta_1 \) we obtain

\[
\frac{dW (m_0 + \mu \Delta_1, \sigma^2 \Delta_1)}{d\Delta_1} = \mathbb{E} \left[ u' \left( m_0 + \mu \Delta_1 + \sigma \sqrt{\Delta_1} z \right) \right] \frac{\sigma^2}{2} \left( \frac{2\mu}{\sigma^2} - R(m_0, \Delta_1) \right),
\]

(5)

where

\[
R(m_0, \Delta_1) \equiv -\frac{\mathbb{E} \left[ u'' \left( m_0 + \mu \Delta_1 + \sqrt{\Delta_1} \sigma z \right) \right]}{\mathbb{E} \left[ u' \left( m_0 + \mu \Delta_1 + \sqrt{\Delta_1} \sigma z \right) \right]}
\]

and where we make use of the fact that \( \mathbb{E} [u'(\cdot) z] = \sigma \sqrt{\Delta_1} \mathbb{E} [u''(\cdot)] \).

The sign of expected marginal utility is therefore determined by the relative size of the ratio \( 2\mu/\sigma^2 \) and \( R(m_0, \Delta_1) \). To interpret these two objects, notice that \( 2\mu/\sigma^2 \) is similar to the Sharpe ratio (Sharpe 19xx). In contrast to the Sharpe ratio, however, the denominator is the variance and not the standard deviation. Notice also that \( R(m_0, \Delta_1) \) is the coefficient of absolute risk aversion for the indirect utility function \( \mathbb{E} [u(\cdot)] \) which is, in general, different from the expected value of the coefficient of absolute risk aversion \( r(\cdot) \). For \( \Delta_1 = 0 \), however, we do have \( R(m_0, 0) = r(m_0) \) and thus

\[
\frac{dW (m_0, 0)}{d\Delta_1} \begin{cases} > 0 & \text{if } m_0 > \hat{m} \\ \leq 0 & \text{if } m_0 \leq \hat{m}, \end{cases}
\]

(6)

recalling that \( \hat{m} \) denotes the largest income level \( m \) for which \( r(m) = 2\mu/\sigma^2 \). Intuitively, the manager prefers a small step to the status quo if and only if he is sufficiently wealthy, in which case his coefficient of absolute risk aversion is sufficiently small.

We can now establish our first proposition which characterizes the manager’s optimal action in the first period.

PROPOSITION 1. The manager’s optimal first period action is unique and given by

\[
a_1^* = \begin{cases} a_0 + \Delta(m_0) & \text{if } m_0 \geq \hat{m} \\ a_0 & \text{if } m_0 < \hat{m}, \end{cases}
\]

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where $\Delta(m_0)$ is implicitly defined by $R(m_0, \Delta(m_0)) = 2\mu/\sigma^2$ and satisfies $\Delta(\hat{m}) = 0$ and $\Delta'(m_0) > 0$ for all $m_0 \geq \hat{m}$.

The proposition establishes that there is a threshold level of status quo income above which the manager engages in experimentation and below which he prefers to stick to the status quo. And it establishes that if the manager does engage in experimentation, his experiment is larger, the higher is his status quo income. Higher status quo income therefore unambiguously favors experimentation in the first period.

5.2 The Second and Subsequent Periods

We just saw that if $m_0 \leq \hat{m}$, the manager takes the status quo action in the first period and learns nothing further about the mapping. In any subsequent period, the manager then faces the same problem as in the first, and takes the status quo action again. Consequently, for the remainder of the paper we suppose that $m_0 \geq \hat{m}$ holds.

After experimenting in the first period, the manager learns an additional point in the mapping and now knows two actions: the status quo action $a_0$ and the manager’s first period action $a_1^*$. From (3) it is clear that any action on the bridge between $a_0$ and $a_1^*$ is dominated by one of the ends as it delivers a lower expected income with uncertainty. Thus, if the manager is to experiment further, he continues to search in the same direction to the right. In forming beliefs about what to expect to the right of $a_1^*$, however, the income at $a_1^*$ is the only relevant information (as the Brownian motion possesses the Markov property). Thus, in deciding whether further experimentation is preferred to action $a_1^*$, the manager faces the exact same problem as he did in the first period.

The second period differs from the first period in two key respects, however. First, the income at $a_1^*$ will almost surely be different from the status quo income, probably having increased but also potentially decreased. In fact, the income at $a_1^*$ could be so low as deter further experimentation altogether. From Proposition 1 we know that this is the case if $m_1^* \leq \hat{m}$. The second difference between the periods is that, should the manager decide not to experiment, he has two known actions from which to choose. He may repeat action $a_1^*$ or he may reverse course and revert to the status quo action $a_0$. Reversing course is clearly optimal if second period income is so low as to fall below $\hat{m}$. Surprisingly, the manager will choose to reverse course and stop experimenting even for income levels above $\hat{m}$ (but below $m_0$). At these income levels the manager would prefer to experiment rather than stay at $a_1^*$, yet that is not his only option and he prefers the certainty of the status quo income over further experimentation. This possibility is captured in the following lemma.
LEMMA 3. There exists a threshold level of income $\tilde{m}(m_0) \in (\tilde{m}, m_0)$ such that

$$\begin{align*}
u(m_0) &= W(\tilde{m}(m_0) + \mu \Delta(\tilde{m}(m_0)), \sigma^2 \Delta(\tilde{m}(m_0))),
\end{align*}$$

(7)

where $\Delta(\tilde{m}(m_0)) > 0$. Moreover, the derivative of $\tilde{m}(m_0)$ satisfies $0 < \tilde{m}'(m_0) \leq 1$.

This implies that if first period income $m_1^*$ is equal to $\tilde{m}(m_0)$, the manager is indifferent between the status quo and further experimentation. It then follows that the manager strictly prefers experimentation to the status quo if $m_1^* > \tilde{m}(m_0)$ and that he strictly prefers the status quo to the constrained optimal action if $m_1^* < \tilde{m}(m_0)$. Thus, for income levels $m_1^*$ strictly between $\tilde{m}(m_0)$ and $\tilde{m}$, learning stops and the manager reverts to the status quo even though the marginal return from engaging in further search is positive.

The problem the manager faces in any period $t > 2$ is very similar to the one he faces in period 2. The only difference is that in any period $t > 2$, the manager does not necessarily compare his expected utility from engaging in further search with his utility from the status quo, as he does in the second period. Instead, the manager compares his expected utility from engaging in further search with his utility from whatever known action generates the largest income level, which may be the status quo action or some other known action.

To state the proposition that characterizes the manager’s optimal action in all periods $t \geq 2$, let $\overline{m}_t$ denote the largest known income level in period $t$, that is, let

$$\overline{m}_t = \max\{m_0, m_1^*, m_2^*, ..., m_{t-1}^*\}.$$  

Also, let $\overline{a}_t$ denote the action that generates $\overline{m}_t$, that is, let

$$\overline{a}_t \in \{a_0, a_1^*, a_2^*, ..., a_{t-1}^*\}$$ such that $m(\overline{a}_t) = \overline{m}_t.$

And finally, recall that $r_t$ denote the right-most known action in period $t$. We can then state our next proposition.

PROPOSITION 2. The manager’s optimal action in period $t \geq 2$ is unique and given by

$$a_t^* = \begin{cases} 
  r_t + \Delta(m(r_t)) & \text{if } m(r_t) > \tilde{m}(\overline{m}_t) \\
  a(\overline{m}_t) & \text{if } m(r_t) \leq \tilde{m}(\overline{m}_t)
\end{cases},$$

where $\Delta(m)$ is the $\Delta$ that solves $R(m, \Delta) = 2\mu/\sigma^2$ and $\tilde{m}(\cdot)$ is defined in (7).

In any period $t \geq 2$, the manager therefore engages in search if and only if the income level $m(r_t)$ associated with the largest previously taken action $r_t$ is above a threshold $\tilde{m}(\overline{m}_t)$, where the threshold is increasing in the largest known income level $\overline{m}_t$. 

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Together Propositions 1 and 2 characterize the optimal search rule and show that it takes a simple form. Figuratively, the manager’s exploration of the rugged performance landscape depends on his starting point. If that point is too low, he just stays put. If his starting point is sufficiently high, he starts exploring the rugged landscape by taking discrete steps towards the right. He continues this rightward march indefinitely unless his income falls off a sufficiently large cliff. If this were to happen, continuing the search bears too much risk for the manager and he instead reverts to the highest peak he discovered during his exploration. Strikingly, this peak is only a local peak in expectation. The manager does not know if a better action lies in his vicinity and he does not find it worthwhile to find out. Moreover, the manager believes with probability one that better actions exists far to the right. Nevertheless, he chooses to not seek out better performance as despite the allure of higher income, the threat of his efforts only lowering his performance is too great and the risk compels him to stop searching altogether.

To conclude this section, we turn to the implications of relaxing the assumption that the coefficient of absolute risk aversion crosses $2\mu/\sigma^2$. The next proposition confirms our earlier claim that the solution to the manager’s problem is then either trivial or does not exist.

**PROPOSITION 3.** If the coefficient of absolute risk aversion $r(m)$ satisfies $r(m) > 2\mu/\sigma^2$ for all $m \in \mathbb{R}$, the manager does not engage in search in the first period or any subsequent period. If, instead, $r(m) < 2\mu/\sigma^2$ for all $m \in \mathbb{R}$, an optimal action does not exist. Finally, if $r(m) = 2\mu/\sigma^2$ for all $m \in \mathbb{R}$, then in any period $t$ the manager is indifferent between the status quo and any action to its right.

For $r(m) > 2\mu/\sigma^2$ for all $m \in \mathbb{R}$ the manager is too risk averse to engage any risk, regardless of the wealth level. The result follows from the definition of $\tilde{m}$ in Proposition 1. On the other hand, if $r(m) < 2\mu/\sigma^2$ the manager is sufficiently risk tolerant to engage risk for any wealth level. However, as this implies that

$$-E\left[u''\left(m_0 + \mu \Delta_1 + \sqrt{\Delta_1}sz\right)\right] < 2\mu/\sigma^2 E\left[u'\left(m_0 + \mu \Delta_1 + \sqrt{\Delta_1}sz\right)\right].$$

it follows from (5) that marginal expected utility is strictly positive for all $\Delta_1 \geq 0$ and an optimum doesn’t exist. Finally, when $r(m) = 2\mu/\sigma^2$ for all $m \in \mathbb{R}$, the manager is indifferent whether to undertake risk at every wealth level, and an analogous argument to the previous case establishes the result.
6 Persistent Performance Differences

A key feature of the optimal search rule is that it depends on the manager’s status quo income. A natural question then is whether two managers with different status quo income levels can expect their incomes to converge or diverge over time. The answer is not immediate since there are forces that go in either direction: on the one hand, higher status quo income favors experimentation in the first period; on the other hand, however, it also makes it more tempting to stop experimentation and revert to the status quo in subsequent periods. The next proposition shows that on average the desire to experiment dominates.

PROPOSITION 4. Suppose there are two managers, Manager $H$ and Manager $L$. The production function of Manager $k = L, H$, is characterized by status quo income $m_0^k$, drift $\mu$, and variance $\sigma^2$, where $m_0^H > m_0^L > \bar{m}$. Then

$$E_1[m_t^*(m_0^H) - m_t^*(m_0^L)] > m_0^H - m_0^L \text{ for all } t = 1, 2, \ldots,$$

where $E_1[\cdot]$ are the expectations taken at the beginning of the first period.

It is easy to show that there are realizations of the production functions for Managers $H$ and $L$ that are consistent with any dynamic: convergence, divergence, overtaking, and so on. The proposition, however, shows that on average, income levels diverge over time. In particular, it shows that if Manager $H$’s status quo income is one dollar above Manager $L$’s, then Manager $H$’s expected income will be strictly more than one dollar above Manager $L$’s expected income in every subsequent period. On average, income therefore does not converge over time. Instead, it diverges.

Proposition 4 does not explain the origin of the initial difference between firms $L$ and $H$. It does explain why a difference that does exist is likely to persist and why it is expected to grow over time. These properties are consistent with the existence of persistent performance differences in organizations. The proposition is silent on the growth rate in income differences. Simulations show that this can be large, with even small initial differences in wealth growing into significant gaps over a relatively short number of periods.\(^4\) In Section 8 we show how coordination problems– and equilibrium multiplicity–can explain the origin of performance differences, even when firms are otherwise identical.

\(^{4}\)Simulations are available from the authors upon request. We employed a matlab code with the utility function: $u(m) = \alpha m - e^{-\beta m}$, for $\alpha = \beta = 1$. 

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7 The Risk of Imitation

Organizations typically do not operate in isolation. This naturally raises the question of imitation and why, if performance differences emerge, the trailing firms simply not imitate the market leaders. An outstanding puzzle in organizations is that successful imitation occurs less frequently than we might expect. A puzzling aspect of this failure, as noted in the introduction, is that it persists despite the strategies of market leaders often being easily observed. We offer an explanation to this puzzle based on complementarities in organizational decision making.

To allow for complementarities, we extend the model to allow the manager to take two actions, denoted by $a^A_t$ and $a^B_t$ for period $t$. These actions produce total income

$$m_t = m^A (a^A_t) + m^B (a^B_t),$$

where $m^A (a^A)$ and $m^B (a^B)$ are the production functions for the two actions. Each production function is an independently realized path of a Brownian motion with drift $\mu$ and variance $\sigma^2$. The status quo actions are given by $a^A_0 = a^B_0 = a_0 = 0$ and each status quo action generates status quo income $m^A (0) = m^B (0) = \frac{m_0}{2}$, where we divide by two for notational convenience.

To enable imitation, we then add a second firm, whose actions on the two dimensions are denoted $b^A$ and $b^B$. We then impose two requirements consistent with the discussion in Milgrom and Roberts (1995). First, we suppose that dimension A represents a common, industry-wide determinant of firm performance, whereas dimension B is an idiosyncratic, firms-specific factor. For instance, a competitor may copy perfectly Lincoln Electric’s piece rate labor contract but within the new firm’s culture, this choice will produce a very different outcome to what it produces at Lincoln Electric. This formulation is consistent with the frequent observation in practice that firms in the same industry with strategies that are indistinguishable nevertheless produce very different performance.

The second requirement we impose is that the two dimensions of choice are complementary. In fact, to allow use of the results from the previous section, we assume that the actions are strict complements. The manager therefore always has to set the actions equal to each other, that is, he has to set $a^A_t = a^B_t = a_t$. Specifically, the manager’s problem in the first period—before contemplating imitation—is now given by

$$W (m_0 + 2\Delta_1 \mu, 2\Delta_1 \sigma^2) = E \left[ u \left( m_0 + 2\mu \Delta_1 + \sigma \sqrt{2\Delta_1} z \right) \right],$$

where $\Delta_1 = a_1 - a_0$ denotes the size of the manager’s first period step. Notice that this expression is equivalent to the manager’s first period expected utility in our baseline model if the drift were
given by \(2\mu\) and the variance were given by \(2\sigma^2\). We can therefore use Proposition 1 to determine the manager’s optimal decisions in the first period.

In formulating strategy, a manager must now incorporate both the history of his own actions and information gleaned from the performance of the other firm. Our interest is in the conditions under which imitation is profitable and drives convergence in the performance of firms. The general analysis of this question depends on the particular histories of each firm and is analytically complicated. To capture the essential forces at work, we consider the special case of imitation at the first period. Specifically, for manager A we suppose his income satisfies \(m_0 \leq \hat{m}\) such that, in isolation, he would choose not to experiment. For manager B we suppose that his status quo actions are \(d\) units to the right of A’s, \(b^A = b^B = d > 0\), and that his income on dimension \(A\) is \(D\) units higher, \(m^A(d) = D + m_0/2\).

The question, then, is how knowledge of B’s action on dimension \(A\) affects decision making by manager A? The trade-off for manager A is that while B’s performance provides valuable information on the common dimension, it offers neither help nor guidance on the second dimension. And because of the complementarity of actions, if manager A is compelled to imitate his competitor, he must simultaneously move his action on dimension \(B\) from the safety of the status quo to a risky alternative. Proposition 5 shows how this trade-off depends on the distance between firms in both the performance space and describes the nature of imitation when it does occur.

**PROPOSITION 5.** *The manager prefers the status quo if and only if*

\[
d \geq \hat{d},
\]

*where \(\hat{d} > 0\) is increasing in \(D\) and decreasing in \(\sigma^2\). Moreover, if the manager departs from the status quo he imitates completely and chooses \(a^A_1 = a^B_1 = d\).*

Imitation, therefore, is all or nothing. Should the information from the other firm cause the manager to depart from what he would choose in isolation, it must be that he prefers to use all of the available information available and imitate precisely. Thus, imitation, when it does occur, leads to convergence in firm performance.

Yet the key finding of Proposition 5 is that imitation does not always occur. If the competing firm is too far from the manager in the action space then imitation is not undertaken. Even though the manager could precisely and fully replicate his competitor’s performance on the industry-wide dimension, the change in strategy necessary to achieve this is so great that the risk generated on the second dimension is too great a cost to bear. This implies that imitation fails when one firm is ‘too far ahead’ from the other, where the relative distance is measured in the action space rather
than performance space. In fact, the greater the distance between firms in the performance space, all else equal, the more attractive is imitation.

A barrier to imitation is also provided by the complexity of the business environment. However, as imitation is all or nothing, it is only the complexity of the firm-specific action that determines the profitability of imitation. As this level decreases imitation becomes more attractive, and at the limit of $\sigma^2 = 0$ the manager will always find it optimal to imitate the other firm.

Collectively, the conditions in Proposition 5 imply that imitation is to be expected in industries that have less complex firm-specific components of strategy, and in industries where the band of managerial practices and strategies used is relatively narrow. Although this suggests a competitive advantage in obtaining a lead in such industries, this must be tempered by the reality that strategic novelty in such industries is commensurately more difficult to obtain.

A final observation on Proposition 5 is that while imitation is not always chosen in equilibrium, it is always possible. Our explanation for the failure of imitation does not depend, therefore, on observability or implementation, but rather it depends on risk. In a parallel to the results in previous sections where we saw that risk inhibits search and experimentation and ensures that performance differences endure and grow, it is again risk that explains why imitation is not more prevalent that we might expect.

8 Coordination Failures and Decentralized Search

Suppose there are two managers, $A$ and $B$. In any period $t = 1, 2, \ldots$ the managers make decisions $a_t^A \in \mathbb{R}$ and $a_t^B \in \mathbb{R}$ and then realize their incomes $m_t^A$ and $m_t^B$. In particular, their incomes are given by

$$m_t^A = m(a_t^A) - \frac{1}{2} \delta (a_t^A - a_t^B)^2$$

and

$$m_t^B = m(a_t^B) - \frac{1}{2} \delta (a_t^A - a_t^B)^2,$$

where $\delta \geq 0$ is a parameter that measures the importance of coordination between the managers’ decisions. The function $m(\cdot)$ is once again the realized path of a Brownian motion with drift $\mu$ and variance $\sigma^2$ and for which $m_0 = m(0)$. Notice that this function is the same for both managers. The managers are therefore learning about the same production function.

The managers know that the production function is generated by a Brownian motion with drift $\mu$ and variance $\sigma^2$ and that $m(0) = 0$. Moreover, in any period $t$, the managers know the actions that they took in previous periods and the income that these actions generated. managers $A$ and
Therefore have the same information set

\[ I_t = \{ \mu, \sigma^2, (0, m_0), (a_1^A, m_1^A), (a_1^B, m_1^B), \ldots, (a_{t-1}^A, m_{t-1}^A), (a_{t-1}^B, m_{t-1}^B) \} \] .

And they have the same utility function \( u(\cdot) \) that satisfies the same properties described above.

We distinguish between centralized learning—in which case the managers coordinate their actions—and decentralized learning—in which case the managers take their actions independently and simultaneously. If learning is centralized, the optimal learning rule is the same as the one we derived above. If learning is decentralized, however, there is a key difference, which we derive below.

8.1 The First Period

The managers never find it optimal to take actions to the left of the status quo, just as in our baseline model. Suppose therefore that \( a_1^A \geq a_0^A \) and \( a_1^B \geq a_0^B \) and let \( \Delta_1^A = a_1^A - a_0^A \) and \( \Delta_1^B = a_1^B - a_0^B \) denote the size of each manager’s first period step. We can then write Manager A’s expected utility as

\[
W^A \left( m_0 + \mu \Delta_1^A - \frac{1}{2} \delta (\Delta_1^A - \Delta_1^B)^2, \sigma^2 \Delta_1^A \right) = E \left[ u \left( m_0 + \mu \Delta_1^A - \frac{1}{2} \delta (\Delta_1^A - \Delta_1^B)^2 + \sigma \sqrt{\Delta_1^A} z \right) \right],
\]

where \( z \) is again a random variable drawn from the standard normal distribution. The definition of Manager B’s expected utility is analogous. The managers’ first period problem is then given by

\[
\max_{\Delta_1^A \geq 0} W^A \left( m_0 + \mu \Delta_1^A - \frac{1}{2} \delta (\Delta_1^A - \Delta_1^B)^2, \sigma^2 \Delta_1^A \right)
\]

and

\[
\max_{\Delta_1^B \geq 0} W^B \left( m_0 + \mu \Delta_1^B - \frac{1}{2} \delta (\Delta_1^A - \Delta_1^B)^2, \sigma^2 \Delta_1^B \right).
\]

The solution to this problem gives the managers’ optimal first period actions, which we characterize in the next proposition.

PROPOSITION 4. The managers’ optimal first period actions are unique and given by

\[ a_1^{A*} = a_1^{B*} = a_1^* \]

where \( a_1^* \) is the optimal action in the baseline model defined in Proposition 1.

The optimal first period actions are therefore the same as in our baseline model. Notice that this implies that there cannot be any coordination failures in the first period, even if the actions are very complementary, that is, even if \( \delta \) is very large.
To understand why coordination failures cannot arise in the first period, suppose that \( \delta \) is very large and that \( A_1^A = A_1^B = 0 \). Notice that if \( A_1^A = A_1^B \), the effect of a marginal increase in the size of a manager’s action on expected utility is second order. As long as \( m_0 > \bar{m} \), Manager A has an incentive to unilaterally increase the size of his action by a small amount. But once A has increased his action, Manager B benefits from increasing her action by the same amount. And once B has increased her action by the same amount, A benefits by unilaterally increasing her action by a further small amount. This process continues until each manager’s action is equal to the optimal action in the single-action model, at which point neither manager has an incentive to unilaterally change his or her action.

### 8.2 The Second and Subsequent Periods

Consider now the second period. As in the baseline model, it is never optimal to take an action that is either strictly to the left of the status quo or strictly between the status quo and the optimal first period action \( a_1^* \). And as in the first period of this model, the two managers will always take the same action. In contrast to either setting, however, the managers may now get stuck in a coordination failure.

To understand why coordination failures can arise in the second period, suppose that \( m_1^* \) is just marginally larger than the threshold income level \( \bar{m} (m_0) \). We saw above that if \( m_1^* > \bar{m} (m_0) \), then in the baseline model the optimal second period action is given by \( a_1^* + \Delta^* (m_1^*) > a_1^* \), where \( \Delta^* (m_1^*) \) is defined in (??). Suppose now that in the second period Manager A takes action \( a_1^* + \Delta^* (m_1^*) \). Since the managers benefit from coordination, it is then optimal for Manager B to take the same action. It is therefore an equilibrium for each manager to take action \( a_1^* + \Delta^* (m_1^*) \). Suppose now, however, that Manager A takes the status quo action \( a_0 \). If \( \delta \) is large enough, it is then optimal for Manager B to take the same action. It is therefore also an equilibrium for each manager to take the status quo action, in which case the managers are stuck in a coordination failure.

To understand the conditions under which coordination failures can arise in the second period, suppose that \( m_1^* > \bar{m} (m_0) \) and let \( \Delta^{k*} (m_1^*, 0) \) denote the solution to

\[
\max_{\Delta_2^A \geq 0} W^A \left( m_0 + \mu \Delta_2^A - \frac{1}{2} \delta (\Delta_2^A)^2, \sigma^2 \Delta_2^A \right)
\]

and

\[
\max_{\Delta_2^B \geq 0} W^B \left( m_0 + \mu \Delta_2^B - \frac{1}{2} \delta (\Delta_2^B)^2, \sigma^2 \Delta_2^B \right)
\]

In the appendix we show that there then exists a unique income level \( \bar{m} (m_0, \delta) \) such that

\[
u(m_0) = W^K \left( \bar{m} (m_0, \delta) + \mu \Delta^{k*} (\bar{m} (m_0, \delta)) - \frac{1}{2} \delta (\Delta^{k*} (\bar{m} (m_0, \delta)))^2, \sigma^2 \Delta^{k*} (\bar{m} (m_0, \delta)) \right)
\]

(9)
for $k = A, B$. In words, there exists a unique income level such that if one manager takes the status quo action, the other is indifferent between also taking the status quo action and engaging in search. Notice that $\tilde{m}(m_0, 0) = \tilde{m}(m_0)$. Moreover, we show in the appendix that while $\tilde{m}(m_0, \delta)$ is strictly increasing in $\delta$, it is always strictly smaller than $m_0$. For any $\delta > 0$ there therefore exists a region $(\tilde{m}(m_0, \delta), m_0)$ such that if $m^*_t$ is within that region, there are two equilibria. In one of those equilibria, both managers take the status quo action. And in the other, both managers take action $a^*_t + \Delta^*(m^*_t)$, which is the optimal second period action in the baseline model.

As in the baseline model, the problem faced by the managers in periods $t \geq 2$ is very similar to the first period problem. To characterize the optimal actions in periods $t \geq 2$, notice first that it is never optimal for managers $A$ and $B$ to take different actions. We can therefore once again use $m_t$ to denote the largest income level known in period $t$. Similarly, we can once again let $a(m_t)$ denote the action associated with $m_t$. And finally, we once again let $r_t$ denote the right-most known action in period $t$. We then have the following proposition.

PROPOSITION 5. The managers’ optimal actions in period $t \geq 2$ are given by

$$
a^*_t = a^*_t = \begin{cases} 
    r_t + \Delta^* (m(r_t)) & \text{if } m(r_t) > \tilde{m}(m_t, 0) \\
    a(m_t) & \text{if } m(r_t) \leq \tilde{m}(m_t, 0)
\end{cases}
$$

where $\Delta^*(m(r_t)) > 0$ and $\tilde{m}(m_t, \delta)$ are defined in (??) and (9) and where $\tilde{m}(m_t, \delta)$ is strictly increasing in $\delta$ and satisfies $\tilde{m}(m_t, \delta) < m_t$ for any $\delta \geq 0$.

In any period $t$, multiple equilibria therefore arise if the income level associated with the right-most action $r_t$ is strictly between $\tilde{m}(m_t, 0)$ and $\tilde{m}(m_t, \delta)$. We already noted that the critical income level $\tilde{m}(m_t, \delta)$ is strictly increasing in $\delta$. In the appendix we further show that as $\delta \to \infty$, $\tilde{m}(m_t, \delta) \to m_t$. This implies that as coordination becomes very important, coordination failures become pervasive. In particular, the only way to avoid coordination failures for sure is for each period’s income level to be larger than the previous period’s income level.
9 Appendix

Recall that
\[ R(m, \Delta) \equiv -\frac{\mathbb{E}[w''(m + \mu \Delta + \sqrt{\Delta} \sigma z)]}{\mathbb{E}[w'(m + \mu \Delta + \sqrt{\Delta} \sigma z)]} \]
and that \( R(m, 0) \) is equal to the coefficient of absolute risk aversion \( r(m) \). For the proofs below it is convenient to also define
\[ P(m, \Delta) \equiv -\frac{\mathbb{E}[w'''(m + \mu \Delta + \sqrt{\Delta} \sigma z)]}{\mathbb{E}[w''(m + \mu \Delta + \sqrt{\Delta} \sigma z)]} \] (10)
and
\[ T(m, \Delta) \equiv -\frac{\mathbb{E}[w'''(m + \mu \Delta + \sqrt{\Delta} \sigma z)]}{\mathbb{E}[w''(m + \mu \Delta + \sqrt{\Delta} \sigma z)]} . \] (11)

Notice that \( P(m, 0) \) is equal to the coefficient absolute prudence \(-u'''(m)/u''(m)\) (Kimball 1990) and that \( T(m, 0) \) is equal to the coefficient of absolute temperance \(-u'''(m)/u''(m)\) (Gollier and Pratt 1996). We can now prove the following lemma.

**Lemma A1.** For any \( m \in \mathbb{R} \) and \( \Delta \geq 0 \) we have
\[ R(m, \Delta) \leq P(m, \Delta) \leq T(m, \Delta) . \]
Moreover, the first inequality is strict if either \( \Delta > 0 \) or \( \Delta = 0 \) and \( r'(m) < 0 \), where \( r(m) \) is the coefficient of absolute risk aversion.

**Proof of Lemma A1:** (i.) Suppose first that \( \Delta = 0 \). Differentiating the coefficient of absolute risk aversion, we get
\[ r'(m) = R(m, 0) (R(m, 0) - P(m, 0)) . \]
As we mentioned above, Kimball (1993) shows that standard risk aversion implies decreasing absolute risk aversion, that is, \( r'(m) \leq 0 \). It then follows from the above expression that \( R(m, 0) \leq P(m, 0) \). Moreover, this inequality is strict if absolute risk aversion is strictly decreasing. Similarly, differentiating the coefficient of absolute prudence, we get
\[ p'(m) = P(m, 0) (P(m, 0) - T(m, 0)) . \]
Kimball (1993) shows that in a setting such as ours, standard risk aversion is equivalent to decreasing absolute prudence, that is, \( p'(m) \leq 0 \) for all \( m \in \mathbb{R} \). It then follows from the above expression that \( P(m, 0) \leq T(m, 0) \).
(ii.) Suppose now that $\Delta > 0$. Property 5 in Meyer (1987) shows that decreasing absolute risk aversion implies $R(m, \Delta) \leq P(m, \Delta)$ and Result 1 in Eichner and Wagener (2003) shows that decreasing absolute prudence implies $P(m, \Delta) \leq T(m, \Delta)$ (see also page 116 in Gollier 2001). All we need to do therefore is to show that our assumption that $\frac{r(m)}{m}$ crosses $\frac{1}{2} = \frac{1}{2}$, and thus $r(m) < 0$ for some $m$, implies $R(m, \Delta) < P(m, \Delta)$. For this purpose, notice that $R(m, \Delta) < P(m, \Delta)$ is equivalent to

$$E \left[ u'(M + \sqrt{V}z) \right] E \left[ u''(M + \sqrt{V}z) \right] - E \left[ u''(M + \sqrt{V}z) \right] > 0,$$

where $M = m + \mu \Delta$ and $V = \Delta \sigma^2$. We can rewrite this inequality as

$$E \left[ u'(M + \sqrt{V}z) \right] E \left[ u''(M + \sqrt{V}z)z \right] - E \left[ u'(M + \sqrt{V}z) \right] E \left[ u''(M + \sqrt{V}z) \right] > 0,$$

where we use the facts that $E \left[ u'(M + \sqrt{V}z) \right] = \sqrt{V}E \left[ u'(M + \sqrt{V}z) \right]$ and $E \left[ u''(M + \sqrt{V}z) \right] = \sqrt{V}E \left[ u''(M + \sqrt{V}z) \right]$. We can follow the same argument as in the proof of Property 5 in Meyer (1987) to show that this inequality holds. In particular, let $z^*$ satisfy

$$z^* \int_{-\infty}^{\infty} u'(M + \sqrt{V}z) dF(z) = \int_{-\infty}^{\infty} u'(M + \sqrt{V}z) z dF(z),$$

where $F(z)$ is the cumulative density function of the standard normal distribution. We can then rewrite the left-hand side of (13) as

$$\int_{-\infty}^{\infty} u'(M + \sqrt{V}z) dF(z) \int_{-\infty}^{\infty} r(M + \sqrt{V}z) u'(M + \sqrt{V}z) (z^* - z) dF(z),$$

where $r(M + \sqrt{V}z)$ is the coefficient of absolute risk aversion. The first integral is strictly positive. To sign the second integral, notice that

$$\int_{-\infty}^{\infty} u'(M + \sqrt{V}z) (z^* - z) dF(z) = 0,$$

and that the integrand changes sign from positive to negative once. Since $r(M + \sqrt{V}z)$ is everywhere decreasing and strictly decreasing for at least some income levels, the second integral in (14) is strictly positive. The overall expression in (14) is therefore strictly positive, which completes the proof.

**Proof of Lemma 1:** This lemma is proven in "Hilfsatz" 4.3 in Schneeweiss (1966) and Theorem 1 in Chipman (1973).
Proof of Lemma 2: As we mentioned above, Kimball (1993) shows that in a setting such as ours, standard risk aversion is equivalent to decreasing absolute prudence. The lemma then follows from Theorem 3 in Chipman (1973) and Theorem 2 in Lajeri and Nielsen (2000).

Proof of Proposition 1: The first-order condition for the manager’s problem is given by

$$
\frac{dW(m_0 + \mu \Delta_1, \Delta_1 \sigma^2)}{d\Delta_1} = E [u'(\cdot)] \frac{\sigma^2}{2} \left( \frac{2\mu}{\sigma^2} - R(m_0, \Delta_1) \right) \begin{cases} 
0 & \text{if } \Delta_1 > 0 \\
\leq 0 & \text{if } \Delta_1 = 0.
\end{cases}
$$

With this condition in mind, we now prove the optimal actions for and such that (i.) \( m_0 < \hat{m}_l \), where \( \hat{m}_l \) denotes the smallest \( m \) such that \( r(m) = 2\mu/\sigma^2 \), (ii.) \( m_0 > \hat{m} \), where \( \hat{m} \) denotes the largest \( m \) such that \( r(m) = 2\mu/\sigma^2 \), and (iii.) \( m_0 \in [\hat{m}_l, \hat{m}] \). We then conclude the proof by performing the comparative static that is summarized in the proposition.

(i.) Optimal action for \( m_0 < \hat{m}_l \): In this case, \( dW(m_0, 0)/d\Delta_1 < 0 \). Since expected utility \( W(m_0 + \mu \Delta_1, \Delta_1 \sigma^2) \) is concave in \( \Delta_1 \geq 0 \) it then follows that the status quo is the uniquely optimal action.

(ii.) Optimal action for \( m_0 > \hat{m} \): In this case, \( dW(m_0, 0)/d\Delta_1 < 0 \). If an optimal action exists, it is therefore strictly to the right of the status quo.

To prove that an optimal action does exist, it is sufficient to show that there is a \( \Delta_1 > 0 \) such that \( E[u(m_0 + \Delta_1 + \sigma \sqrt{\Delta_1} z)] < u(m_0) \). For this purpose, consider a status quo income level \( \hat{m} \) such that \( r(\hat{m}) > 2\mu/\sigma^2 \). For such an income level the manager strictly prefers the status quo to any action that is strictly to the right of the status quo. Now let \( k(\Delta_1) \) denote the "compensating premium" that would make the manager indifferent between, on the one hand, taking the status quo action and, on the other hand, receiving \( k(\Delta_1) \) and taking an action that is a distance \( \Delta_1 \geq 0 \) to the right of the status quo. Formally, \( k(\Delta_1) \) is given by the \( k \) that solves

$$
E\left[u\left(m + k + \Delta_1 + \sigma \sqrt{\Delta_1} z\right)\right] = u(m) \quad \text{for } \Delta_1 \geq 0,
$$

where the Implicit Function Theorem ensures that \( k(\Delta_1) \) exists. Implicitly differentiating this expression, we get

$$
\frac{dk(\Delta_1)}{d\Delta_1} = \frac{\mu \sigma^2}{2} \left( R(m + k, \Delta_1) - \frac{2\mu}{\sigma^2} \right).
$$

Notice that this derivative is strictly positive for \( \Delta_1 = 0 \). Differentiating again we get

$$
\frac{d^2k(\Delta_1)}{d\Delta_1^2} = R(m + k, \Delta_1) \left( \frac{\sigma^2}{2} \right)^2 \left[ \left( R(m + k, \Delta_1) - P(m + k, \Delta_1)\right)^2 + P(m + k, \Delta_1) (T(m + k, \Delta_1) - P(m + k, \Delta_1)) \right].
$$
Lemma A1 implies that this expression is positive for all $\Delta_1 \geq 0$.

Consider now any $m_0 > \bar{m}$. Since $k(\Delta_1)$ is strictly increasing and convex, there exists a $\Delta_1 > 0$ such that $m_0 = m + k(\Delta_1)$. We then have

$$E\left[u\left(m_0 + \Delta_1 + \sigma \sqrt{\Delta_1}z\right)\right] = u(m) < u(m_0),$$

where the equality follows from the definition of $k(\Delta_1)$ and the inequality from the fact that $k(\Delta_1) > 0$. This implies that an optimal action exists.

Finally, we need to show that the optimal action is unique. For this purpose, we differentiate (15) to obtain

$$\frac{d^2W(\cdot, \cdot)}{d\Delta^2_1} = \mu^2 \left[ E\left[u''(\cdot)\right] + 2 \left( \frac{\sigma^2}{2\mu} \right) E\left[u'''(\cdot)\right] + \left( \frac{\sigma^2}{2\mu} \right)^2 E\left[u''''(\cdot)\right] \right],$$

where we used the facts that $E[u'(\cdot)z] = \sigma \sqrt{\Delta_1} E[u''(\cdot)]$ and $E[u''(\cdot)z] = \sigma \sqrt{\Delta_1} E[u'''(\cdot)]$. We can then use the definitions of $P(m_0, \Delta_1)$ and $T(m_0, \Delta_1)$ in (10) and (11) to rewrite this expression as

$$\frac{d^2W(\cdot, \cdot)}{d\Delta^2_1} = -\mu^2 E[u'(\cdot)] R(m_0, \Delta_1) \left( \frac{\sigma^2}{2\mu} \right)^2 \left[ \left( \frac{2\mu}{\sigma^2} - P(m_0, \Delta_1) \right)^2 + P(m_0, \Delta_1) (T(m_0, \Delta_1) - P(m_0, \Delta_1)) \right].$$

Lemma A1 implies that this expression is negative for any $\Delta_1 \geq 0$. This confirms that expected utility is concave. Moreover, Lemma A1 implies that the above expression is strictly negative for any $\Delta_1 > 0$ that satisfies the first-order condition (15). This, in turn, implies that the optimal action is unique.

(iii.) Optimal action for $m_0 \in [\bar{m}_1, \bar{m}]$: In this case, $dW(m_0, 0)/d\Delta_1 = 0$. The status quo is therefore an optimal action. Moreover, it follows from Lemma A1 and the second derivative in (17) that the status quo is the unique optimum.

(iv.) Comparative statics for any $m_0 \geq \bar{m}$: Above we showed that for any $m_0 \geq \bar{m}$, the uniquely optimal action is given by $a^*_1 = a_0 + \Delta(m_0)$, where $\Delta(m_0)$ is the $\Delta_1 \geq 0$ that solves

$$\frac{dW(m_0 + \mu \Delta_1, \Delta_1 \sigma^2)}{d\Delta_1} = E\left[u'(\cdot)\right] \frac{\sigma^2}{2} \left( \frac{2\mu}{\sigma^2} - R(m_0, \Delta_1) \right) = 0.$$ 

Implicitly differentiating this expression we get

$$\frac{d\Delta(m_0)}{dm_0} = \frac{2\mu}{\sigma^2} \left( \frac{2\mu}{\sigma^2} - P(m_0, \Delta_1) \right)^2 + P(m_0, \Delta_1) (T(m_0, \Delta_1) - P(m_0, \Delta_1)) > 0,$$

where the inequality follows from the first-order condition (18) and Lemma A1. \[\blacksquare\]
There exists a threshold level of income $\tilde{m}(m_0) \in (\hat{m}, m_0)$ such that

$$u(m_0) = W(\tilde{m}(m_0) + \mu \Delta(\tilde{m}(m_0)), \sigma^2 \Delta(\tilde{m}(m_0))),$$

(19)

where $\Delta(\tilde{m}(m_0)) > 0$. Moreover, the derivative of $\tilde{m}(m_0)$ satisfies $0 < \tilde{m}'(m_0) \leq 1$.

**Proof of Lemma 3:** We first show that there exists a unique $\tilde{m}(m_0) \in (\hat{m}, m_0)$ such that

$$u(m_0) = E\left[u\left(\tilde{m}(m_0) + \mu \Delta(\tilde{m}(m_0)) + \sqrt{\Delta(\tilde{m}(m_0))}\sigma z\right)\right].$$

(20)

For this purpose, notice that

$$E\left[u\left(\hat{m} + \mu \Delta(\hat{m}) + \sqrt{\Delta(\hat{m})}\sigma z\right)\right] = u(\hat{m}) < u(m_0)$$

and

$$E\left[u\left(m_0 + \mu \Delta(m_0) + \sqrt{\Delta(m_0)}\sigma z\right)\right] > u(m_0).$$

Expected utility $E\left[u\left(m + \mu \Delta(m) + \sqrt{\Delta(m)}\sigma z\right)\right]$ is therefore strictly less than $u(m_0)$ for $m = \hat{m}$, strictly larger than $u(m_0)$ for $m = m_0$. To show the existence of a unique $\tilde{m}(m_0)$ it is therefore sufficient to show that expected utility $E\left[u\left(m + \mu \Delta(m) + \sqrt{\Delta(m)}\sigma z\right)\right]$ is strictly increasing in $m \in [\hat{m}, m_0]$. Applying the Envelope Theorem we obtain

$$\frac{dE\left[u\left(m + \mu \Delta(m) + \sqrt{\Delta(m)}\sigma z\right)\right]}{dm} = E\left[u'\left(m + \mu \Delta(m) + \sqrt{\Delta(m)}\sigma z\right)\right] > 0$$

which completes the proof of the existence of a unique $\tilde{m}(m_0) \in (\hat{m}, m_0)$.

To prove the comparative statics, we implicitly differentiate (20). Once again applying the Envelope Theorem, we have

$$\frac{d\tilde{m}(m_0)}{dm_0} = \frac{u'(m_0)}{E\left[u'\left(\tilde{m} + \mu \Delta(\tilde{m}) + \sqrt{\Delta(\tilde{m})}\sigma z\right)\right]} > 0,$$

where the inequality follows from non-satiation.

To show that $d\tilde{m}/dm_0 \leq 1$ we need to establish that (20) implies

$$u'(m_0) \leq E\left[u'\left(\tilde{m} + \mu \Delta(\tilde{m}) + \sqrt{\Delta(\tilde{m})}\sigma z\right)\right]$$

or, equivalently,

$$v(m_0) \geq E\left[v\left(\tilde{m} + \mu \Delta(\tilde{m}) + \sqrt{\Delta(\tilde{m})}\sigma z\right)\right],$$

(21)
where we define \( v(m_0) \) as the utility function \( v(m_0) = -u'(m_0) \). Notice that (20) implies (21) if an manager with utility \( v(\cdot) \) is more risk averse than an manager with utility function \( u(\cdot) \). It is therefore sufficient to show that
\[
-\frac{u''(m)}{u''(m)} \geq -\frac{u''(m)}{u'(m)} \text{ for all } m \in \mathbb{R},
\]
where the LHS is the coefficient of absolute risk aversion associated with \( v(\cdot) \) and the RHS is the one associated with \( u(\cdot) \). This inequality is satisfied since the utility function \( u(\cdot) \) satisfies decreasing absolute risk aversion, which completes the proof.

**Proof of Proposition 2:** Follows immediately from the discussion in the text.

**Proof of Proposition 3:** Follows immediately from the discussion in the text.

**Proof of Proposition 4:** Suppose that Manager \( L \) engages in optimal search. This is going to generate some income \( m_{1}^{L*} \) in the first period, \( m_{2}^{L*} \) in the second, and so on. Now take any period \( T \) and let \( \tau \) denote the largest \( t \in [1, T] \) in which Manager \( L \) engaged in search. Note that since \( m_{0}^{L} > \hat{m} \) Manager \( L \) engages in search in the first period and thus \( \tau \in [1, T] \). We can now write the income Manager \( L \) realized each period as
\[
m_{t}^{L*} = \begin{cases} 
m_{t-1}^{L*} + \mu \Delta (m_{t-1}^{L*}) + \sigma \sqrt{\Delta (m_{t-1}^{L*})} Z_t & \text{for } t = 1, \ldots, \tau \\
\mu \Delta (m_{\tau}^{L*}) + \sigma \sqrt{\Delta (m_{\tau}^{L*})} Z_\tau & \text{for } t = \tau + 1, \ldots, T,
\end{cases}
\]
where \( Z_t \) is the realization of a random variable \( z_t \) that is drawn from a standard normal distribution and where we streamline our notation by defining \( m_{0}^{L*} \equiv m_{0}^{L} \).

Consider now Manager \( H \). Since \( m_{0}^{H} > \hat{m} \) this manager also engages in search in the first period. Suppose now that if Manager \( H \) engages in search in a period \( t = 1, \ldots, \tau \) he happens to realize the same \( Z_t \) that Manager \( L \) realized. We will show that it must then be the case that
\[
E_{T+1} [m_{T+1}^{H} - m_{T+1}^{L*}] \geq m_{0}^{H} - m_{0}^{L},
\]
where the inequality is strict for some values of \( Z_1, \ldots, Z_\tau \). Since this result holds for any \( T \) and any \( Z \)’s, it implies (8).

To show (22), we first need to introduce two definitions. For the first definition, consider some period \( t \) in which both managers find it optimal to engage in search. We know from above that each manager’s income is given by
\[
m_{t}^{k*} = m_{t-1}^{k*} + \mu \Delta (m_{t-1}^{k*}) + \sigma \sqrt{\Delta (m_{t-1}^{k*})} Z_t \quad \text{for } k = H, L
\]
We then define
\[
\tilde{z}_t (m^H_{t-1}, m^L_{t-1}) \equiv - \frac{\mu (\Delta (m^H_{t-1}) - \Delta (m^L_{t-1}))}{\sigma \left( \sqrt{\Delta (m^H_{t-1})} - \sqrt{\Delta (m^L_{t-1})} \right)}
\] 
(23)
as the value of $Z_t$ such that $m^H_t - m^L_t = m^H_{t-1} - m^L_{t-1}$. For the second definition, recall that an manager engages in search in period $t + 1$ if and only if
\[
m_t^* > \bar{m}(\bar{m}_t),
\]where $\bar{m}_t = \max \left[ m_0, m_1^*, ..., m_{t-1}^* \right]$. We then define
\[
\tilde{z}_t (m^L_{t-1}, \bar{m}^L_t) \equiv - \frac{m^L_{t-1} + \mu \Delta (m^L_{t-1}) - \bar{m}(\bar{m}^L_{t})}{\sigma \sqrt{\Delta (m^L_{t-1})}}
\]as the value of $Z_t$ such that
\[
m_t^* = m^L_{t-1} + \mu \Delta (m^L_{t-1}) + \sigma \sqrt{\Delta (m^L_{t-1})} Z_t = \bar{m}(\bar{m}^L_{t}).
\]Since Manager $L$ engages in search in periods $2, ..., \tau$ it must be that
\[
Z_t > \tilde{z}_t (m^L_{t-1}, \bar{m}^L_t) \quad \text{for all } t = 1, ..., \tau - 1.
\]In Lemma A2 below we show that if (25) holds then it must be that
\[
\tilde{z}_t (m^L_{t-1}, \bar{m}^L_t) > \max \left[ \tilde{z}_t (m^H_{t-1}, \bar{m}^H_t), \tilde{z}_t (m^H_{t-1}, m^L_{t-1}) \right] \quad \text{for all } t = 1, ..., \tau.
\]To see the implications of this result, suppose first that $\tau = T$, in which case Manager $L$ is searching in all periods up to and including period $T$. Since $Z_t > \tilde{z}_t (m^L_{t-1}, \bar{m}^L_t)$ for all $t = 1, ..., T - 1$, it follows from (26) that
\[
Z_t > \tilde{z}_t (m^H_{t-1}, \bar{m}^H_t) \quad \text{and} \quad Z_t > \tilde{z}_t (m^H_{t-1}, m^L_{t-1}) \quad \text{for all } t = 1, ..., T - 1.
\]Together with the fact that $m^H_0 > \bar{m}$, the first inequality implies that Manager $H$ also engages in search in all periods up to and including period $T$. And the second inequality implies that
\[
m^H_{T-1} - m^L_{T-1} > m^H_{T-2} - m^L_{T-2} > ... > m^H_0 - m^L.
\]Finally, since (26) holds for $t = T$ it must bet that either (i.) $Z_T > \tilde{z}_T (m^L_T, \bar{m}^L_T)$, (ii.) $Z_T < \tilde{z}_T (m^H_T, \bar{m}^H_T)$, or (iii.) $Z_T \in (\tilde{z}_T (m^H_T, \bar{m}^H_T), \tilde{z}_T (m^L_T, \bar{m}^L_T))$. We will show next that (22) holds in any one of those three cases:
Case (i): If $Z_T > \bar{z}_T(m_T^{H*}, \overline{m}_T^L)$ both managers search in period $T + 1$. We then have

\[
E_{T+1} [m_{T+1}^{H*} - m_{T+1}^{L*}] = m_T^{H*} + \mu \Delta (m_T^{H*}) - m_T^{L*} - \mu \Delta (m_T^{L*}) \geq m_T^{H*} - m_T^{L*} > m_0^H - m_0^L,
\]

where the first inequality follows from the fact that the optimal experiment is strictly increasing in income. To derive the second inequality, notice that since $Z_T > \bar{z}_T(m_T^{H*}, m_T^{L*})$ we have $m_T^{H*} - m_T^{L*} > m_{T-1}^{H*} - m_{T-1}^{L*}$. The second inequality then follows from (25).

Case (ii): If $Z_T < \bar{z}_T(m_T^{H*}, \overline{m}_T^H)$ neither manager engages in search in period $T$. We then have

\[
E_{T+1} [m_{T+1}^{H*} - m_{T+1}^{L*}] = \overline{m}_T^H - \overline{m}_T^L > m_0^H - m_0^L.
\]

To see the inequality, let $\tau^k \in \{0, 1, ..., T - 1\}$ denote the period in which the income of Manager $k = H, L$ peaked, that is, in which $m_{\tau^k}^{H*} = \overline{m}_T^{H*}$ (where $\tau^k = 0$ is the case in which income was below status quo income in $t = 1, 2, ..., T - 1$). Suppose first that $\overline{\tau}^H = \tau^L$. Then

\[
\overline{m}_T^H - \overline{m}_T^L = m_{\tau^H}^{H*} - m_{\tau^L}^{L*} = m_0^H - m_0^L \text{ if } \overline{\tau} = 0.
\]

and

\[
\overline{m}_T^H - \overline{m}_T^L = m_{\tau^H}^{H*} - m_{\tau^L}^{L*} > m_0^H - m_0^L \text{ if } \overline{\tau} > 0,
\]

where the inequality follows from (27). Suppose next that $\overline{\tau}^H \neq \tau^L$. Then

\[
\overline{m}_T^H - \overline{m}_T^L > m_{\tau^H}^{H*} - m_{\tau^L}^{L*} > m_0^H - m_0^L,
\]

where the first inequality follows from $m_T^{H*} > m_{\tau^H}^{H*}$ and $\overline{m}_T^L = m_{\tau^L}^{L*}$ and the second inequality follows from (27).

Case (iii): If $Z_T \in (\bar{z}_T(m_T^{H*}, \overline{m}_T^H), \bar{z}_T(m_T^{L*}, \overline{m}_T^L))$ Manager $H$ engages in search in period $T + 1$ but Manager $L$ does not. Since Manager $H$ prefers engaging in search to realizing his previous peak $\overline{m}_T^H$ it must be that $E_{T+1} [m_{T+1}^{H*}] > \overline{m}_T^H$. We therefore have

\[
E_{T+1} [m_{T+1}^{H*} - m_{T+1}^{L*}] = \overline{m}_T^H - \overline{m}_T^L \geq m_0^H - m_0^L,
\]

where the second inequality follows from our discussion in Case (ii) above.

To complete the prove, suppose that $\tau < T$. Since $\tau$ is the last period in which Manager $L$ engaged in search, we have $m_{\tau}^{L*} = \overline{m}_T^L$ for all $t \geq \tau + 1$. Since $Z_t > \bar{z}_t(m_{t-1}^{L*}, \overline{m}_t^L)$ for all $t = 1, ..., \tau - 1$, it follows from (25) that

\[
Z_t > \bar{z}_t(m_{t-1}^{L*}, \overline{m}_t^L) \quad \text{and} \quad Z_t > \bar{z}_t(m_{t-1}^{H*}, m_{t-1}^{L*}) \text{ for all } t = 1, ..., \tau - 1.
\]
Together with the fact that $m_0^H > \bar{m}$, the first inequality implies that Manager $H$ also engages in search in all periods up to and including period $\tau$. And the second inequality implies that
\begin{equation}
m_{t-1}^H - m_t^L > m_{t-2}^H - m_{t-1}^L > \ldots > m_0^H - m_0^L. \tag{29}
\end{equation}
Finally, since (26) holds for $t = \tau$ it must be that either (a.) $Z_\tau < \tilde{z}_\tau (m_\tau^H, \bar{m}_\tau^H)$ (b.) $Z_\tau \in (\tilde{z}_\tau (m_\tau^H, \bar{m}_\tau^H), \bar{z}_\tau (m_\tau^L, \bar{m}_\tau^L))$. We will show next that (22) holds in either of those cases:

Case (a.): If $Z_\tau < \tilde{z}_\tau (m_\tau^H, \bar{m}_\tau^H)$ then Manager $H$ also does not search in periods $t = \tau + 1, \ldots, T$. We then have
\[
E_{T+1} [m_{T+1}^{H*} - m_{T+1}^L] = \bar{m}_T^H - \bar{m}_T^L \geq m_0^H - m_0^L,
\]
where the inequality follows from our discussion in Case (ii.) above.

Case (b.): if $Z_\tau \in (\tilde{z}_\tau (m_\tau^H, \bar{m}_\tau^H), \bar{z}_\tau (m_\tau^L, \bar{m}_\tau^L))$ then Manager $H$ does engage in search in period $\tau + 1$ and, possibly, in period $T + 1$. Since, in period $T + 1$, Manager $H$ can guarantee himself $\bar{m}_T^H$ it must be that $E_{T+1} [m_{T+1}^{H*}] \geq \bar{m}_T^H$. We therefore have
\[
E_{T+1} [m_{T+1}^{H*} - m_{T+1}^L] \geq \bar{m}_T^H - \bar{m}_T^L \geq m_0^H - m_0^L,
\]
where, once again, the second inequality follows from our discussion in Case (ii.) above. \hfill \blacksquare

**LEMMA A2.** If
\[
Z_t > \tilde{z}_t (m_{t-1}^L, \bar{m}_t^L) \text{ for all } t = 1, \ldots, \tau - 1.
\]
then
\[
\tilde{z}_t (m_{t-1}^L, \bar{m}_t^L) > \max [\tilde{z}_t (m_{t-1}^H, \bar{m}_t^H), \bar{z}_t (m_{t-1}^H, m_{t-1}^L)] \text{ for all } t = 1, \ldots, \tau. \tag{31}
\]

**Proof of Lemma A2:** To prove this lemma, we first show that (31) holds for $t = 1$. We then show that if (31) holds for $1, \ldots, x$, where $1 < x \leq \tau - 1$, then it also holds for $t = x + 1$. Together these facts imply that (31) holds for $t = 1, \ldots, \tau$ as claimed in the lemma.

Suppose first then that $t = 1$. From the definitions of $\tilde{z}_t (\cdot)$ and $\bar{z}_t (\cdot)$ in (23) and (24) we have
\[
\tilde{z}_1 (m_0^L, \bar{m}_0^L) - \tilde{z}_1 (m_0^H, \bar{m}_0^L) = \frac{\bar{m} (m_0^L) + \mu \sqrt{\Delta (m_0^L)} \sqrt{\Delta (m_0^H) - m_0^L}}{\sigma \sqrt{\Delta (m_0^L)}}. \tag{32}
\]
and
\[
\tilde{z}_1 (m_0^L, \bar{m}_0^L) - \tilde{z}_1 (m_0^H, \bar{m}_0^H) = \frac{m_0^H - m_0^L + \bar{m} (m_0^L) - \bar{m} (m_0^H)}{\sigma \sqrt{\Delta (m_0^H)}} \tag{33}
\]
\[
+ \frac{\sqrt{\Delta (m_0^H)} - \sqrt{\Delta (m_0^L)}}{\sigma \sqrt{\Delta (m_0^L)} \sqrt{\Delta (m_0^H)}} \left( \bar{m} (m_0^L) + \mu \sqrt{\Delta (m_0^L)} \sqrt{\Delta (m_0^H) - m_0^L} \right).
\]

31
To see that (32) is strictly positive notice that
\[
\tilde{m} (m_0^L) + \mu \sqrt{\Delta (m_0^L)} \sqrt{\Delta (m_0^H) - m_0^L} > 0,
\]
where the first inequality follows from \( \Delta (m_0^H) > \Delta (m_0^L) \) and the second follows from \( m_0^L > \tilde{m} (m_0^L) \). To see the third inequality, recall that \( \tilde{m} (m_0^L) \) is defined as the income level at which the manager is indifferent between receiving \( m_0^L \) and a normally distributed gamble that pays \( \tilde{m} (m_0^L) + \mu \Delta (\tilde{m} (m_0^L)) \) on average and has a strictly positive variance \( \Delta (\tilde{m} (m_0^L)) \sigma^2 \). Since the manager is risk averse, it must then be that \( \tilde{m} (m_0^L) + \mu \Delta (\tilde{m} (m_0^L)) > m_0^L \).

To show that (33) is also strictly positive, consider first the second term on the RHS of (33). Since \( \Delta (m_0^H) > \Delta (m_0^L) \) it follows from (34) that this term is strictly positive. Consider next the first term on the RHS of (33). This term has to be weakly positive since \( m_0^H > m_0^L \) and \( d\tilde{m} (m)/dm \in (0, 1) \). We therefore have \( \tilde{z}_1 (m_0^L, m_0^L) > \max [\tilde{z}_1 (m_0^H, m_0^H), \tilde{z}_t (m_0^H, m_0^L)] \).

Suppose now that (31) holds for \( t = 1, \ldots, x \), where \( 1 < x \leq \tau - 1 \). We will show that (31) then also holds for \( t = x + 1 \). For this purpose, notice first that if (31) holds for \( t = 1, \ldots, x \), then it follows from (30) that (i.) both managers are engaging in search in periods \( t = 1, \ldots, x + 1 \) and (ii.) it must be that
\[
m_{x+1}^H - m_{x+1}^L > m_x^H - m_x^L > \ldots > m_0^H - m_0^L. \tag{35}
\]
Furthermore, we know from the definitions of \( \tilde{z}_t (\cdot) \) and \( \tilde{z}_t (\cdot) \) in (23) and (24) that
\[
\tilde{z}_{x+1} (m_x^L, m_x^L) - \tilde{z}_{x+1} (m_x^H, m_x^L) = \frac{\tilde{m} (m_{x+1}^L) + \mu \sqrt{\Delta (m_{x+1}^L)} \sqrt{\Delta (m_x^H) - m_x^L}}{\sigma \sqrt{\Delta (m_x^L)}}.
\]
To see that this expression is strictly positive, notice that
\[
\tilde{m} (m_{x+1}^L) + \mu \sqrt{\Delta (m_{x+1}^L)} \sqrt{\Delta (m_x^H) - m_x^L} > 0,
\]
where the first inequality follows from \( m_{x+1}^L \geq m_x^L \), the second from \( \Delta (m_x^H) > \Delta (m_x^L) \), and the third from \( m_x^L > \tilde{m} (m_x^L) \). To derive the last inequality, recall that \( \tilde{m} (m_x^L) \) is defined as the income
level at which the manager is indifferent between receiving $m_x^L$ and a normally distributed gamble that pays $\bar{m} (m_x^L) + \mu \Delta (\bar{m} (m_x^L))$ on average and has a strictly positive variance $\Delta (\bar{m} (m_x^L)) \sigma^2$.

Since the manager is risk averse, it must then be that $\bar{m} (m_x^L) + \mu \Delta (\bar{m} (m_x^L)) - m_x^H$. We therefore have that if (31) holds for $t = 1, \ldots, x$, then $\bar{z}_{x+1} (m_x^L, m_x^L) > \bar{z}_{x+1} (m_x^H, m_x^L)$.

Next, we know from the definition of $\bar{z}_k$ in (24) that

$$
\bar{z}_{x+1} (m_x^L, m_x^L) - \bar{z}_{x+1} (m_x^H, m_x^L) = \frac{m_x^H - m_x^L + \bar{m} (m_x^L) - \bar{m} (m_x^H)}{\sigma \sqrt{\Delta (m_x^L)}} \\
+ \frac{\sqrt{\Delta (m_x^H)} - \sqrt{\Delta (m_x^L)}}{\sigma \sqrt{\Delta (m_x^L)} \sqrt{\Delta (m_x^H)}} \left( \bar{m} (m_x^L) + \mu \sqrt{\Delta (m_x^L)} \sqrt{\Delta (m_x^H)} - m_x^L \right).
$$

(37)

We know from (36) that the first term on the RHS is strictly positive. To show that the second term is weakly positive, we first show that

$$m_x^H - m_x^L \geq m_x^H - m_x^L = \bar{m} (m_x^L) - \bar{m} (m_x^L),$$

where the inequality follows from (35) and the equality follows from the definition of $\bar{m}^H$ and $\bar{m}^L$. Suppose next that $\bar{m}^H \neq \bar{m}^L$. Then

$$m_x^H - m_x^L \geq m_x^H - m_x^L > m_x^L - m_x^L = \bar{m} (m_x^L) - \bar{m} (m_x^L),$$

where the first inequality follows from (35), the second follows from $m_x^L > m_x^L$, and the equality follows from the definition of $\bar{m}^H$ and $\bar{m}^L$. We therefore have $m_x^H - m_x^L \geq \bar{m} (m_x^L) - \bar{m} (m_x^L)$ as claimed above.

Finally, notice that

$$\bar{m} (m_x^L) - \bar{m} (m_x^L) \geq \bar{m} (m_x^L) - \bar{m} (m_x^L),$$

where the inequality follows from $\bar{m}^H > \bar{m}^L$ and $d\bar{m} (m) / dm \in (0, 1]$. We therefore have

$$m_x^H - m_x^L \geq \bar{m} (m_x^L) - \bar{m} (m_x^L) \geq \bar{m} (m_x^L) - \bar{m} (m_x^L)$$

which implies that the second term on the RHS of (37) is weakly positive. □

**Proof of Proposition 5:** We first show that there exists a unique $\tilde{d} > 0$ such that

$$u (m_0) = E \left[ u \left( m_0 + D + \mu \tilde{d} + \sigma \sqrt{\tilde{d}} \right) \right],$$

(38)

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in which case the manager is indifferent between the status quo and imitation. Suppose first that
\(m_0 + D \leq \hat{m}\). Then it follows from (6) that \(dW(m_0 + D, 0)/dd \leq 0\). Since \(E\left[u\left(m_0 + D + \mu d + \sigma \sqrt{d}z\right)\right]\)
is concave in \(d\) (which follows immediately from Lemma 2) this proves existence of \(\hat{d}\) if \(m_0 + D \leq \hat{m}\).

Suppose now that \(m_0 + D > \hat{m}\). Then it follows from (6) that \(dW(m_0 + D, 0)/dd > 0\). Moreover, we know from Proposition 1 that there exists a finite \(d\) that maximizes \(W(m_0 + D + \mu d, \sigma^2 d)\). Since expected utility is concave this proves existence of \(\hat{d}\) if \(m_0 + D > \hat{m}\). Finally, notice that whether \(m_0 + D \leq \hat{m}\) or \(m_0 + D > \hat{m}\), expected utility is decreasing in \(d\) at \(\hat{d}\), that is,

\[
\mu E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right] + \frac{\sigma^2}{2} E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right] < 0. \tag{39}
\]

This fact will help sign the comparative statics to which we turn next. Specifically, by implicitly differentiating (38) we obtain

\[
\frac{d\hat{d}}{dD} = -\frac{E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right]}{\mu E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right] + \frac{\sigma^2}{2} E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right]} > 0
\]

and

\[
\frac{d\hat{d}}{d\sigma} = -\frac{E\left[\sqrt{d} u''(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z)\right]}{\mu E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right] + \frac{\sigma^2}{2} E\left[u'\left(m_0 + D + \mu \hat{d} + \sigma \sqrt{\hat{d}}z\right)\right]} < 0,
\]

where the signs follow immediately from non-satiation, risk aversion, and (39).

References


