Markets with Multidimensional Private Information*

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Abstract

This paper explores price formation in environments with multidimensional private information. Asset sellers are informed both about their need to raise cash and about the quality of the asset they are selling. Asset buyers have rational expectations about the distribution of assets for sale at different prices. We find that there are many equilibria of this model and that standard signaling game refinements do not reduce the multiplicity problem. Under an additional behavioral restriction, we pin down a unique equilibrium. This equilibrium has partial pooling: identical assets sell for different prices, depending on the seller's discount factor; while conversely different assets sell for the same price. Sellers who set a higher price are less likely to succeed at selling. The equilibrium allocation depends on the joint distribution of seller and asset characteristics, and not just the support of that distribution.

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1 Introduction

This paper develops a model of trade in an asset market with adverse selection. The economy is populated by a measure 1 of risk-neutral investors who live for two periods. At the start of period 1, each investor is endowed with one unit of a perishable consumption good and one asset that produces a unit of the consumption good in period 2. Different assets produce different amounts of the consumption good δ. Investors are heterogenous in their discount factor β between periods 1 and 2. At the beginning of the period each investor observes the quality of his asset (the amount of consumption good it will produce) and his discount factor. Next, there is trade of the period 1 consumption good for assets. Investors may use their consumption good to buy assets, sell their asset for the consumption good, engage in both activities, or simply live in autarky. We allow investors to buy or sell at any price, forming beliefs about the probability that they will be able to trade at that price and about the composition of assets offered for sale at that price. Trade is rationed by the short side of the market at any price, with all traders on the long side of the market equally likely to be trade.

Our goal is to characterize the set of equilibria of this model. Towards that end, we define a key model primitive, the expected quality of an asset δ conditional on the owner's continuation value v, which in turn is the product of the owner's discount factor β and the asset quality δ. We assume throughout our analysis that this function is continuous and strictly increasing in v. Under this parameter restriction, we prove that an equilibrium always exists, but we find that the set of equilibria is large and difficult to characterize. Moreover, standard equilibrium refinements in signaling games (e.g. Cho and Kreps, 1987; Banks and Sobel, 1987) are not useful for reducing the set of equilibria.

We prove instead that there is a unique equilibrium in which (i) all investors with the same continuation value use the same strategy when setting sale prices, so on-the-equilibrium-path behavior is the same; and (ii) buyers believe that any all investors with the same continuation value are equally likely to select any sale price not chosen in equilibrium, so off-the-equilibrium-path behavior is the same as well.

In the unique equilibrium that survives this refinement, investors with the lowest continuation value sell their asset with probability 1. If the expected quality of an asset conditional on the investor having the lowest continuation value is positive, such investors set a positive price. Moreover, investors with a somewhat higher continuation value set a higher price and sell with a lower probability, while investors with a much higher continuation value never sell their assets. In such an equilibrium all investors with the same continuation value sell their assets at the same price. An investor is therefore uncertain about exactly what quality asset he will purchase at a given price. Finally, if the expected quality of an asset conditional on the investor having the lowest continuation value is zero, no asset is sold for a positive price in equilibrium.

A shift in the joint distribution of asset quality and preferences can cause a collapse in trade in this environment, even if neither marginal distribution changes. In particular, if the expected asset quality of an investor with the lowest continuation value is zero, then all trade must collapse.

The equilibrium of this model with multidimensional private information differ from our previous work in which investors' discount factors are observable (Guerrieri and Shimer, forthcoming). In that model, we found that there is a unique fully separating equilibrium and that assets of higher quality trade at higher price in less liquid markets. The predictions of the two models differ along at least four dimensions. First, with multidimensional private information there is price dispersion for assets of the same quality and heterogeneous assets selling for the same price. In our prior work, there was a one-to-one mapping from asset quality to price. Second, the equilibrium payoffs in this paper are affected by the joint distribution of discount factors and asset quality, while in our prior work, equilibrium payoffs only depended on the support of the distribution and the relative supply of assets. Third, with multidimensional private information some investors both buy and sell assets. In contrast, with private information only about asset quality, investors only participate on one side of the market. Finally, we find that typically a continuum of equilibria exist in this environment and therefore introduce a refinement to reduce the set of equilibria.

Our notion of equilibrium builds on Guerrieri, Shimer and Wright (2010). To our knowledge, Chang (2012) is the only previous paper that has explored multidimensional private information in that sort of environment. There are several important differences between the results in the two papers. First, Chang looks at an environment in which the role of an investor as a buyer or seller is determined exogenously. We allow investors to choose whether to buy assets, sell assets, do both, or do neither, an important possibility in more realistic environments. Second, Chang assumes that sellers are heterogeneous while buyers are homogeneous. Moreover, all buyers value any asset more than any seller does, so all trades are socially beneficial. This ensures that in equilibrium, all assets are sold with a positive probability. In our model, investors are heterogeneous, the decision to buy and sell is endogenous, and in equilibrium some assets are transferred from investors who value them more to investors who value them less. As a result, we find that some investors choose not to attempt to sell their assets in equilibrium.[1] Third, under an analogous parameter restriction

[1]Formally we model this as investors setting a high price at which they know they will be unable to sell their assets.

to ours, Chang only characterizes one equilibrium, while we prove that in our economy there can generically be a continuum of equilibria. Fourth, Chang (2012) characterizes equilibria when the parameter restriction fails. The current version of our paper does not.

Numerous previous papers (e.g. Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010; Chiu and Koeppl, 2011; Tirole, 2012) have developed models of adverse selection in which all trade occurs at a single price. Those papers do not allow investors to consider trading at a different price. In contrast, this paper builds on our previous work, (Guerrieri, Shimer and Wright, 2010; Guerrieri and Shimer, forthcoming), allowing trade at any price but recognizing that sellers who demand a high price may be rationed. This implies that in any equilibrium with trade, trade occurs at a range of prices.

The paper proceeds as follows. Section 2 lays out the basic model. We analyze the equilibrium when there is symmetric information in Section 3. We define our notion of equilibrium in our main model with multidimensional private information in Section 4. In Section 5 we establish by construction that our model exhibits a huge number of equilibria. In Section 6 we propose a refinement of equilibrium based on the notion that investors with identical preferences behave identically. We establish uniqueness of equilibrium under this additional restriction. The current version of the paper simply sketches the proof.

2 Model

The economy lasts for two periods, $t = 1, 2$. It is populated by a unit measure of risk-neutral investors with heterogeneous discount factors $\beta \in [\underline{\beta}, \bar{\beta}] \subseteq \mathbb{R}_+$. Each investor is endowed in period 1 with one unit of a consumption good and one asset that gives a dividend in period 2. Assets are heterogenous in the amount $\delta \in [\underline{\delta}, \bar{\delta}] \subseteq \mathbb{R}_+$ of the period 2 consumption good they produce as a dividend. Both consumption goods and assets are divisible. Consumption goods are perishable and must be consumed within the period and consumption must be nonnegative in each period.

At the beginning of period 1, each investor observes his type, that is, his discount factor β, and the quality of his asset δ. Next, there is a market in which period 1 consumption goods and assets are exchanged. Each investor makes independent buying and selling decisions and so may engage in trade on both sides of the market, one side, or none. We assume that investors can only buy assets using the consumption goods that he holds at the start of the period and so must consume any consumption goods he gets from selling his asset.[2] After the market meets, investors consume any remaining period 1 consumption good, c_1. In period 2,

[2]Other assumptions are possible here. While they would change some of our calculations, we not believe that changing this constraint would alter our main results.

each investor consumes the dividend generated by the assets he holds in that period, c_2. An investor with discount factor β seeks to maximize $\mathbb{E}(c_1 + \beta c_2)$, where expectations recognize that the investor may be uncertain about the whether he will succeed in buying and selling assets and about the quality of the assets that he buys.

Let $G : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \mapsto [0, 1]$ denote the initial joint distribution of discount factors and endowed asset quality, so $G(\beta, \delta)$ is the measure of investors who have a discount factor no more than β and are endowed with an asset with dividend no more than δ. We assume G is atomless and let g denote the associated density with $g(\beta, \delta) = 0$ if $(\beta, \delta) \neq [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}]$.

Formally we assume that for any (β, δ) with $g(\beta, \delta) > 0$ there are many investors with this discount factor and asset quality. Informally we identify investors by the pair (β, δ). It is convenient to define an investor (β, δ)'s continuation value as $v = \beta\delta$ and to denote the lowest and highest continuation values as $\underline{v} = \underline{\beta}\underline{\delta}$ and $\bar{v} = \bar{\beta}\bar{\delta}$. The assumption that the support of G is rectangular is for notational convenience only and can easily be relaxed.

We analyze two different versions of this model. First, we study a benchmark model with symmetric information. Second, we study a model where investors are privately informed about both their asset quality and their discount factor, but they are allowed to choose different prices.

3 Symmetric Information

Let us start by introducing the benchmark economy where information is complete. All investors observe the future dividend of all assets and the discount factor of all investors. In this environment, we look for the competitive equilibrium.

In a competitive equilibrium, different assets sell for different prices. An equilibrium is then described by a price schedule $p : [\underline{\delta}, \bar{\delta}] \mapsto R_+$, where $p(\delta)$ denotes the price of an asset with dividend δ. Each investor (β, δ) takes this price schedule as given and decides whether to sell their asset, whether to use their consumption good to buy assets, and what type of asset to buy. Let $\mathbb{I}_s(\beta, \delta)$ be an indicator function which takes value 1 if (β, δ) sells his asset. Let $d_b(\beta, \delta)$ denote the dividend of the asset that (β, δ) buys using his consumption good, with $d_b(\beta, \delta) = \varnothing$ indicating that the investor consumes. Then a competitive equilibrium satisfies the following three conditions:

Definition 1 *An equilibrium is given by functions* $p : [\underline{\delta}, \bar{\delta}] \mapsto \mathbb{R}_+$, $\mathbb{I}_s : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \times \mathbb{R}_+ \mapsto \{0, 1\}$, *and* $d_b : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \times \mathbb{R}_+ \mapsto [\underline{\delta}, \bar{\delta}] \cup \varnothing$ *where the functions satisfy the following conditions:*

1. *Optimal Selling Decision: given* $p(\delta)$*, for all* (β, δ)

$$\mathbb{I}_s(\beta, \delta) = \begin{cases} 1 \\ & p(\delta) \gtrless \beta\delta; \\ 0 \end{cases}$$

2. *Optimal Buying Decision: given* $p(\delta)$*, for all* (β, δ')*,*

$$d_b(\beta, \delta') \in \arg\max_{\delta} \frac{\delta}{p(\delta)}$$

if $\max_\delta \beta\delta/p(\delta) > 1$ *and* $d_b(\beta, \delta') = \varnothing$ *otherwise;*

3. *Market Clearing: for each* δ*,* $p(\delta)\, d\mu_s(\delta) = d\mu_b(\delta)$*, where*

$$\mu_s(\delta) \equiv \int_{\underline{\delta}}^{\delta} \int_{\underline{\beta}}^{\bar{\beta}} \mathbb{I}_s(\beta, \delta') g(\beta, \delta')\, d\beta\, d\delta'$$

$$\mu_b(\delta) \equiv \int_{d_b(\beta,\delta') \le \delta} \int_{\underline{\beta}}^{\bar{\beta}} g(\beta, \delta') d\beta\, d\delta'$$

are the measure of asset worse than δ *for sale and the measure of consumption goods used to buy assets worse than* δ*, respectively.*

The first condition requires that (β, δ) sells his asset if the price exceeds the value he places on holding onto the asset, $\beta\delta$. The second condition requires that (β, δ') buys an asset with dividend δ if this is the asset with the highest dividend-price ratio and if his discount factor exceeds the price-dividend ratio. This follows immediately from the tradeoff between consuming or using his consumption good to buy $1/p(\delta)$ assets with dividend δ. The final equilibrium condition implies that the amount of consumption goods used to purchase assets with dividend δ is equal to the product of the number of such assets offered for sale and the sale price, so the market for each type of asset clears.

It is straightforward to prove that the price function must satisfy $p(\delta) = \hat{\beta}\delta$ for some $\hat{\beta} \in [0, 1]$. All investors with discount factor $\beta > \hat{\beta}$ buy any type of asset using their consumption good (since all assets have the same price-dividend ratio) and do not sell their asset. All investors with discount factor $\beta < \hat{\beta}$ sell their asset.

The market clearing condition determines the threshold $\hat{\beta}$. With the characterization in the previous paragraph, this reduces to the following single market clearing condition:

$$\int_{\underline{\delta}}^{\bar{\delta}} \int_{\hat{\beta}}^{\bar{\beta}} g(\beta, \delta)\, d\beta\, d\delta = \hat{\beta} \int_{\underline{\delta}}^{\bar{\delta}} \int_{\underline{\beta}}^{\hat{\beta}} \delta g(\beta, \delta)\, d\beta\, d\delta. \tag{1}$$

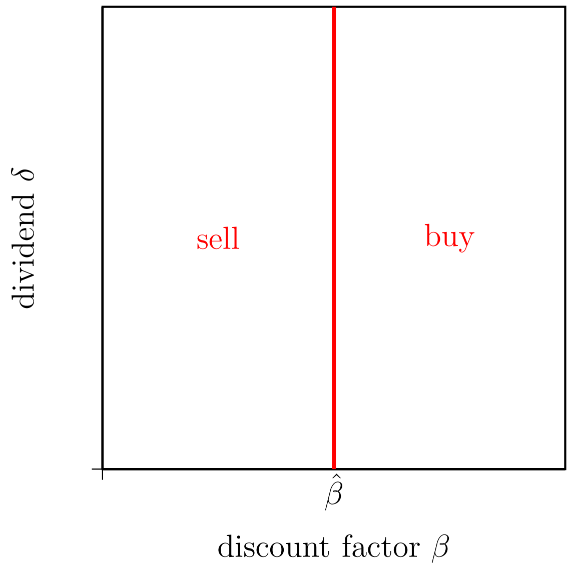

Figure 1: Competitive equilibrium with symmetric information

The left hand side is the consumption good held by patient investors while the right hand side is the cost of purchasing the assets held by the impatient investors.

We summarize this analysis in the next proposition:

Proposition 1 *In a competitive equilibrium, $p(\delta) = \hat{\beta}\delta$ for all δ, where $\hat{\beta}$ solves 1. Moreover, any investor (β, δ) sells his asset if and only if $\beta < \hat{\beta}$ and uses consumption good to buy assets otherwise.*

Figure 1 represents the competitive equilibrium under symmetric information in the space (β, δ). The equilibrium can be characterized by the cutoff $\hat{\beta}$ so that all the investors with $\beta > \hat{\beta}$ always buy and $\beta < \hat{\beta}$ always sell irrespective of the type of asset that they have.

4 Multidimensional Private Information

We now introduce multidimensional private information. The environment is the same as before, except that we assume that both the asset's quality and the degree of impatience of an investor are his own private information. Our equilibrium notion builds on Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (forthcoming).

At the beginning of the period, each investor (β, δ) knows his discount factor β and the quality of his asset δ. A continuum of markets characterized by a price $p \in R_+$ may open up. Each investor makes an independent buying and selling decision. On the buying side, he has to decide whether to consume his unit of the consumption good or to use it to buy assets and, if he buys assets, he has to decide at which price. On the selling side, he has to choose whether to sell his asset or not and, if he sells, he has to decide at which price. We assume that each asset and each unit of consumption good can be brought to only one market, so an effort to sell an asset at a price p is also a commitment not to sell the asset at any other price.

In making their optimal trading decisions, investors must form beliefs about the trading probability and the type of asset for sale at any potential price, even those not offered in equilibrium. Let $\Theta(p)$ denote the market tightness associated to price p, that is, the ratio of the amount of the consumption good that buyers want to use to buy at price p, relative to the cost of the assets that sellers want to sell at price p. If $\Theta(p) < 1$, there is not enough goods to buy all the assets for sale at price p and the sellers are randomly rationed. If instead $\Theta(p) > 1$, there is more goods than needed to buy all the assets for sale at price p and the buyers are randomly rationed. Specifically, a seller who chooses to trade at price p expects to sell with probability $\min\{\Theta(p), 1\}$. Similarly, a buyer who decides to trade at price p expects to buy with probability $\min\{\Theta(p)^{-1}, 1\}$. A seller who is rationed keeps his asset and in period 2 eats the dividend produced by it. A buyer who is rationed eats his consumption goods in period 1.

In addition, let $\Delta(p)$ denote buyers' belief about the average dividend among the assets offered for sale at a price p. If some assets are sold at a price p, these beliefs must be consistent with the quality of assets offered for sale. Otherwise, as long as the buyer-seller ratio $\Theta(p)$ is finite, buyers' belief $\Delta(p)$ must be reasonable in the sense that there must be some set of sellers with average asset quality $\Delta(p)$ who find it weakly optimal to set the price p. This assumption restricts the set of possible equilibria by ruling out equilibria that are sustained by weird beliefs about markets that are inactive.

4.1 Equilibrium Definition

We are now ready to define an equilibrium. We let $p_s(\beta, \delta)$ denote the optimal sale price for investor (β, δ) and $p_b(\beta, \delta)$ denote his optimal buy price. We do not offer investors an explicit option not to sell their asset or not to buy an asset, but instead note that in equilibrium, sellers (buyers) can assure that outcome by setting a sufficiently high (low) price. We also impose without loss of generality that investors use pure strategies.

Definition 2 *An equilibrium is four functions* $p_s : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, $p_b : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$, $\Delta : \mathbb{R}_+ \mapsto [\underline{\delta}, \bar{\delta}]$ *satisfying the following conditions:*

1. *Optimal Selling Decision: given* Θ*, for all* (β, δ)

$$p_s(\beta, \delta) \in \arg\max_{p \geq \beta\delta} \min\{\Theta(p), 1\}(p - \beta\delta);$$

2. *Optimal Buying Decision: given* Θ *and* Δ*, for all* (β, δ)

$$p_b(\beta, \delta) \in \arg\max_{p \geq 0} \min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Delta(p)}{p} - 1\right);$$

3. *Beliefs: For all* $p \in \mathbb{R}_+$ *with* $\Theta(p) < \infty$*,*

   (a) *if there exists a* (β, δ) *with* $p_s(\beta, \delta) = p$*,* $\Delta(p) = \mathbb{E}(\delta | p_s(\beta', \delta') = p)$*;*

   (b) *otherwise there exists a* (β_1, δ_1) *with* $\delta_1 \leq \Delta(p)$*,* $p \geq \beta_1\delta_1$*, and*

$$\min\{\Theta(p_s(\beta_1, \delta_1)), 1\}\big(p_s(\beta_1, \delta_1) - \beta_1\delta_1\big) = \min\{\Theta(p), 1\}\big(p - \beta_1\delta_1\big);$$

   and similarly a (β_2, δ_2) *with* $\delta_2 \geq \Delta(p)$*,* $p \geq \beta_2\delta_2$*, and*

$$\min\{\Theta(p_s(\beta_2, \delta_2)), 1\}\big(p_s(\beta_2, \delta_2) - \beta_2\delta_2\big) = \min\{\Theta(p), 1\}\big(p - \beta_2\delta_2\big);$$

4. *Market Clearing: for all* $p \geq 0$*,* $d\mu_b(p) = p\,\Theta(p)\,d\mu_s(p)$*, where*

$$\mu_s(p) \equiv \iint_{p_s(\beta,\delta) \leq p} g(\beta, \delta)\,d\delta\,d\beta \text{ and } \mu_b(p) \equiv \iint_{p_b(\beta,\delta) \leq p} g(\beta, \delta)\,d\delta\,d\beta$$

   are the measure of assets for sale at prices below p *and the measure of goods used to purchase assets at prices below* p*. Moreover, if there exists a* (β, δ) *with* $p_s(\beta, \delta) = p$ *and* $\Theta(p) > 0$*, then there exists a* (β', δ') *with* $p_b(\beta', \delta') = p$*; and if there exists a* (β, δ) *with* $p_b(\beta, \delta) = p$ *and* $\Theta(p) < \infty$*, then there exists a* (β', δ') *with* $p_s(\beta', \delta') = p$*.*

The first condition requires that investors make optimal selling decisions. Each seller (β, δ) must set an optimal price for her asset.[3] A seller who sets a price p only succeeds in selling with probability $\Theta(p)$ if $\Theta(p) < 1$. In this event, he gets p units of the consumption good in period 1 but gives up δ units of the consumption good in period 2, which he values at $\beta\delta$. If he fails to sell, he gains nothing. We impose for expositional convenience the restriction that sellers never set a price below their continuation value $\beta\delta$.[4]

[3] There is no loss of generality in assuming that he attempts to sell the asset. Attempting to sell at any price $p \geq \beta\delta$ always weakly dominates not selling the asset.

[4] It is never strictly optimal for a seller (β, δ) to set a price $p < \beta\delta$, and is only weakly optimal if $\Theta(p) = 0$ and $\Theta(p') = 0$ for all $p' \geq \beta\delta$.

The second condition requires that investors make optimal buying decisions. Each buyer (β, δ) sets an optimal price for buying assets.[5] A buyer who sets a price p only succeeds in buying with probability $\min\{\Theta(p)^{-1}, 1\}$. In this event, he gives up a unit of the consumption good and gets $1/p$ assets, each of which he anticipates will produce dividend $\Delta(p)$ next period. If he fails to buy, he gains nothing.

The first part of the third condition imposes that beliefs are consistent with the observed trading patterns whenever possible. If at least one seller sets a price p, then the expected dividend must be the average among the sellers who set that price. The second part of this condition describes beliefs at prices that nobody sets. Intuitively, we require that buyers must be able to rationalize the expected dividend as coming from some probability distribution over sellers, each of whom has a continuation value $\beta\delta$ less than the price and finds setting this price to be weakly optimal. This means that there must either be some investor with dividend $\Delta(p)$ who finds it optimal to set the price p, or that there must be both an investor with a higher-quality asset and an investor with a lower-quality asset, both of whom find this price optimal. In the latter case, appropriate weights on those two investors justify the expectation $\Delta(p)$.[6]

Finally, the last condition imposes market clearing. It requires that the buyer-seller ratio $\Theta(p)$ at any price p is equal to the ratio of the measure of buyers purchasing at price p to the product of the price and the measure of sellers selling at that price. The last piece of this condition ensures that this holds even if both measures are zero yet a finite number of buyers or sellers sets price p. For notational convenience alone, we do not impose that the buyer-seller ratio is exactly equal to $\Theta(p)$ in this case.

4.2 Parameter Restriction

Let $\Gamma(v) \equiv \mathbb{E}(\delta|\beta\delta = v)$ denote the expected dividend conditional on an investor's continuation value $\beta\delta = v$. It is straightforward to prove that

$$\Gamma(v) \equiv \frac{\int_{\underline{\delta}}^{\bar{\delta}} g\left(\frac{v}{\delta}, \delta\right) d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \frac{1}{\delta} g\left(\frac{v}{\delta}, \delta\right) d\delta},$$

a function of the joint density g, a model primitive. We focus our analysis on the case where the following restriction holds:

[5]We prove below that in any equilibrium with trade, $\Theta(p) = \infty$ at sufficiently low prices p. Therefore buyers can always be sure to consume in period 1 by setting a low price.

[6]In our previous research (Guerrieri, Shimer and Wright, 2010; Guerrieri and Shimer, forthcoming), the analogous condition defined a probability distribution over seller types at each price p. None of the results in this paper would change if we used that definition, but the one we use here is simpler to apply.

Assumption 1 *Assume Γ is continuous and increasing.*

A distribution function that satisfies our assumption is $G(\beta, \delta) = \beta^\alpha \delta^{\alpha+k}$ with $\underline{\beta} = \underline{\delta} = 0$ and $\bar{\beta} = \bar{\delta} = 1$, $\alpha > 0$, and $\alpha + k > 0$. Then

$$\Gamma(v) = \frac{k\big(1 - v^{k+1}\big)}{(1+k)\big(1 - v^k\big)},$$

which is continuous and increasing. We will use this example to illustrate some of our results.

5 Examples of Equilibria

This section constructs some equilibria to illustrate the types of outcomes that are feasible in this environment. We do not attempt a comprehensive characterization of the full set of equilibria.

5.1 Partially-pooling Equilibria

We start by looking for an equilibrium in which every investor sets a sale price that is a strictly increasing function of her continuation value, $p_s(\beta, \delta) = P(\beta\delta)$. Higher prices are associated with a lower buyer-seller ratio, ensuring that every investor is willing to set the appropriate price. On the other hand, higher expected quality compensates buyers for higher prices, so investors are willing to purchase assets at a range of different prices. We call these equilibria partially pooling because any two investors with the same continuation value set the same price, but investors with different continuation values set distinct prices.

One partially-pooling equilibrium is characterized by a discount factor for the marginal buyer, $\hat{\beta} \in (\underline{\beta}, \bar{\beta}]$; we determine $\hat{\beta}$ in equation (2) below. Given this, observe that $\hat{\beta}\Gamma(\underline{v}) > \underline{\beta}\underline{\delta} = \underline{v}$ and let $\bar{p} > \underline{v}$ be the smallest solution to $\bar{p} = \hat{\beta}\Gamma(\bar{p})$, so $\hat{\beta}\Gamma(v) > v$ for all $v \in [\underline{v}, \bar{p})$. Then we can construct an equilibrium as follows:

- Investors with a continuation value $\beta\delta \in [\underline{v}, \bar{p}]$ set sell price

  $$p_s(\beta, \delta) = P(\beta\delta) \equiv \hat{\beta}\Gamma(\beta\delta) \geq \beta\delta,$$

  while those with $\beta\delta > \bar{p}$ set any price $p \geq \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$.

- The buyer-seller ratio satisfies

  $$\Theta(P(v)) = \exp\left(-\int_{\underline{v}}^{v} \frac{\hat{\beta}\Gamma'(v')}{\hat{\beta}\Gamma(v') - v'} dv'\right)$$

for $v \in [\underline{v}, \bar{p}]$, with $\Theta(p) = \infty$ if $p < \hat{\beta}\Gamma(\underline{v})$ and $\Theta(p) = 0$ if $p > \bar{p}$. Given this buyer-seller ratio, it is easy to verify that the sell prices $p_s(\beta, \delta)$ satisfy the first part of the definition of equilibrium, optimal selling decision.

- Expected asset quality satisfies $\Delta(P(v)) = \Gamma(v)$ for $v \in [\underline{v}, \bar{p}]$. In addition, $\Delta(p)$ is arbitrary at $p < P(\underline{v})$, while $\Delta(p) \leq p/\hat{\beta}$ for $p > \bar{p}$. Given the sell prices $p_s(\beta, \delta)$, these satisfy the third part of the definition of equilibrium, beliefs. In addition, given the buyer-seller ratio and these beliefs, the second part of the definition of equilibrium, optimal buying decision, implies that all investors with $\beta > \hat{\beta}$ set any buy price $p_b(\beta, \delta) \in [P(\underline{v}), P(\bar{p})]$; and all investors with $\beta < \hat{\beta}$ set any buy price $p_b(\beta, \delta) < P(\underline{v})$.

- Asset purchases equal asset sales:

$$\int_{\underline{\delta}}^{\bar{\delta}} \int_{\hat{\beta}}^{\bar{\beta}} g(\beta, \delta) \, d\beta \, d\delta \equiv \int_{\underline{v}}^{\bar{p}} P(v)\Theta(P(v))h(v)dv, \tag{2}$$

  where $h(v) \equiv \int_{\underline{\delta}}^{\bar{\delta}} \frac{1}{\delta} g(v/\delta, \delta) \, d\delta$ is the density of continuation values. This pins down $\hat{\beta}$. The left hand side is the expenditures by buyers while the right hand side is the cost of purchasing the assets sold by sellers. Finally, buyers with $\beta > \hat{\beta}$ set prices in appropriate numbers so as to satisfy the fourth part of the definition of equilibrium, market clearing.

Figure 2 illustrates investors' behavior in this equilibrium. Investors are divided into four groups. Patient investors with a high quality asset buy other assets. Impatient investors with a low quality asset sell their asset. There are also patient investors with a low quality asset who sell their asset and buy other assets; and somewhat impatient investors with a high quality asset who neither buy nor sell asset but simply consume their endowment in each period.

We say this equilibrium is partially pooling because all investors with the same continuation value charge a common price, but investors with different continuation values offer different prices. The partial pooling equilibrium necessarily has some inefficiency. For example, there are some investors with the highest possible discount factor who sell their asset, transferring it to an investor with a lower discount factor.

Under some conditions, there is a continuum of equilibria in which the sale price is a continuous, strictly increasing function of an investor's continuation value. If the lowest asset quality is less than the average asset quality held by the investor with the lowest continuation value, $\underline{\delta} < \Gamma(\underline{v})$, there is a continuum of partially pooling equilibria:

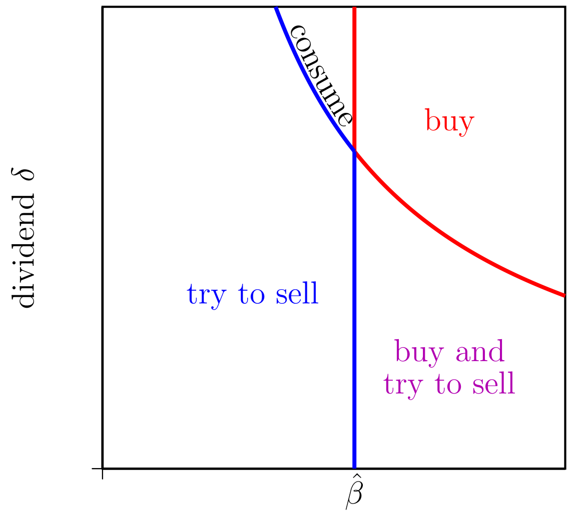

Figure 2: Behavior in partial pooling equilibrium.

Proposition 2 *Impose Assumption 1. Fix $\hat{\beta} \in [\underline{\beta}, \bar{\beta}]$. Define*

$$\hat{\theta} \equiv \frac{\int_{\underline{\delta}}^{\bar{\delta}} \int_{\hat{\beta}}^{\bar{\beta}} g(\beta, \delta)\, d\beta\, d\delta}{\int_{\underline{v}}^{\bar{p}} \hat{\beta}\Gamma(v) \exp\left(-\int_{\underline{v}}^{v} \frac{\hat{\beta}\Gamma'(v')}{\hat{\beta}\Gamma(v')-v'} dv'\right) h(v) dv}, \tag{3}$$

where $h(v) \equiv \int_{\underline{\delta}}^{\bar{\delta}} \frac{1}{\delta} g(v/\delta, \delta)\, d\delta$ is the density of continuation values. If $1 \geq \hat{\theta} \geq \frac{\hat{\beta}\underline{\delta}-\underline{v}}{\hat{\beta}\Gamma(\underline{v})-\underline{v}}$, then there exists an equilibrium characterized by three thresholds $\underline{p} \leq \hat{p} < \bar{p}$ satisfying:

- $\underline{p} \equiv \hat{\theta}\hat{\beta}\Gamma(\underline{v}) + (1-\hat{\theta})\underline{v}$
- $\hat{p} \equiv \hat{\beta}\Gamma(\underline{v})$
- *$\bar{p}$ is the smallest solution to $\bar{p} = \hat{\beta}\Gamma(\bar{p})$*

In any such equilibrium,

- *Any investor (β, δ) with $\beta\delta = \underline{v}$ is indifferent about selling his asset at any price $p \in [\underline{p}, \hat{p}]$. At these prices, the sale probability is $\Theta(p) = \hat{\theta}(\hat{p} - \underline{v})/(p - \underline{v})$.*
- *Any investor (β, δ) with $\beta\delta \in (\underline{v}, \bar{p})$ sells his asset at the price $P(\beta\delta) = \hat{\beta}\Gamma(\beta\delta)$. In equilibrium he sells with probability*

$$\Theta(P(v)) = \hat{\theta} \exp\left(-\int_{\underline{v}}^{v} \frac{\hat{\beta}\Gamma'(v')}{\hat{\beta}\Gamma(v') - v'} dv'\right).$$

- *Any investor* (β, δ) *with* $\beta\delta \geq \bar{p}$ *is indifferent about selling his asset at any price* $p \geq \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$*. In equilibrium he sells with probability* $\Theta(p) = 0$*, except possibly* $\Theta(\bar{p}) > 0$.

- *Any investor* (β, δ) *with* $\beta > \hat{\beta}$ *is indifferent about buying assets at any price* $p \in [\hat{p}, \bar{p})$*. Any investor with* $\beta < \hat{\beta}$ *buys with probability* 0 *at a price* $p < \underline{p}$.

Proof. To construct an equilibrium, we must first define the selling prices for all investors. Assume $p_s(\beta, \delta) = P(\beta\delta) \equiv \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$, strictly increasing in $\beta\delta$. It is straightforward to verify that these sale prices are weakly optimal for all sellers given the function $\Theta(p)$ and so satisfy the first equilibrium condition.

Turn next to the belief function $\Delta(p)$. Since $P(v)$ is strictly increasing, a buyer who purchases at a price $P(v)$ knows that the asset was sold by a seller with continuation value v and hence expects the quality of the asset to be $\Gamma(v)$ from part 3(a) of the definition of equilibrium: $\Delta(P(v)) = \Gamma(v)$ for all $v \in [\underline{v}, \bar{v}]$. At lower prices, $\Delta(p) = \underline{\delta}$ for $p < \hat{p}$ is consistent with part 3(b) of the definition of equilibrium. At higher prices, $\Delta(p) = \Gamma(\bar{p})$ for $p > P(\bar{v})$ is consistent with part 3(b) of the definition of equilibrium.

Given these beliefs, it is straightforward to construct expected return from purchasing at different prices. If $\beta > \hat{\beta}$, this is maximized by buying at any price $p \in [\hat{p}, \bar{p}]$, while otherwise this is maximized by buying at any price $p < \underline{p}$.

The final equilibrium condition is the market clearing condition, which we can write as

$$\int_{\underline{\delta}}^{\bar{\delta}} \int_{\hat{\beta}}^{\infty} g(\beta, \delta)\, d\beta\, d\delta = \int_{\underline{v}}^{\hat{p}} P(v)\Theta(P(v))h(v)dv.$$

The functional forms allow us to solve this for $\hat{\theta}$, so equation (3) implies that the market clears. By allocating the purchases of investors with $\beta > \hat{\beta}$ appropriately across markets, we can then get all other markets to clear at the appropriate buyer-seller ratio. ■

An example illustrates the multiplicity of partial pooling equilibria. Assume $G(\beta, \delta) = \beta\delta^2$ on $[0,1]^2$. Then $\underline{v} = 0$ and $\Gamma(v) = \frac{1+v}{2}$. In this case, $\underline{p} \in [0, \frac{1}{2}\hat{\beta}]$, $\hat{p} = \frac{1}{2}\hat{\beta}$, and $\bar{p} = \hat{\beta}/(2 - \hat{\beta})$. We can solve explicitly for the buyer-seller ratio:

$$\Theta(p) = \begin{cases} \infty & \text{if } p \in [0, \underline{p}) \\ \underline{p}/p & \text{if } p \in \big(\underline{p}, \hat{p}\big) \\ (\underline{p}/\hat{p})\big(2\hat{\beta}^{-2}(\hat{\beta} - (2 - \hat{\beta})p)\big)^{\frac{\hat{\beta}}{2-\hat{\beta}}} & \text{if } p \in [\hat{p}, \bar{p}) \\ 0 & \text{if } p \in [\bar{p}, \infty), \end{cases}$$

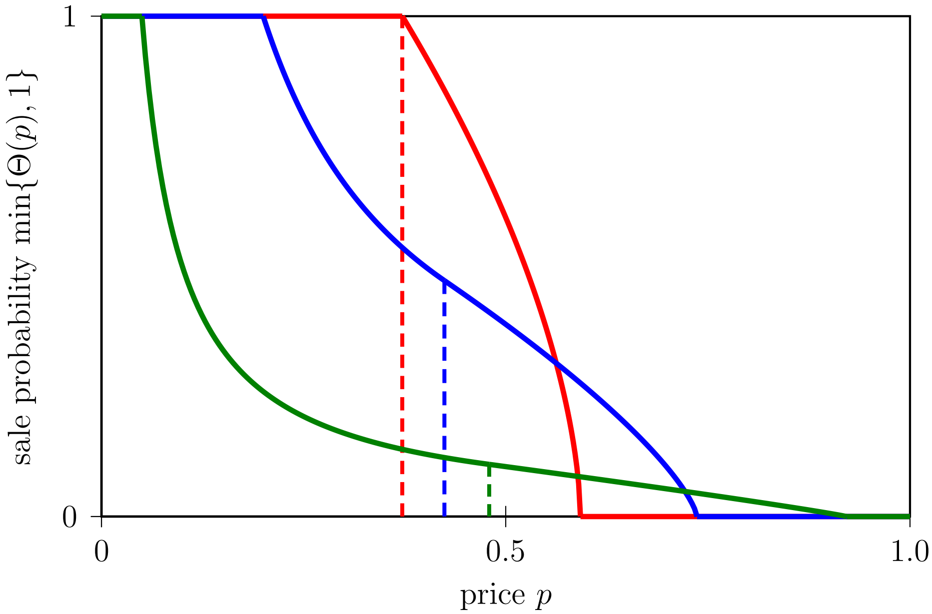

Figure 3: This illustrates three different equilibrium buyer-seller ratios with $G(\beta,\delta) = \beta\delta^2$. The red line corresponds to the case of $\underline{p} = \hat{p} = 0.3720$, which implies $\hat{\beta} = 0.7441$. The blue line has $\underline{p} = 0.2$, which implies $\hat{\beta} = 0.8483$. The green line has $\underline{p} = 0.05$ and so $\hat{\beta} = 0.9592$. The dashed lines indicate the value of $\hat{p}$ in each equilibrium.

where we take advantage of the closed form to solve the integral explicitly. The market clearing condition then implies, after a mess of algebra, that

$$1 - \hat{\beta} = \underline{p}\frac{\hat{\beta}(12 - 7\hat{\beta})}{12 - 7\hat{\beta} + \hat{\beta}^2}.$$

It is easy to show that $\hat{\beta}$ falls monotonically from 1 when $\underline{p} = \hat{\theta} = 0$ to 0.7441 when $\underline{p} = \hat{p} = 0.3720$ and $\hat{\theta} = 1$. Any value of $\hat{\theta}$ (or $\underline{p}$) in this range corresponds to an equilibrium.

A curious feature of these equilibria is that the probability of selling is higher at high prices but lower at low prices in "less liquid" equilibria with lower θ. Indeed, the trading probability at the lower bound $\hat{p} = \frac{1}{2}\hat{\beta}$ is unambiguously lower with $\hat{\beta}$ is larger (since $\underline{p}$ is lower), while the upper bound $\frac{\hat{\beta}}{2-\hat{\beta}}$ is unambiguously increasing and so the trading probability rises at those prices. We illustrate this in Figure 3 by indicating the function $\Theta(p)$ in three different equilibria, all consistent with the parameterization $G(\beta,\delta) = \beta\delta^2$.

5.2 One-Price Equilibrium

We next construct equilibria in which all trade takes place at a single price. A one-price equilibrium is characterized by two numbers, the trading price p_1 and the identity of the marginal buyer $\hat{\beta} \in [\underline{\beta}, \bar{\beta}]$. We find two equations that characterize these variables and construct an associated equilibrium.

In a one-price equilibrium, an investor can purchase an asset at any price greater than or equal to p_1 and can sell an asset at any price less than or equal to p_1:

$$\Theta(p) = \begin{cases} \infty \\ 1 \\ 0 \end{cases} \Leftrightarrow p \lesseqgtr p_1.$$

It follows that an investor with a continuation value $\beta\delta < p_1$ will choose to sell for p_1 while any other investor will set a higher sale price. This means that the average quality of assets for sale at p_1 is

$$\Delta(p_1) = \frac{\int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p_1/\beta} \delta g(\beta, \delta)\, d\delta\, d\beta}{\int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p_1/\beta} g(\beta, \delta)\, d\delta\, d\beta}.$$

Turn next to the buying decision. An investor with a discount factor β is willing to buy an asset for p_1 if $\beta\Delta(p_1) > p_1$. The marginal buyer then satisfies

$$\hat{\beta}\Delta(p_1) = p_1.$$

Substituting for $\Delta(p_1)$ gives

$$\hat{\beta}\frac{\int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p_1/\beta} \delta g(\beta, \delta)\, d\delta\, d\beta}{\int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p_1/\beta} g(\beta, \delta)\, d\delta\, d\beta} = p_1, \tag{4}$$

one equation relating the two key endogenous objects. The second equation comes from market clearing,

$$\int_{\hat{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{\bar{\delta}} g(\beta, \delta) d\delta d\beta = p_1 \int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p_1/\beta} g(\beta, \delta)\, d\delta\, d\beta, \tag{5}$$

where the left hand side is the amount of the period 1 consumption good held by investors with discount factors greater than $\hat{\beta}$ and the right hand side is the cost of buying the assets held by investors with continuation value less than p_1.

Given values of p_1 and $\hat{\beta}$ that solve equations (4) and (6), we can construct a one-price equilibrium by specifying investors' buy and sell prices and beliefs. We choose the sell price to influence buyers' beliefs appropriately:

$$p_s(\beta,\delta) = \begin{cases} p_1 \\ \bar{\beta}\delta \end{cases} \Leftrightarrow \beta\delta \lesseqgtr p_1.$$

Buyers' beliefs follow from this price setting behavior:

$$\Delta(p) = \begin{cases} \underline{\delta} & \text{if } p < p_1 \\ p_1/\hat{\beta} & \text{if } p = p_1 \\ p/\bar{\beta} & \text{if } p \in (p_1, \bar{v}] \\ \bar{\delta} & \text{if } p > \bar{v} \end{cases}$$

Note that beliefs are arbitrary at the lowest prices $p < p_1$ and highest prices $p > \bar{v}$, since no seller sets these prices and many sellers find the highest prices weakly optimal.

Now since $\hat{\beta} \leq \bar{\beta}$, buyers prefer purchasing a quality $\Delta(p_1)$ asset for p_1 rather than a quality $\Delta(p)$ asset for $p \in (p_1, \bar{v}]$. And since $\Delta(\bar{v}) = \Delta(p)$ for all $p > \bar{v}$, they also prefer spending p_1 rather than any higher price. Finally, since $\Theta(p) = \infty$ for all $p < p_1$, any buyer with $\beta > \hat{\beta}$ prefers buying for sure at price p_1 instead of bidding a lower price, failign to buy, and consuming her endowment. It follows that

$$p_b(\beta,\delta) = \begin{cases} 0 \\ p_1 \end{cases} \Leftrightarrow \beta \lesseqgtr \hat{\beta}.$$

These beliefs therefore support the proposed equilibrium.

Figure 4 represents a one-price equilibrium. The equilibrium looks superficially similar to the equilibrium in which the sale price is a continuously increasing function of the continuation value. The buying decision is only affected by the patience of the investor, that is, all the investors that are patient enough (with $\beta > \hat{\beta}$) buy assets and all the investors who are more impatient do not buy assets. The selling decision is affected by both the patience of the investor and the quality of her asset. In equilibrium, some investors both buy and sell assets while others, patient investors with a fairly good asset, neither buy not sell but instead just consume their endowment in each period. There is one important difference between the two types of equilibria, however. In the one-price equilibrium, any investor who wants to sell for p_1 is successful in equilibrium. When the sale price is a continuously increasing function of the continuation value, many investors attempt to sell but are unsuccessful.

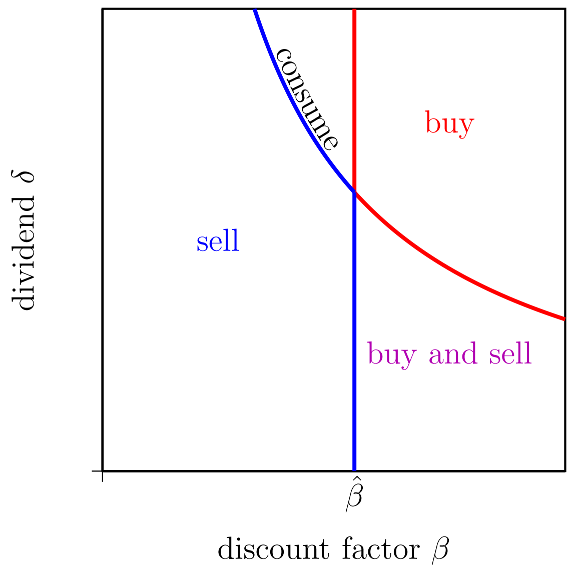

Figure 4: One-price equilibrium.

We again illustrate this equilibrium through a parametric example. Assume $G(\beta, \delta) = \beta\delta^2$ on $[0, 1]^2$. Then equations (4) and (6) reduce to

$$\hat{\beta}\frac{3 - p_1^2}{3(2 - p_1)} = p_1,$$
$$1 - \hat{\beta} = p_1^2(2 - p_1),$$

uniquely determining $p_1 = 0.426$ and $\hat{\beta} = 0.714$.

In general, a continuum of one price equilibria may exist, but in almost every equilibrium there is rationing at the equilibrium trading price. Equilibria are now characterized by three numbers, the equilibrium trading price p_1, the probability of trade at that price $\theta_1 \in [0, 1]$, and the discount factor of the marginal buyer $\hat{\beta}$. The marginal buyer's indifference condition (4) is unchanged, while the market clearing condition (6) adjusts to reflect the reduced probability of trade:

$$\int_{\hat{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{\bar{\delta}} g(\beta, \delta) d\delta d\beta = p_1\theta_1 \int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p_1/\beta} g(\beta, \delta)\, d\delta\, d\beta. \tag{6}$$

In our parametric example, we find that for every price $p_1 \in [0.426, 0.634]$, there is a unique $\theta_1 = \frac{3-6p_1+2p_1^2}{p_1^2(2-p_1)(3-p_1)^2} \in [0, 1]$ and $\hat{\beta} = \frac{3p_1(2-p1)}{3-p_1^2} \in [0, 1]$ that solves these equations.

To support such an equilibrium, we must again specify investors' strategies and beliefs.

Let $p_0 = \theta_1 p_1 + (1-\theta_1)\underline{v}$. Then the buyer-seller ratio is

$$\Theta(p) = \begin{cases} \infty & p < p_0 \\ \frac{p_0 - \underline{v}}{p - \underline{v}} \;\Leftrightarrow & p \in [p_0, p_1] \\ 0 & p > p_1. \end{cases}$$

This implies that an investor with the lowest continuation value $\underline{v}$ is indifferent between all prices $p \in [p_0, p_1]$, investors with continuation values $v \in (\underline{v}, p_1]$ find price p_1 optimal, and all other investors set any higher continuation value. We again assume

$$p_s(\beta, \delta) = \begin{cases} p_1 \\ \bar{\beta}\delta \end{cases} \Leftrightarrow \beta\delta \lesseqgtr p_1.$$

Given this behavior, buyers' beliefs are unchanged as well,

$$\Delta(p) = \begin{cases} \underline{\delta} & \text{if } p < p_1 \\ p_1/\hat{\beta} & \text{if } p = p_1 \\ p/\bar{\beta} & \text{if } p \in (p_1, \bar{v}] \\ \bar{\delta} & \text{if } p > \bar{v}. \end{cases}$$

Finally, if

$$\theta_1 \geq \frac{(\hat{\beta} - \underline{\beta})\underline{\delta}}{p_1 - \underline{\beta}\underline{\delta}}, \tag{7}$$

then the dividend-price ratio is higher at p_1 than at p_0, $\Delta(p_1)/p_1 > \Delta(p_0)/p_0$. This is enough to ensure that investors' optimal buy strategy is still

$$p_b(\beta, \delta) = \begin{cases} 0 \\ p_1 \end{cases} \Leftrightarrow \beta \lesseqgtr \hat{\beta},$$

unchanged from before. Note that since $\hat{\beta}\underline{\delta} > p_1 = \hat{\beta}\Delta(p_1)$, condition (7) always holds when θ_1 is sufficiently close to 1 but it will fail for small values of θ_1 if $\underline{\delta} > 0$. In that case buyers would prefer to pay p_0 and get the lowest quality asset, and so there is no such one-price equilibrium.

Much of the literature on adverse selection in financial markets assumes that all trade occurs at a common price p_1 (see, for example, Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010). In such papers, the environment is set up in such a way that a seller cannot even consider selling his assets at a price different than p_1. These papers also assume that sellers can sell with probability 1 at price p_1. While our equilibrium concept is consistent with this particular one-price equilibrium, it is consistent with a continuum of other one-price equilibria and with many other equilibria.

5.3 n Price Equilibrium

In addition to the continuum of one-price equilibria, we find an n-dimensional continuum of n-price equilibria. Denote the prices by $p_1 < \cdots < p_n$; in equilibrium all trade occurs at these prices. Also let $\theta_1 > \cdots > \theta_n$ denote the buyer-seller ratios at these prices, with $\theta_1 \in (0, 1]$. A seller is dissuaded from setting a higher price by a lower sale probability. This will induce sellers with a higher continuation value to set a weakly higher price. Let $v_1 < \cdots v_n$ denote the n critical continuation values who are indifferent between neighboring prices. Finally, let $\hat{\beta}$ denote the discount factor of the marginal buyer. This gives us a total of $3n + 1$ variables.

We find that these must satisfy $2n + 1$ equations. First, the marginal seller must be indifferent between the relevant prices:

$$\theta_i(p_i - v_i) = \theta_{i+1}(p_{i+1} - v_i) \text{ for } i \in \{1, \ldots, n-1\}$$

and $p_n = v_n$. This gives us a total of n equations. Second, the buyer with a discount factor $\hat{\beta}$ must be indifferent about purchasing at any price:

$$\hat{\beta}\Delta(p_i) = p_i \text{ for } i \in \{1, \ldots, n\},$$

where

$$\Delta(p_i) = \frac{\int_{\underline{\beta}}^{\bar{\beta}} \int_{v_{i-1}/\beta}^{v_i/\beta} \delta g(\beta, \delta)\, d\delta\, d\beta}{\int_{\underline{\beta}}^{\bar{\beta}} \int_{v_{i-1}/\beta}^{v_i/\beta} g(\beta, \delta)\, d\delta\, d\beta}$$

is the expected asset quality for an investor with continuation value $\beta\delta \in (v_{i-1}, v_i)$. This gives n more equations. Third, the goods market must clear:

$$\int_{\hat{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{\bar{\delta}} g(\beta, \delta) d\delta d\beta = \sum_{i=1}^{n} \left(p_i \theta_i \int_{\underline{\beta}}^{\bar{\beta}} \int_{p_{i-1}/\beta}^{p_i/\beta} g(\beta, \delta)\, d\delta\, d\beta \right),$$

which gives one more equation.

Using these equations, we can again construct an equilibrium, at least if θ_1 is not too small. First, the optimal sale price satisfies

$$p_s(\beta,\delta) = p_i \text{ if } v_{i-1} < \beta\delta < v_i$$

where $v_0 = \underline{v}$. In addition, we assume $p_s(\beta,\delta) = \bar{\beta}\delta$ if $\beta\delta > v_n$. Buyers beliefs are clearly defined at the prices $p_1, \dots, p_n$ and at $p \in (v_n, \bar{\beta}\bar{\delta}]$. At still higher prices we set $\Delta(p) = \bar{\delta}$ while at all other prices $p < v_n$, we set $\Delta(p) = \underline{p}$. Finally, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < p_0 \\ \frac{\theta_i(p_i - v_i)}{p - v_i} \text{ if} & p \in [p_i, p_{i+1}],\ i \in \{0, \dots, n-1\} \\ 0 & p > p_n \end{cases}$$

where $\theta_0 = 1$ and $p_0 = \theta_1 p_1 + (1-\theta_1)v_0$. For this to be an equilibrium, a buyer must prefer to purchase at any price p_i, $i \in \{1, \dots, n\}$, rather than at p_0. This requires $\hat{\beta}\underline{\delta} \leq p_0$, an inequality that always holds if $\underline{\delta} = 0$ or if $\hat{\theta}_1$ is sufficiently small.

We return again to our parametric example $G(\beta,\delta) = \beta\delta^2$. First suppose $\theta_1 = 1$. We find that for any value of $\theta_2 \in [0, 0.832125]$, it is possible to construct an equilibrium with trade at two prices. Higher values of θ_2 are associated with lower values of p_1 (falling from 0.426254 to 0.371291), lower values of $p_2 = v_2$ (falling from 0.526550 to 0.446197), lower values of v_1 (falling from 0.426254 to 0), and higher values of $\hat{\beta}$ (rising from 0.714062 to 0.742583). It does not seem possible to construct equilibria with $\theta_2 > 0.832125$, because the system of equations would imply $v_1 < 0$. For lower values of θ_1, there is a smaller interval of θ_2 corresponding to an equilibrium, but the interval always exists.

Qualitatively an n-price equilibrium looks very similar to the partially-pooling equilibrium. Investors with higher continuation values set weakly higher sale prices and sell with a weakly lower probability. Indeed, we suspect that in the limit as n converges to infinity, the functions $\Theta(p)$ and $\Delta(p)$ in any n price equilibrium will be close to their values in some partially-pooling equilibrium in the sense of the sup-norm.

5.4 Mixed Equilibrium

Equilibria may feature a mix of mass points and continuous distributions. For example, investors with continuation values in some interval $[\underline{v}, v_1)$ may set a common price p_1, while investors with continuation values in a higher interval $[v_1, v_2]$ may set prices $P(v)$ that are strictly increasing in the continuation value. To ensure buyers are willing to pay all these prices, we require that $\hat{\beta}\Gamma(v) = P(v)$ and $\hat{\beta}\Delta(p_1) = p_1$. Since $\Delta(p_1) < \Gamma(v_1)$, this implies $p_1 < P(v_1)$, so there is also a gap in the distribution.

The same logic implies that investors with continuation value in the next interval (v_2, v_3) may set a common price p_3 satisfying $\hat{\beta}\Gamma\Delta(p_3) = p_3$; and since $\Delta(p_3) > \Gamma(v_2)$, $p_3 > P(v_2)$. Tgys again there is a gap in the distribution followed by another mass point. Mass points are followed by a gap, which in turn may be followed by another mass point or by a continuous distribution. Continuous distributions are followed by a gap and then a mass point. Numerous configurations are possible.

5.5 Off-the-Equilibrium-Path Beliefs

We stress that the multiplicity of equilibria highlighted here are not driven by off-the-equilibrium-path beliefs. In particular, although a price is a signal of quality in this model, our multiplicity of equilibria is not driven by the usual sources in a signaling game.[7] Instead, multiple equilibria can arise even when trade takes place at all relevant prices in equilibrium.

First suppose that $\underline{\beta} = \underline{\delta} = 0$. In many of our equilibria, the investor with the lowest continuation value $\underline{v}$ is indifferent over a range of different prices $[p_0, p_1]$. We typically specified that the investor charged the price $p_s(\underline{\beta}, \underline{\delta}) = p_1$, while no investor sets a lower sale price. This left the off-the-equilibrium path beliefs $\Delta(p)$ free for $p \in [p_0, p_1)$ and we closed the model by specifying that buyers believe that only investors with the lowest continuation value, $\underline{\delta} = 0$, set those prices, $\Delta(p) = 0$ if $p < p_1$.

Other assumptions are possible including one that eliminates off-the-equilibrium path beliefs. For example, we may assume that

$$p_s(0, \delta) = \begin{cases} \hat{\beta}\delta & \text{if } \hat{\beta}\delta \in [p_0, p_1] \\ p_0 & \text{if } \hat{\beta}\delta \notin [p_0, p_1] \end{cases}$$

An investor rationally expects that $\Delta(p) = p/\hat{\beta}$ for all $p \in (p_0, p_1]$. In addition, $\Delta(p_0) > p/\hat{\beta}$, but $\Theta(p_0) = \infty$ so that buyers cannot profit at this price. These modifications imply that $p_b(\beta', \delta') \in (p_0, p_1]$ is optimal if $\beta' > \hat{\beta}$; these prices are no longer off-the-equilibrium path.

Now drop the assumption that $\underline{\beta} = \underline{\delta} = 0$. In this case there is a unique $\underline{\delta}$ associated with the lowest continuation value, $\Gamma(\underline{\beta}\underline{\delta}) = \underline{\delta}$. Our logic from before indicates that there is a unique partially-pooling equilibrium in this case. In an n-price equilibrium with $\theta_1 < 1$, buyers rationally believe that only the investor with the lowest continuation value, and therefore the lowest asset quality, would find it optimal to charge a price $p \in (p_0, p_1)$. The

[7] Part 3(b) of the definition of equilibrium eliminates the usual source of multiplicity: investors do not attempt to sell at a particular price because they believe that there are no buyers at that price. And investors do not attempt to buy at that price because they believe that only low quality assets are sold at that price. We preclude these beliefs unless investors with a low quality asset have the strongest incentive to sell at that price, as in Cho and Kreps (1987) and Banks and Sobel (1987).

beliefs we have specified therefore seem correct.

Alternatively, consider an n-price equilibrium in which investors with value v_i are indifferent over a range of prices $[p_i, p_{i+1}]$. Now it is possible to get all these prices charged in equilibrium, validating buyers' beliefs. Assume that $p_s(v_i/\delta, \delta) = \hat{\beta}\delta$ if $\hat{\beta}\delta \in [p_i, p_{i+1}]$, with $p_s(v_i/\delta, \delta) = p_{i+1}$ otherwise. Such selling prices are optimal by construction and leave buyers with the belief that $\Delta(p) = p/\hat{\beta}$ for all $p \in (p_i, p_{i+1})$. Buyers are therefore willing to buy at each such price again eliminating any need to refer to off-the-equilibrium-path beliefs.

6 Equilibrium Selection

6.1 Assumptions

Since off-the-equilibrium-path beliefs do not restrict behavior, we turn instead to two related restrictions on equilibrium behavior. First, we assume that any two investor with the same continuation value set the same sale price:

Assumption 2 *There exists a function $P : [\underline{v}, \bar{v}] \mapsto \mathbb{R}_+$ such that $p_s(\beta, \delta) = P(\beta\delta)$ for all (β, δ).*

Second we impose an additional constraint on how buyers rationalize the quality of assets available at a price that is not charged in equilibrium. If they believe that some investor (β, δ) would be inclined to offer that price, then they must believe that all investors with the same continuation value would be equally inclined to offer that price:

Assumption 3 *For all $p \in \mathbb{R}_+$ with $\Theta(p) < \infty$,*

(a) if there exists a $v \in [\underline{v}, \bar{v}]$ with $P(v) = p$, $\Delta(p) = \mathbb{E}(\delta | P(v') = p)$;

(b) otherwise there exists a v_1 with $\Gamma(v_1) \leq \Delta(p)$, $p \geq v_1$, and

$$\min\{\Theta(P(v_1)), 1\}(P(v_1) - v_1) = \min\{\Theta(p), 1\}(p - v_1);$$

and similarly a v_2 with $\Gamma(v_2) \geq \Delta(p)$, $p \geq v_2$, and

$$\min\{\Theta(P(v_2)), 1\}(P(v_2) - v_2) = \min\{\Theta(p), 1\}(p - v_2).$$

Assumption 3(a) is unchanged from part 3(a) of the definition of equilibrium and included only for expositional convenience. Assumption 3(b) imposes an additional and important restriction on beliefs.

One way to think about this restriction is to imagine what would happen if a single buyer offered a price p that was not previously offered in the market. Some sellers would respond by offering some assets at that price, driving down the buyer-seller ratio until it achieved the value $\Theta(p)$ described in part 3(b) of the definition of equilibrium and in Assumption 3(b). At this buyer-seller ratio, only a small number of investors would find it optimal to offer assets at that price. The assumption states that if buyers believe that some seller with continuation value v offers assets at that price with some probability, then he must believe that all sellers with the same continuation value will offer assets at that price with the same probability. This means that the average quality of assets offered by sellers with continuation value v at any price that they find optimal is $\Gamma(v)$, a tighter restriction than imposed in part 3(b) of the definition of equilibrium.

6.2 Sketch of Proof

We prove that there is a unique equilibrium under Assumptions 2 and 3, the partially-pooling equilibrium described at the start of Section 5.1.

The structure of the proof works as follows. First, a simple revealed preference argument establishes that for any two investors $v_1 < v_2$, $P(v_1) \leq P(v_2)$. Then optimal selling decisions imply

$$\begin{aligned} &\min\{\Theta(P(v_1)), 1\}(P(v_1) - v_1) \geq \min\{\Theta(P(v_2)), 1\}(P(v_2) - v_1) \\ \text{and } &\min\{\Theta(P(v_2)), 1\}(P(v_2) - v_2) \geq \min\{\Theta(P(v_1)), 1\}(P(v_1) - v_2), \end{aligned}$$

since v_1 weakly prefers $P(v_1)$ to $P(v_2)$ and conversely for v_2. The left hand sides are non-negative by assumption and this ensures that the right hand side of the first inequality is strictly positive: $P(v_2) \geq v_2 > v_1$. We can therefore multiply inequalities to get

$$(P(v_1) - v_1)(P(v_2) - v_2) \geq (P(v_1) - v_2)(P(v_2) - v_1) \Leftrightarrow (P(v_2) - P(v_1))(v_2 - v_1) \geq 0,$$

which proves that $P(v_2) \geq P(v_1)$.

Now if price is a weakly increasing function of the continuation value, then we can invert this to get that the continuation value is a weakly increasing correspondence of the price. That is, let $V(p)$ denote the continuation value that finds the price p weakly optimal, possibly set-valued. Take any $p_1 < p_2$, $v_1 \in V(p_1)$, and $v_2 \in V(p_2)$. Then $v_1 \leq v_2$.

Next use the buyer's beliefs. This is given by $\Delta(p) = \Gamma(V(p))$ if $V(p)$ is a singleton, while otherwise there exists a $v_1 \in V(p)$ and $v_2 \in V(p)$ with $\Gamma(v_1) < \Delta(p) < \Gamma(v_2)$. That is, buyers' beliefs are an increasing function of prices, with jumps in the belief function corresponding to prices where multiple continuation values charge the same price and flats in the belief function corresponding to a range of prices that one buyer continuation value finds optimal. This result would not hold in the absence of Assumptions 2 and 3 and is central to the most of the equilibria in Section 5.

Finally, we can rule out both jumps and flats in the belief function. If $\Delta(p)$ jumps at p (so multiple continuation values find this sale price optimal), then buyers would prefer to pay $p + \varepsilon$ rather than p to get a discretely better asset. If $\Delta(p)$ is constant between p_1 and p_2 (so one continuation value v_1 finds this range of prices optimal), then buyers would prefer to pay $p_1 - \varepsilon$ rather than $p_2 + \varepsilon$, with a small fall in asset quality more than offset by the sharp drop in price.

References

Banks, Jeffrey S. and Joel Sobel, "Equilibrium Selection in Signaling Games," *Econometrica*, 1987, *55* (3), 647–661.

Chang, Briana, "Adverse Selection and Liquidity Distortion in Decentralized Markets," 2012. University of Wisconsin Mimeo.

Chari, V.V., Ali Shourideh, and Ariel Zetlin-Jones, "Adverse Selection, Reputation and Sudden Collapses in Secondary Loan Markets," 2010. University of Minnesota Mimeo.

Chiu, Jonathan and Thorsten V. Koeppl, "Market Freeze and Recovery: Trading Dynamics under Optimal Intervention by a Market-Maker-of-Last-Resort," 2011. Bank of Canada Mimeo.

Cho, In-Koo and David M. Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 1987, *102* (2), 179–221.

Daley, Brendan and Brett Green, "Waiting for News in the Dynamic Market for Lemons," 2010. Duke Mimeo.

Eisfeldt, Andrea L., "Endogenous Liquidity in Asset Markets," *Journal of Finance*, 2004, *59* (1), 1–30.

Guerrieri, Veronica and Robert Shimer, "Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality," *American Economic Review*, forthcoming.

_ , _ , and Randall Wright, "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 2010, *78* (6), 1823–1862.

Kurlat, Pablo, "Lemons, Market Shutdowns and Learning," 2009. MIT Mimeo.

Tirole, Jean, "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning," *American Economic Review*, 2012, *102* (1), 29–59.