Externalities and Fundamental Nonconvexities: A Reconciliation of Approaches to General Equilibrium Externality Modelling and Implications for Decentralization

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Externalities and Fundamental Nonconvexities: A Reconciliation of Approaches to General Equilibrium Externality Modeling and Implications for Decentralization*

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Abstract

By distinguishing between producible and nonproducible public goods, we are able to propose a general equilibrium model with externalities that distinguishes between and encompasses both the Starrett [1972] and Boyd and Conley [1997] type external effects. We show that while nonconvexities remain fundamental whenever the Starrett type external effects are present, these are not caused by the type discussed in Boyd and Conley. Secondly, we find that the notion of a "public competitive equilibrium" for public goods found in Foley [1967, 1970] allows a decentralized mechanism, based on both price and quantity signals, for economies with externalities, which is able to restore the equivalence between equilibrium and efficiency even in the presence of nonconvexities. This is in contrast to equilibrium notions based purely on price signals such as the Pigouvian taxes.

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by

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1. Introduction

Arrow [1970] perceived the market failure associated with externalities as a problem of incomplete markets. He showed that the equivalence between a competitive equilibrium and a Pareto optimum can be restored if markets for external effects can be created. However, employing Arrow's framework, where the commodity space is extended to include the rights to generate externalities as additional commodities, Starrett [1972] demonstrated that the presence of detrimental production externalities creates fundamental nonconvexities in the technology sets of firms. He considers an example where increases in the level of an externality reduces the maximum output a firm can produce, given the levels of all inputs. But, the maximum output of the firm, for any given level of inputs, is assumed never to fall below zero, even in the face of an unlimited amount of the externality (the firm always has the option of shutting down production). This implies that the frontier of the technology is either asymptotic to the axis reserved for the externality or coincides with it after a critical level of the externality, where the maximum output has fallen to zero, has been reached. As is well known, when the convexity assumption fails, the existence of a competitive equilibrium becomes questionable.[1]

A question then arises about the possibility of existence of some other alternative decentralized mechanism (for a definition, see the footnote below) that will, in the presence of externalities, ensure the equivalence between the underlying equilibrium concept and Pareto optimality.[2] A popular candidate is the one associated with Pigovian taxes,

[1] In this context, the nonconvexity implies that, at all positive prices of a right to generate a detrimental externality, there exists no solution to the profit maximization of a firm facing the detrimental externality for, if the (personalized) price that a firm receives from the generator of an externality is positive, then shutting down production and supplying an arbitrarily large positive amount of the externality rights to the generator is both technologically feasible and profitable for the firm. On the other hand, a non-positive price creates an excess demand for externality rights.

[2] Roughly, employing the terminology of Calsamiglia [1977], a mechanism is *decentralized* if the response of an agent to messages or signals received depends only on that agent's characteristics. This reflects the initial dispersion of knowledge of the the economic environment, so that "each agent knows only his component of the environment" and "all the information concerning the rest of the agents has to be come via formal messages."

attributable to Pigou [1932] and Baumol [1972]. This can be interpreted as a decentralized mechanism where the government is also an economic agent, whose responses (the determination of the Pigovian taxes on the externality generators and the redistribution of tax revenue) depend on the information (the shadow prices) communicated to it by the agents affected by the externalities. As has been well documented, an equilibrium with Pigouvian taxes is compatible with nonconvex technology sets of the firms facing detrimental externalities, so long as the technologies of these firms are convex in the appropriate subspaces.[3] However, the problem with the Pigovian tax mechanism is that, while any Pareto optimum can be decentralized as a Pigovian tax equilibrium, the reverse is not true. Baumol and Bradford [1972] showed that, if the detrimental effects of externalities on victim firms are sufficiently large, the *aggregate* technology set of the economy could well be nonconvex. In such a nonconvex economy, although the first order conditions of Pareto optimality would hold at a Pigovian tax equilibrium determined by government chosen levels of Pigovian taxes, the second order conditions for even a local Pareto optimum may fail. Thus, an arbitrary Pigovian tax equilibrium may not be efficient, *unless*, we restrict the class of economies to those where the externalities are weak enough to ensure convexity of the social transformation set, as is done in Hurwicz [1999].[4] In general, Hurwicz [1999] shows the impossibility of the existence of finite-dimensional decentralized mechanisms that guarantee Pareto optimality in the presence of externalities, for all economic environments (including nonconvex ones).[5]

More recently, however, Boyd and Conley [1997], henceforth referred to as BC, and Conley and Smith [2002] have challenged the fundamentality of nonconvexities for *real* economies with externalities. They argue that nonconvexities are fundamental to the Arrow/Starrett framework because it does not seem to offer a method of placing reasonable bounds on the extent to which the victim firms can observe the externality (sell externality rights to the generators). In real economies, BC argue, there are natural limits to the extent to which externalities can be generated. For example, the capacity of land, water, and air to absorb wastes and pollution is really not unlimited. According to them, nonconvexities with externalities are no longer fundamental in a model that treats the externality absorption capacity of the economy as a bounded resource, which has different qualitative values for different agents. Thus, they propose a decentralized

[3] See, *e.g.,* Starrett, Baumol and Bradford [1972], and Hurwicz [1999].

[4] A more general notion of a Pigou-Baumol equilibrium is formulated in Tulkens and Schoumaker [1975]. This equilibrium concept (a Nash equilibrium) includes cases, where even the first order conditions of optimality may not hold at the the government chosen level of Pigovian taxes (disagreement equilibria). The issue in this paper is to design a decentralizable resource allocation process, for convex environments, that moves the economy from a disagreement equilibrium to a Pareto optimum through a sequence of Pigou-Baumol equilibria obtained by adjusting the Pigou taxes.

[5] Hurwicz shows this in a manner analogous to the case of increasing returns in Calsamiglia [1977].

mechanism in the spirit of Coase [1960] for convex environments, where the endowment of this capacity is bounded and distributed among agents who trade them.[6] They prove the equivalence between an equilibrium and a Pareto optimum.[7]

This paper aims to make two contributions. Firstly, we propose a general equilibrium model with externalities, which distinguishes between and encompasses both the Arrow/Starrett and BC type external effects. The key to constructing the general model lies in distinguishing between producible and nonproducible public goods. It is the latter kind of goods that are the objects of concern in BC. In BC, they include *scarce* economic resources such as land, air, water, etc., that provide a means of disposal for producible public goods such as pollution and wastes. As BC argue, nonconvexity is not fundamental to them. Arrow/Starrett's concern, on the other hand, is with the producible public goods. Unlike the nonproducible natural resources, when the markets for rights to generate producible public goods (such as emission of green house gases) are created and these rights are traded, there may be net additions (or deletions/abatement) to the already existing stocks (the endowments) in the economy. We show that to the extent external effects caused by producible public goods are prevalent in the economy, Starrett type technological nonconvexities will remain fundamental.

Thus, we are once again confronted with Hurwicz's negative result for nonconvex economies. A second objective of this paper, thus, is to propose a decentralized mechanism that permits autonomy of decision making based on both current market price and quantity signals, incorporates (to some extent) the Coasian component in the equilibrium proposed by BC, and restores the equivalence between equilibrium and efficiency even in the presence of nonconvexities. This equilibrium concept is motivated by the notion of a "public competitive equilibrium" discussed in Foley [1967, 1970]. In Foley's mechanism, the demand for public goods is collectively determined and financed (at current market prices) by a (decentralized) unanimity rule, while their supply is determined by profit maximization at prevailing market prices.

The remaining paper is arranged as follows. In section 2, we set up a general equilibrium model of externalities. Section 3 derives the Arrow/Starrett model and the BC model as special cases of the general model and discusses the nonconvexities and market failures associated with certain producible public goods. This is done by providing axioms that distinguish between public goods that are by-products (such as pollution) and joint-products (such as national defense) of production. We show that,

[6] For a collection of these agents (the consumers in BC model), this capacity may well be a public good (*i.e.,* a good of collective consumption), so that the underlying equilibrium has the Lindahl property.

[7] Conley and Smith also prove the existence of the Coasian equilibrium in a more general economy that includes consumption externalities as well. See also Hurwicz [1995] and Varian [1995].

abstracting from issues such as thin markets and free riding, markets will typically fail in the case of by-product public goods–both detrimental and beneficial; while they could be functional for the joint product public goods. In section 4, we define a collective consumption equilibrium and prove its welfare properties. We conclude in section 5.

2. A General Model of Externalities.

There are two types of commodities in the economy. Firstly, there are ordinary commodities possessing the rivalness property of private goods. K is the index set for such commodities, which will be indexed by k. Then there are goods which are jointly consumed (that is, having public good or non-rivalness property in consumption.) We further classify the public goods into nonproducible and producible public goods. Nonproducible public goods are those whose supply is fixed by the availability of their resources. Some imperfect examples of these goods include forest cover (at least in the short run), a water body, etc. They will be will be indexed by l belonging to an index set L. Producible public goods are those whose whose supply can be augmented by production beyond their respective resource availability. M is the index set of these producible public goods, which will be indexed by m. Later, we will also distinguish between joint-product producible public goods (national defense is a classic example) and by-product producible public goods (for example, pollution and nectar produced in the apple blossoms of an orchard neighboring a bee-keeping farm).

There are three types of agents in the economy: (i) consumers, who are indexed by h that belongs to an index set H, (ii) firms for which the goods in L and M are public goods. These are indexed by i that belongs to an index set I, and (iii) firms for which goods in L are standard inputs (having rivalness property: the total use by all firms of these goods is the sum of individual uses) and which produce the goods in M. These are indexed by j that belongs to an index set J.[8]

The endowment vector of ordinary commodities is denoted by ω and of nonproducible public goods is denoted by $\eta + \sigma$. The initial stock of producible public goods is ξ. We assume that $\omega \in \mathbf{R}^K_+$, $\eta + \sigma \in \mathbf{R}^L_+$, and $\xi \in \mathbf{R}^M_+$.

For all $i \in I$, the technology of firm i is denoted by $Y^i \subseteq \mathbf{R}^K \times \mathbf{R}^L_+ \times \mathbf{R}^M_+$, and its production vector is denoted by $y^i = \langle o^i, n^i, e^i \rangle \in \mathbf{R}^K \times \mathbf{R}^L_+ \times \mathbf{R}^M_+$. The net production of ordinary commodities is $o^i \in \mathbf{R}^K$. The consumption (use) of nonproducible public

[8] Focusing on production externalities alone is not a restriction. The model can be generalized to include consumption externalities. Following the framework of Milleron (where he distinguishes between outputs (+) and inputs (-) of public goods for every firm), we can also extend the model to include bilateral (reciprocal) externalities, where firms can both generate and be victims to externalities. We refrain from doing so here in order to keep the notation tractable and the exposition simple enough for studying the qualitative distinction between being a consumer and being a producer of public goods.

goods by i is denoted by $n^i \in \mathbf{R}_+^L$, and its consumption of producible public goods is denoted by $e^i \in \mathbf{R}_+^M$.

For all $j \in J$, the technology of firm j is denoted by $Y^j \subseteq \mathbf{R}^K \times \mathbf{R}_-^L \times \mathbf{R}^M$ and its net output vector is denoted by $y^j = \langle o^j, d^i, z^j \rangle \in \mathbf{R}^K \times \mathbf{R}_-^L \times \mathbf{R}_+^M$. The net production of ordinary commodities is $o^j \in \mathbf{R}^K$. The use by firm j of the nonproducible goods in index set L is $d^j \in \mathbf{R}_-^L$. The net addition to the stock of producible public goods in the economy by firm j is denoted by $z^j \in \mathbf{R}^M$.

The consumption set of consumer $h \in H$ is denoted by $X^h \subseteq \mathbf{R}_+^{K+L+M}$. A consumption bundle is denoted by $x^h = \langle \tilde{o}^h, \tilde{n}^h, \tilde{e}^h \rangle \in \mathbf{R}_+^{K+L+M}$, where $\tilde{o}^h$ is the gross consumption of ordinary commodities by h and $\tilde{n}^h$ and $\tilde{e}^h$ are the consumption levels of nonproducible and producible public goods, respectively, by h. The preferences are representable by real valued utility function u^h, for all $h \in H$.

An economy with above specifications will be represented by $E = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, \omega, \eta + \sigma, \xi \rangle$.

Definition: A feasible allocation of the economy $E = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, \omega, \eta + \sigma, \xi \rangle$ is a tuple $\langle (x^h)_h, (y^i)_i, (y^j)_j \rangle$ such that $x^h \in X^h$ for all $h \in H$, $y^i \in Y^i$ for all $i \in I$, $y^j \in Y^j$ for all $j \in J$, and

$$\begin{aligned}
\sum_h \tilde{o}^h &= \sum_i o^i + \sum_j o^j + \omega, \\
\tilde{n}^h &= \sum_j d^j + \eta + \sigma, \ \forall h \in H, \\
n^i &= \sum_j d^j + \eta + \sigma, \ \forall i \in I, \\
\tilde{e}^h &= \sum_j z^j + \xi, \ \forall h \in H, \text{ and} \\
e^i &= \sum_j z^j + \xi, \ \forall i \in I.
\end{aligned} \tag{2.1}$$

A reason for external effects to arise in this economy is the inability of agents to voluntarily choose the levels of their respective consumption or use of all commodities. This will be true, for example, if in a market based economy, competitive markets for all commodities did not exist. In the case of public goods, absence of external effects requires the existence of personalized markets, where each consumer or user of the good trades alone with the generator(s) of the good. External effects associated with public goods will be the primary focus of this paper.[9]

[9] Though the model proposed in this paper can easily be extended to encompass, also, externalities created by absence of markets for goods having the rivalness feature.

We now provide a private ownership structure for the endowments in this economy that can sustain a market based institutional structure with public goods. We assume that the endowment of all private goods is owned by the consumers in H. Let ω be distributed as $(\omega_h)_h$. The endowment of nonproducible public goods is divided between those for whom they are public goods (firms in I and consumers in H) and those for whom they are standard (rival) inputs (firms in J). Because they are public goods for agents in H and I, we assume that each agent in H and I owns the same amount η of such resources. The remaining amount σ is distributed between agents in J as $(\sigma_j)_j$. Lastly, we assume that each agent in H and I owns the same amount ξ of the initial stock of producible public goods.

We define the restrictions of the technology sets to appropriate subspaces by employing the following notation: For all $i \in I$ and $e^i \in \mathbf{R}^M_+$, define $P^i(e^i) := \big\{\langle o^i, n^i\rangle \in \mathbf{R}^K \times \mathbf{R}^L_+ \big| \ \langle o^i, n^i, e^i\rangle \in Y^i\big\}$. Likewise, we can define the correspondence $P^i(n^i)$, and for all $j \in J$, the correspondence, $P^j(o^j, z^j)$, and so on.

3. A Reconciliation of Approaches to General Equilibrium Externality Modeling, Nonconvexities, and Market Failure.

3.1. The Boyd and Conley Model.

In the BC model the goods in L include environmental resources like air, water, land, which provide a means of disposal for producible public goods in M such as pollution. Goods in L have alternative uses for different users. For agents in H and I they are public goods, while for agents in J they are inputs for disposing of by-products of production. Using these goods for certain purposes, crowds out their availability for other purposes (for example, use of a waterbody for dumping wastes crowds out the amount available for recreation or, as in BC, increase in the pollution content of the air reduces the amount of clean air that is needed for drying clothes by a laundry.)

We argue that the BC model is a special case of (2.1), where consumers in H and producers in I have zero values for the producible public goods in M themselves, but they value the environmental resources in L in their clean state.[10] If markets for goods in L existed, then the opportunity cost of enjoying the environmental resources would be the income foregone by not selling them to firms in J. An analogy may make it clearer. Time endowment is usually assumed to be a nonproducible resource. Consumers allocate it between leisure and labor. The wage rate reflects the opportunity cost of enjoying leisure. Labor time can be used by firms to produce several different outputs for which consumers have different intrinsic values, and hence, there are

[10] This is captured in Assumption BC, below.

different markets associated with the output produced out of labor time. Consumers buy these outputs at prices (which are distinctly different from their wages) that reflect their marginal valuation of these commodities. Similarly, a collectively owned piece of land can be allocated to recreational use, dumping wastes, making buildings, etc. The opportunity cost of its recreational use is the income foregone by not selling it to firms that can use it as an input for dumping wastes or building houses, etc. Further, the firms can use it to dump several different types of wastes (*e.g.,* biodegradable and non-biodegradable wastes), each of which has its own intrinsic (negative) value for consumers.

Assumption BC, below, implies that consumers' preferences and technologies of the producers in I are independent of the levels of the producible public goods in M. We state Theorem BC without proof. It says that, under Assumption BC, we need only consider the projection of economy E to the space of commodities in K and L to identify all the feasible states of E. This is because, given Assumption BC, the amounts of the producible public goods, produced by firms in J at any feasible state of such a projected economy, are consistent with the (unprojected) consumers' preferences and technologies of the firms in I.[11]

Assumption BC (Zero Valuation of Goods in M.): For all $h \in H$ if $x^h := \langle \tilde{o}^h, \tilde{n}^h, \tilde{e}^h \rangle \in X^h$, then $\hat{x}^h := \langle \tilde{o}^h, \tilde{n}^h, \hat{\tilde{e}}^h \rangle \in X^h$ and $u^h(x^h) = u^h(\hat{x}^h)$ for all $\hat{\tilde{e}}^h \in \mathbf{R}^M_+$. For all $i \in I$ and for all $e^i, \hat{e}^i \in \mathbf{R}^M_+$, we have $P^i(e^i) = P^i(\hat{e}^i)$.

Theorem BC: *Under Assumption BC, if (i) $\langle \langle \tilde{o}^h, \tilde{n}^h \rangle_h, \langle o^i, n^i \rangle_i, \langle o^j, d^j \rangle_j \rangle$ is such that $\langle \tilde{o}^h, \tilde{n}^h \rangle, \langle o^i, n^i \rangle_i,$ and $\langle o^j, d^j \rangle_j$ are in the projections of X^h, Y^i and Y^j to the space of commodities in K and L for all $h \in H$, $i \in I$, and $j \in J$, (ii) the first, second, and the third conditions of (2.1) hold, and (iii) there exists a z^j for all $J \in J$ such that $\langle o^j, d^j, z^j \rangle \in Y^j$, then $\langle \langle \tilde{o}^h, \tilde{n}^h, \sum_j z^j + \xi \rangle_h, \langle o^i, n^i, \sum_j z^j + \xi \rangle_i, \langle o^j, d^j, z^j \rangle_j \rangle$ is a feasible state of E.*

3.2. The Arrow/Starrett Model.

On the other hand, we claim that the Arrow/Starrett model is concerned primarily with external effects created by goods in M. We derive a variant of this model, a special case of (2.1), where the goods in L are not of any intrinsic value to consumers in H and firms in I, that is, changes in the consumption levels of these resources, do not affect the welfare or production possibilities of these agents (this is Assumption AS1), and the external effects (the producible public goods) comprise of observations by each consumer in H and each firm in I of the ordinary commodities produced by all firms in J (these

[11] Note, no problem is posed if the endowment of any good in L is unbounded.In a market economy, in a general equilibrium, the price of that commodity will be zero.

are, precisely, the Arrovian commodities). This implies that $M = JK$, and that, in an institutional structure with complete markets, we would need $JK(H+I)$ personalized (artificial) markets for the public goods in M. This is reflected in Assumption AS2.

Assumption AS1 (Zero Intrinsic Valuation of Goods in L.): For all $h \in H$, if $x^h := \langle \tilde{o}^h, \tilde{n}^h, \tilde{e}^h \rangle \in X^h$, then $\hat{x}^h := \langle \tilde{o}^h, \hat{\tilde{n}}^h, \tilde{e}^h \rangle \in X^h$ and $u^h(x^h) = u^h(\hat{x}^h)$ for all $\hat{\tilde{n}}^h \in \mathbf{R}^L_+$. For all $i \in I$ and for all $n^i, \hat{n}^i \in \mathbf{R}^L_+$, we have $P^i(n^i) = P^i(\hat{n}^i)$.

Assumption AS2: $|M| = JK$ and for all $j \in J$, if $y^j \in Y^j$ then $z^j = \langle 0^K, \dots, 0^K, o^j, 0^K \dots, 0^K \rangle \in \mathbf{R}^M$.

We state Theorem AS without proof. It says that, under Assumptions AS1 and AS2, we need consider only the projection of the economy E to the space of commodities in K and M, to identify all the feasible states of E. This is because, under Assumptions AS1 and AS2, at any feasible state of such a projected economy, the amounts of the nonproducible public goods available to consumers and firms in I, after the implied usage by firms in J, are consistent with the (unprojected) consumers' preferences and technologies of the firms in I.

Theorem AS: *Under Assumptions AS1 and AS2 if (i) $\langle \langle \tilde{o}^h, \tilde{e}^h \rangle_h, \langle o^i, e^i \rangle_i, \langle o^j, z^j \rangle_j \rangle$ is such that $\langle \tilde{o}^h, \tilde{e}^h \rangle, \langle o^i, e^i \rangle_i$, and $\langle o^j, z^j \rangle_j$ are in the projections of X^h, Y^i and Y^j to the space of commodities in K and M for all $h \in H$, $i \in I$, and $j \in J$, (ii) the first, fourth, and the fifth conditions of (2.1) hold, and (iii) there exists a d^j for all $j \in J$ such that $\langle o^j, d^j, z^j \rangle \in Y^j$, then $\langle \langle \tilde{o}^h, \sum_j d^j + \eta + \sigma, \tilde{e}^h, \rangle_h, \langle o^i, \sum_j d^j + \eta + \sigma, e^i \rangle_i, \langle o^j, d^j, z^j \rangle_j \rangle$ is a feasible state of E.*

3.3. Nonconvexities.

To the extent the set M of producible public goods is not empty, we show, below, that the nonconvexities discussed in Starrett [1972] and Starrett and Zeckhauser [1974] remain pertinent.

3.3.1. Technological Nonconvexities for Firms in I.

For the firms in I, we distinguish between detrimental and beneficial producible public goods. Intuitively, $m \in M$ is detrimental (beneficial) if increases in its level contract (expand) the set of production possibilities of all other commodities.

Definition: The good $m \in M$ is a detrimental (beneficial) producible public good for firm $i \in I$ if $P^i(e^i) \subseteq P^i(\bar{e}^i)$ $\forall$ $\bar{e}^i, e^i \in \mathbf{R}^M_+$ such that $\bar{e}^i_m < e^i_m$ ($\bar{e}^i_m > e^i_m$) and $\bar{e}^i_{m'} = e^i_{m'}, \forall\, m' = 1, \dots, M$, with $m' \neq m$, and *Boundary* $(P^i(e^i)) \subset$ *Boundary* $(P^i(\bar{e}^i))$ for some $\bar{e}^i, e^i \in \mathbf{R}^M_+$ such that $\bar{e}^i_m < e^i_m$ ($\bar{e}^i_m > e^i_m$) and $\bar{e}^i_{m'} = e^i_{m'}, \forall\, m' = 1, \dots, M$, with $m' \neq m$.

We now present a set of assumptions regarding the technologies of firms in I. Notable among these, is Assumption I2. This assumption, on the one hand, provides a general unboundedness condition on the technologies of firms in I in the direction of goods in M and, on the other, is a generalization of Starrett's view that the damage done by detrimental producible public goods is finite: precisely, in his model, shutting down production of ordinary commodities is an option available to the victim firm at every level of the production externality observed. Thus, this is a compelling assumption on the technologies of firms in I.

Assumption I1 (Convexity): Y^i is convex.

Assumption I2: $\bigcap_{e^i \in \mathbf{R}_+^M} P^i(e^i) \neq \emptyset$.

Assumption I3: $P^i(e^i)$ is closed $\forall\ e^i \in \mathbf{R}_+^M$.

Assumption I4 (Detrimental Producible Public Good): $m \in M$ is a detrimental producible public good for $i \in I$.

Assumption I5 (Beneficial Producible Public Good): $m \in M$ is a beneficial producible public good for $i \in I$.

Otani and Sicilian demonstrated the technological nonconvexities associated with detrimental producible public goods by showing that Assumptions I1, I2, I3, and I4 are inconsistent. We demonstrate these nonconvexities by showing that the restricted profit functions of firms in I are nonconcave in the coordinate direction of a detrimental producible public good (Theorem 2). These restricted profit functions for firms in I will be later employed in defining a decentralized equilibrium for the economy.

For all $e^i \in \mathbf{R}_+^M$, let $B^i(e^i)$ denote the barrier cone of $P^i(e^i)$.[12]

Assumption I6: $P^i(e^i)$ is strongly continuous in $e^i \in \mathbf{R}_+^M$.[13]

Assumption I7: $\forall e^i \in \mathbf{R}_+^M$, we have $P^i(e^i)$ is semi-bounded.[14]

Define the restricted profit function of $i \in I$ as a function of $e^i \in \mathbf{R}_+^M$ and $\langle p_o, p_n^i \rangle \in B^i(e^i)$ as

$$\begin{aligned} \pi = \hat{\Pi}^i(e^i, p_o, p_n^i) := \max_{o^i, n^i}\ & p_o \cdot o^i + p_n^i \cdot n^i \\ & \text{subject to} \\ & \langle o^i, n^i \rangle \in P^i(e^i). \end{aligned} \tag{3.1}$$

The argmax of (3.1) is denoted by the function

$$\langle o^i, n^i \rangle = \langle o^i(p_o, p_e^i, p_n^i),\ n^i(p_o, p_e^i, p_n^i) \rangle. \tag{3.2}$$

The following theorem, proof of which can be found in McFadden [1978], presents some properties of the restricted profit function.

[12] As defined by McFadden [1978], the barrier cone $B^i(e^i)$ of a set $P^i(e^i)$ is the set of all prices p such that $p \cdot x$ is bounded above for all $x \in P^i(e^i)$.

[13] A correspondence $P^i(e^i)$ is strongly continuous in e^i if it is continuous and its asymptotic cone is an upper hemicontinuous correspondence. See McFadden.

[14] The set $P^i(e^i)$ is semi-bounded if *int* $B^i(e^i) \neq \emptyset$, where *int* $B^i(e^i)$ is the interior of the set $B^i(e^i)$.

Theorem 1: *For $i \in I$, under Assumptions I2, I3, I4 (I5), I6, and I7, we have (i) for each $e^i \in \mathbf{R}^M_+$, $\hat{\Pi}^i$ is a finite valued, linear homogeneous, convex, and continuous function of $\langle p_o, p^i_n \rangle \in int\ B^i(e^i)$, (ii) for all $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+} int\ B^i(e^i)$, we have $\hat{\Pi}^i$ is non-increasing (non-decreasing) in $e^i_m \in \mathbf{R}_+$, and $\exists \bar{e}^i, \tilde{e}^i \in \mathbf{R}^M_+$ with $\bar{e}^i_m < \tilde{e}^i_m$ and $\bar{e}^i_{m'} = \tilde{e}^i_{m'}, \forall\ m' = 1, \ldots, M$ and $m' \neq m$ such that $\hat{\Pi}^i(\tilde{e}^i, p_o, p^i_n) < \hat{\Pi}^i(\bar{e}^i, p_o, p^i_n)$ ($\hat{\Pi}^i(\tilde{e}^i, p_o, p^i_n) > \hat{\Pi}^i(\bar{e}^i, p_o, p^i_n)$), and (iii) $\hat{\Pi}^i$ is continuous in $e^i \in \mathbf{R}^M_+$ and jointly continuous in $\langle e^i, p_o, p^i_n \rangle \in \mathbf{R}^M_+ \times \cap_{e^i \in \mathbf{R}^M_+} int\ B(e^i)$.*

We now show that the restricted profit functions will not be concave in the coordinate direction of a detrimental producible public good. In other words, given $\langle p_o, p^i_n \rangle$, the hypograph of the restricted profit function in the space of e^i and π is nonconvex.[15]

Theorem 2: *For all $i \in I$,*

(1) under Assumption I2, given any $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+} int\ B^i(e^i)$, there exists $c \in \mathbf{R}$ such that $\hat{\Pi}^i(e^i, p_o, p^i_n) \geq c$ for all $e^i \in \mathbf{R}^M_+$, and

(2) under Assumptions I2, I3, I4, I6, and I7, for all $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+} int\ B^i(e^i)$, the set $\{\langle e^i, \pi \rangle \in \mathbf{R}^{M+1}_+ | \pi \leq \hat{\Pi}^i(e^i, p_o, p^i_n)\}$ is nonconvex.

Proof:
(1) Under Assumption I2, $\exists \langle \bar{o}^i, \bar{n}^i \rangle \in \bigcap_{e^i \in \mathbf{R}^M_+} P^i(e^i)$. Choose any $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+} int\ B^i(e^i)$. Define $c = p_o \cdot \bar{o}^i + p^i_n \cdot \bar{n}^i$. c has the required property from the definition (3.1) of $\hat{\Pi}^i$.
(2) Suppose not. Then for all $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+} int\ B^i(e^i)$, we have $A := \{\langle e^i, \pi \rangle \in \mathbf{R}^{M+1}_+ | \pi \leq \hat{\Pi}^i(e^i, p_o, p^i_n)\}$ is convex. Assumption I4 and conclusion (ii) of Theorem 1 imply that $\exists \bar{e}^i, \tilde{e}^i \in \mathbf{R}^M_+$ with $\bar{e}^i_m < \tilde{e}^i_m$ and $\bar{e}^i_{m'} = \tilde{e}^i_{m'}, \forall\ m' = 1, \ldots, M$ and $m' \neq m$ such that $\tilde{t} := \hat{\Pi}^i(\tilde{e}^i, p_o, p^i_n) < \hat{\Pi}^i(\bar{e}^i, p_o, p^i_n) =: \bar{t}$. From conclusion (iii) of Theorem 1, $\hat{\Pi}^i$ is continuous in e^i. Hence, $\exists \epsilon > 0$ and $\delta > 0$ such that $\hat{\Pi}^i(e^i, p_o, p^i_n) \in N_\epsilon(\bar{t})$ whenever $e^i \in N_\delta(\bar{e}^i)$ and $\tilde{t} < t,\ \forall t \in N_\epsilon(\bar{t})$. From (1) of this theorem $\exists c \in \mathbf{R}$ such that $\langle e^i, c \rangle \in A\ \forall e^i \in \mathbf{R}^M_+$. Choose $\lambda \in [0,1]$ such that $\overset{*}{t} := \lambda c + (1-\lambda)\bar{t} \in N_\epsilon(\bar{t})$. We can freely choose $\overset{*}{e}{}^i \in \mathbf{R}^M_+$ big enough such that $\tilde{e}^i = \lambda \overset{*}{e}{}^i + (1-\lambda)\bar{e}^i$. Hence, by maintained convexity of set A, we have that $\langle \tilde{e}^i, \overset{*}{t} \rangle \in A$. But this means $\overset{*}{t} \leq \tilde{t}$. This contradicts $\overset{*}{t} \in N_\epsilon(\bar{t})$.■

3.3.2. Nonconvexities In Consumption.

Starrett and Zeckhauser argue that detrimental producible public goods can also be sources of nonconvex preferences for consumers in H. The basic idea is the same

[15] Let $a \in \mathbf{R}^n$. We denote the ϵ neighborhood of $a \in \mathbf{R}^n$ by $N_\epsilon(a) \subseteq \mathbf{R}^n$.

as the nonconvexities in the technology sets of firms in I. For all $h \in H$, if for every level of welfare u there exists a particular combination of ordinary commodities that can maintain the welfare of h at u, independent of the level of goods in M, then it can be shown that the no-worse-than-set of consumer h, corresponding to utility level u, is nonconvex. Here is an example (motivated by Starrett and Zeckhauser) that illustrates this.

Example 1: There are two ordinary commodities, swimming and money spent on all other goods, denoted by $\tilde{o}_1^h$ and $\tilde{o}_2^h$, respectively. There is one producible public good, pollution content of the river, denoted by $\tilde{e}^h$. The consumption set is $X^h = \mathbf{R}_+^3$, and the utility function is

$$u^h = \alpha^h \frac{\tilde{o}_1^h}{(\tilde{e}^h + 1)^2} + \beta^h \tilde{o}_2^h, \ \alpha^h > 0, \beta^h > 0. \tag{3.3}$$

This example shows that the pleasure from any positive amount of swimming is adversely affected by the pollution content in the river. However, if h decides not to swim, then he can maintain any level of welfare, independent of the pollution content of the river, as long as he compensates the loss of pleasure from swimming by consuming appropriate amount of the Hicks-Marshall money.

For all $h \in H$, the upper level sets of u^h are denoted by $\geq_u^h (\overset{*}{u}) := \{\langle \tilde{o}^h, \tilde{n}^h, \tilde{e}^h \rangle \in X^h | \ u^h(\tilde{o}^h, \tilde{n}^h, \tilde{e}^h) \geq \overset{*}{u}\}$. Similarly, for all $h \in H$, the strictly upper level sets of u^h are denoted by $>_u^h (\overset{*}{u}) := \{\langle \tilde{o}^h, \tilde{n}^h, \tilde{e}^h \rangle \in X^h | \ u^h(\tilde{o}^h, \tilde{n}^h, \tilde{e}^h) > \overset{*}{u}\}$. We also define restrictions of these sets to appropriate subspaces, *e.g.,* $\geq_u^h (\overset{*}{u}, \tilde{e}^h) := \{\langle \tilde{o}^h, \tilde{n}^h \rangle \in \mathbf{R}_+^{K+L} | \ \langle \tilde{o}^h, \tilde{n}^h, \tilde{e}^h \rangle \in X^h$ and $u^h(\tilde{o}^h, \tilde{n}^h, \tilde{e}^h) \geq \overset{*}{u}\}$.

We distinguish between detrimental and beneficial producible public goods for consumers. Intuitively, $m \in M$ is detrimental (beneficial) for $h \in H$, if at any level of welfare u, increases in its level requires a contraction (an expansion) in the set of combinations of other commodities in order to maintain the level of welfare at u.

Definition: The good $m \in M$ is a detrimental (beneficial) producible public good for consumer $h \in H$ if for all $u \in u^h(X^h)$, we have $\geq_u^h (u, \tilde{e}^h) \subseteq \geq_u^h (u, \bar{\tilde{e}}^h) \ \forall \ \bar{\tilde{e}}^h, \tilde{e}^h \in \mathbf{R}_+^M$ such that $\bar{\tilde{e}}_m^h < \tilde{e}_m^h$ ($\tilde{e}_m^h > \tilde{e}_m^h$) and $\bar{\tilde{e}}_{m'}^h = \tilde{e}_{m'}^h, \forall \ m' = 1, \ldots, M$, with $m' \neq m$, and $\geq_u^h (u, \tilde{e}^h) \subset \geq_u^h (u, \bar{\tilde{e}}^h)$ for some $\bar{\tilde{e}}^h, \tilde{e}^h \in \mathbf{R}_+^M$ such that $\bar{\tilde{e}}_m^h < \tilde{e}_m^h$ ($\bar{\tilde{e}}_m^h > \tilde{e}_m^h$) and $\bar{\tilde{e}}_{m'}^h = \tilde{e}_{m'}^h, \forall \ m' = 1, \ldots, M$, with $m' \neq m$.

Consider a consumer $h \in H$ and $u \in u^h(X^h)$. We now present the relevant assumptions that can axiomatize nonconvexities caused by externalities in consumption.

Assumption H1 (Convexity): $\geq_u^h (u)$ is convex.

Assumption H2: $\bigcap_{\tilde{e}^h \in \mathbf{R}_+^M} \geq_u^h (u, \tilde{e}^h) \neq \emptyset$.

Assumption H3: $\geq_u^h (u, \tilde{e}^h)$ is closed $\forall \ \tilde{e}^h \in \mathbf{R}_+^M$.

Assumption H4 (Detrimental Producible Public Good): $m \in M$ is a detrimental producible public good for $h \in H$.
Assumption H5 (Beneficial Producible Public Good): $m \in M$ is a beneficial producible public good for $h \in H$.

Theorem 3: *Consider a consumer $h \in H$ and $u \in u^h(X^h)$. Assumptions H1, H2, H3, and H4 are inconsistent.*

Proof. Analogous to Otani and Sicilian for technological nonconvexities.■

As argued by Starrett and Zeckhauser, the nonconvexities caused by detrimental producible public goods for preferences of consumers are not so compelling as technological nonconvexities: precisely, Assumption H2 is not so compelling as demonstrated by the following example, which violates it, but satisfies Assumptions H1, H3, and H4.
Example 2: The consumption set is $X^h = \mathbf{R}^3_+$, and the utility function is

$$u^h = \frac{(\tilde{o}^h_1 + \alpha^h)(\tilde{o}^h_2 + \beta^h)}{(\tilde{e}^h + 1)^2},\ \alpha^h > 0, \beta^h > 0. \tag{3.4}$$

3.4. Market Failure.

We now distinguish between two types of producible public goods: by-product public goods (such as pollution and nectar produced in an apple orchard that aids a neighboring bee-keeping farm) and joint-product public goods (such as national defense). We show that (abstracting from issues such as free riding and thin markets) competitive markets will not exist in the case of both detrimental and beneficial by-product public goods, while they cannot be precluded in the case of joint-product public goods.[16]

Suppose the index set of external effects is partitioned into the sets M_1 and M_2. Likewise, we partition any vector of external effects z into $z_1 \in \mathbf{R}^{M_1}$ and $z_2 \in \mathbf{R}^{M_2}$. For all $k \in K$, denote $A^j(o^j_k) = \{z^j_1 \in \mathbf{R}^{M_1} | \ \exists \langle o^j_1, \ldots, o^j_{k-1}, o^j_{k+1}, \ldots, o^j_K, d^j, z^j_2 \rangle$ such that $\langle o^j, d^j, z^j_1, z^j_2 \rangle \in Y^j\}$. Assumptions J1, J1', and J2 are postulated to capture externalities like pollution. Assumptions J1 and J1' are two alternative assumptions that reflect the fact that disposal of these goods is costly. Assumption J2 captures the fact that they are by-products. This is because, their production (or abatement) is correlated to the production of ordinary commodities and joint- product public goods. Hence, their production is only ancillary or secondary to the production of other commodities. For

[16] A discussion of the distinction between by-products and jointly produced goods can be found in Russell and Murty [2002].

example, emission of green house gases is correlated to the use of fossil fuels, which are used in producing ordinary commodities.

Assumption J1 (Costly Disposal 1): For all $\langle o^j, d^j, z_1^j, z_2^j \rangle \in \mathbf{R}^K \times \mathbf{R}_-^L \times \mathbf{R}^{M_2}$, the sets $P^j(o^j, d^j, z_2^j)$ are bounded below.

Assumption J1' (Costly Disposal 2): For all $\langle o^j, d^j, z_1^j, z_2^j \rangle \in \mathbf{R}^K \times \mathbf{R}_-^L \times \mathbf{R}^{M_2}$ if $\circledast\!z_1^j \geq z_1^j$, then $\langle o^j, d^j, \circledast\!z_1^j, z_2^j \rangle \in Y^j$.

Assumption J2 (By-production of Producible Public Goods): Either there exists $k \in K$ such that $A^j(o_k^j) \subseteq A^j(\bar{o}_k^j)$ whenever $o_k^j \geq \bar{o}_k^j$ or there exists $k \in K$ such that $A^j(\bar{o}_k^j) \subseteq A^j(o_k^j)$ whenever $o_k^j \geq \bar{o}_k^j$.[17]

Definition. For all $j \in J$, the index set $M_1 \subseteq M$ comprises of by-products of Y^j if Assumptions J1 and J2 or Assumptions J1' and J2 are true.

On the other hand, joint-product producible public goods are outputs of firms in J whose disposal is free. These include the standard public goods. Assumption J3 is such a free disposability condition.

Assumption J3 (Joint-production of Producible Public Goods): For all $\langle o^j, d^j, z_1^j, z_2^j \rangle \in Y^j$, if $\circledast\!z_2^j \leq z_2^j$, then $\langle o^j, d^j, z_1^j, \circledast\!z_2^j \rangle \in Y^j$.

Definition. For all $j \in J$, the index set $M_2 \subseteq M$ comprises of joint-products of Y^j if Assumption J3 holds.

Example 3: There is a single ordinary commodity guns (a private good), denoted by o_1, produced as a (main) output of a firm in j, which employs labor o_2 and coal o_3 as inputs. Thus, coal causes the by-product, smoke, denoted by z_1. The firm also produces another (main) joint-product z_2, namely, equipment that is bought for national defense–a public good. In addition, it buys d amount of disposal capacity of the atmosphere to discharge smoke. The technology of the firm is given by

$$Y = \left\{ \langle o, d, z_1, z_2 \rangle \in \mathbf{R}^3 \times \mathbf{R}_- \times \mathbf{R}_+^2 \,\Big|\, \frac{z_2 o_1}{o_2} \leq o_3,\ z_1 \geq (o_3)^2,\ \text{and } d = -z_1 \right\}. \tag{3.5}$$

For all $j \in J$, suppose B^j denotes the barrier cone of Y^j. Define the unrestricted profit function of $j \in J$ for all $\langle p_o, p_d, p_z \rangle \in B^j$ as

$$\begin{aligned} \pi = \Pi^j(p_o, p_d, p_z) := \max_{o^j, d^j, z} \ & p_o \cdot o^j + p_d \cdot (d^j + \sigma^j) + p_z \cdot z \\ & \text{subject to} \\ & \langle o^j, d^j, z^j \rangle \in Y^j. \end{aligned} \tag{3.6}$$

[17] For example, the first condition reflects greater generation of the externality because of greater production or use of output or input k, while the second condition may reflect the fact that some $k \in K$ is helpful for abatement.

Suppose the solution to (3.6) is given by

$$\langle o^j, d^j, z^j \rangle = \langle o^j(p_o, p_d, p_z,),\ d^j(p_o, p_d, p_z),\ z^j(p_o, p_d, p_z) \rangle. \tag{3.7}$$

It can be easily proved that, because of the free disposability assumption $J3$, there will be no solution to (3.6) if price of any good in M_2 is negative. Similarly, if costly disposability, as captured in Assumption $J1'$ holds, then there will be no solution to (3.6) if price of any good in M_1 is positive. We state this in the theorem below.

Theorem 4: *Suppose Assumption J1' is true. There is no solution to (3.6) if there exists $m \in M_1$ such that $p_{zm} > 0$ or if there exists $m \in M_2$ such that $p_{zm} < 0$.*

For all $i \in I$, $p^i_e \in \mathbf{R}^M$ and $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+}$ *int* $B^i(e^i)$, define the unrestricted profit function

$$\pi = \Pi^i(p_o, p^i_e, p^i_n) := \max_{e^i \in \mathbf{R}^M_+} \ p^i_e \cdot (e^i - \xi) + \hat{\Pi}^i(e^i, p_o, p^i_n). \tag{3.8}$$

The following theorem proves Starrett's conclusion that profits of the externality bearing firms are unbounded (*i.e.,* there is no solution to (3.8)) at positive prices of goods in M_1. This is true for both detrimental and beneficial by-product public goods. The reason for this result is Assumption I2, the unboundedness condition on technologies of firms in I when M_1 is not an empty set. This condition implies that, in the case of beneficial by-product public goods, any vector of positive prices for these goods does not belong to the barrier cone of the technology, while in the case of detrimental by-product public goods, the nonconvexities kick in.[18]

Theorem 5: *Under Assumptions of Theorem 1, for all $i \in I$, for all $\langle p_o, p^i_n \rangle \in \cap_{e^i \in \mathbf{R}^M_+}$ int $B^i(e^i)$, there is no solution to (3.8) if there exists $m \in M$ such that $p^i_{em} > 0$.*

Proof: Conclusions of Theorems 1 and 2 hold under Assumptions I2, I3, and I6. Note, problem (3.8) can equivalently be re-written as

$$\begin{aligned} \pi = \Pi^i(p_o, p^i_e, p^i_n) := -p^i_e \cdot \xi + & \max_{\langle a^i, b^i \rangle \in \mathbf{R}^{M+1}_+} \ p^i_e \cdot a^i + b^i \\ & \text{subject to} \\ & b^i \leq \Pi^i(e^i, p_o, p^i_n). \end{aligned} \tag{3.9}$$

[18] A similar result can also be demonstrated in the case of consumers. Under Assumption H2, if faced with a negative price for any good in M_1 (that is, if they are paid to consume such a good), then the consumers find that they can *any* level of welfare with their budget constraint, so that their utility maximization has no solution.

Suppose $\langle \overset{*}{a}^i, \overset{*}{b}^i \rangle$ solves (3.9). Then $\Pi^i(\overset{*}{a}^i, p_o, p_n^i) = \overset{*}{b}^i$. By conclusion (1) of Theorem 2 there exists $c \in \mathbf{R}$ such that $c \leq \Pi^i(e^i, p_o, p_n^i)$, $\forall e^i \in \mathbf{R}_+^M$. In particular, $c \leq \overset{*}{b}$. Let $\mathcal{M}^i := \{m \in M | p_{em}^i > 0\}$. For all $m \in \mathcal{M}$, pick $\bar{e}_m^i \in \mathbf{R}_+$ that solves $\sum_{m \notin \mathcal{M}} p_{em}^i \cdot \overset{*}{a}_m^i + \sum_{m \in \mathcal{M}} p_{em}^i \cdot \bar{e}_m^i + c = \overset{*}{b}^i + p_e^i \cdot \overset{*}{a}^i$. ($\bar{e}_m^i$ for $m \in \mathcal{M}$ exist as under our assumptions $\overset{*}{b}^i - c + p_e^i \cdot \overset{*}{a}^i - \sum_{m \notin \mathcal{M}} p_{em}^i \cdot \overset{*}{a}_m^i \geq 0$.) Now freely choose $\tilde{e}_m^i \gg \bar{e}_m^i$ for all $m \in \mathcal{M}$. Define $\tilde{e}^i$ such that the m^{th} component is $\tilde{e}_m^i$ whenever $m \in \mathcal{M}$ and is $\overset{*}{a}_m^i$ whenever $m \notin \mathcal{M}$. Then by conclusion (1) of Theorem 2, $c \leq \Pi^i(\tilde{e}^i, p_o, p_n^i)$ and $p_e^i \cdot \tilde{e}^i + c > p_e^i \cdot \bar{e}^i + c = \overset{*}{b}^i + p_e^i \cdot \overset{*}{a}^i$. A contradiction to $\langle \overset{*}{a}^i, \overset{*}{b}^i \rangle$ solves (3.9).■

Thus, in the case of $m \in M_1$, we find that, if all firms in J satisfy Assumption J1', then at a price $p_{zm} > 0$ (that is, at a positive price received by firms in J for producing the m^{th} good) they would each like to supply an infinite amount of the m^{th} public good. On the other hand, if price $p_{zm} < 0$ (that is, when the firms in J *pay* a positive price for producing the m^{th} good), then the firms in I receive a positive price for consuming the m^{th} good ($p_{em}^i > 0$). Hence, the firms in I would choose to consume an infinite amount of the m^{th} public good. This demonstrates the market failure in the case of both beneficial and detrimental by-product public goods.

4. Collective Consumption Equilibrium.

To the extent the set M_1 of by-product producible public goods is non empty, we are once again confronted with Hurwicz's negative result for the existence of non-wasteful decentralized mechanisms for nonconvex economies. Failure of competitive markets for goods in M_1 was demonstrated in the previous section (precisely, a market equilibrium cannot exist even if property rights were well established). In fact, any decentralized mechanism based purely on transmission of information regarding shadow prices will not ensure efficiency of the equilibrium in nonconvex economies. This is because, while such mechanisms may ensure that the first order conditions of Pareto optimality hold at an equilibrium, in nonconvex economies the second order conditions for even a local Pareto optimum may fail at the equilibrium. Thus, at these stationary points, there may exist adjustments in the underlying allocations that are Pareto improving, that is, if put to a vote, such adjustments would be unanimously accepted by all consumers.

We feel that the concept of a "public competitive equilibrium" proposed by Foley [1967, 1970] offers great scope for constructing efficient decentralized procedures in the case of producible public goods. Though Foley operated in convex economies, we find that, precisely because his mechanism allows decentralized choice based on both price and quantity signals, it permits a concept of an equilibrium for general (including nonconvex) economies that will always be efficient.

For all $h \in H$, let $B^h(\tilde{e}^h) := \{\langle p_o, \tilde{p}_n^h\rangle \in \mathbf{R}^{K+L} | p_o\tilde{o}^h + \tilde{p}_n^h\tilde{n}^h$ is bounded below for all $\langle \tilde{o}^h, \tilde{n}^h\rangle \in \succeq_{u^h}^h (u, \tilde{e}^h),\ \forall u\}$. For all $\tilde{e}^h \in \mathbf{R}_+^M$, the externality restricted indirect utility function of consumer $h \in H$ is a function of his income $r^h \in \mathbf{R}_+$ and prices $\langle p_o, \tilde{p}_n^h\rangle \in B^h(\tilde{e}^h)$ such that

$$\begin{aligned} u^h = V^h(p_o, \tilde{p}_n^h, r^h, e^h) := \max_{\tilde{o}^h, \tilde{n}^h}\ & u^h(\tilde{o}^h, \tilde{n}^h, e^h) \\ \text{subject to} & \\ \langle \tilde{o}^h, \tilde{n}^h, e^h\rangle \in X^h,\ & \text{and} \\ p_o\tilde{o}^h + \tilde{p}_n^h\tilde{n}^h \le r^h. & \end{aligned} \tag{4.1}$$

Suppose the argmax of (4.1) is given by the functions

$$\langle \tilde{o}^h, \tilde{n}^h\rangle = \langle \tilde{o}^h(p_o, \tilde{p}_n^h, r^h, e^h), \tilde{n}^h(p_o, \tilde{p}_n^h, r^h, e^h)\rangle. \tag{4.2}$$

If u^h is continuous, then V^h is continuous in its arguments.[19] Denote a price system by the vector $p = \langle p_o, (\tilde{p}_n^h)_{h\in H}, (p_n^i)_{i\in I}, p_d, p_z\rangle$.

The equilibrium that will be defined in this section involves a separation of the finance from the production of public goods:[20] an agency (perhaps the government) is formed via the collective action of individual consumers to manage their collective needs of the public good. This agency takes the market prices of the public goods as given and collects contributions from consumers to finance the purchase of these goods from the producers.[21]

Definition. A government proposal relative to the price system p is a profile of individual lump-sum taxes for consumers $T := (T^h)_h \in \mathbf{R}^H$ and levels of externalities $e \in \mathbf{R}_+^M$, denoted by $\langle T, e\rangle$, such that $\sum_h T^h = p_z(e - \xi)$.

We derive a private ownership economy from the economy E. The structure of ownership of resources is as described in section 2 and $\langle \theta^{hi}, \theta^{hj}\rangle_{h,i,j}$ denotes the profile of shares of consumers in the profits of the firms. The private ownership economy will be denoted by $E_{pvt} = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, (\omega^h, \theta^{hi}, \theta^{hj}, \sigma^j)_{h,i,j}, \eta, \xi\rangle$.

[19] See Diewert [1974].

[20] This is true also of most equilibrium concepts in the public goods literature, *e.g.,* the Lindahl equilibrium can be interpreted in terms of both creation of personalized markets for the public goods and of an agency collecting personalized contributions from the consumers of public good to finance the purchase from the producers.

[21] In the context of detrimental external effects, this amounts to the agency choosing the level of rights to pollute to sell to the generating firms and redistributing proceeds to consumers as transfers.

The income of any $h \in H$ at a given price system p, a given distribution of profit shares $\langle(\theta^{hi})_{h,i}, (\theta^{hj})_{h,j}\rangle$, and a government proposal $\langle T, e\rangle$ is

$$\begin{aligned} r^h &= r^h(p, e, T^h) \\ &= p_o\ \omega^h + \tilde{p}_n^h \eta + \sum_i \theta^{hi}\hat{\Pi}^i(p_o, p_n^i, e) + \sum_j \theta^{hj}\Pi^j(p_o, p_d, p_z) - T^h,\ \forall h, \end{aligned} \tag{4.3}$$

where, for all $i \in I$ and $j \in J$, $\hat{\Pi}^i(p_o, p_n^i, e)$ and $\Pi^j(p_o, p_d, p_z)$ are defined as in (3.1) and (3.6), respectively. The externality restricted indirect utility function of consumer $h \in H$ is obtained from (4.1) as $u^h = V^h(p_o, \tilde{p}_n^h, e, r^h(p, e, T^h))$.

Budget proposals relative to the current system of prices are offered to all consumers by the collective action. A budget proposal is accepted iff it is not unanimously rejected by all the consumers in favor of all other government budget proposal relative to the current system of prices.

Definition. A government budget proposal $\langle T, e\rangle$ relative to price system p, is unanimously rejected in favor of another government budget proposal $\langle \overset{*}{T}, \overset{*}{e}\rangle$ relative to price system p if, for all $h \in H$, we have

$$V^h(p, e, r^h(p, e, T^h)) < V^h(p, \overset{*}{e}, r^h(p, \overset{*}{e}, \overset{*}{T}{}^h)). \tag{4.4}$$

Thus, the collective action expresses the collective demand for the public goods by the consumers at the prevailing market prices for these goods. It does this by aggregating over individual consumer preferences by using the unanimity criteria. As Malinvaud [1985] observes, "the economy will preserve some degree of decentralization with the consumers, the firms, and the 'public authority' acting in a relatively autonomous way". He has aptly called this class of equilibria, "politico-economic equilibria."

Definition. A collective consumption equilibrium (CCE) of $E_{pvt} = \langle(X^h, u^h)_h, (Y^i)_i, (Y^j)_j, (\omega^h, \theta^{hi}, \theta^{hj}, \sigma^j)_{h,i,j}, \eta, \xi\rangle$ is a configuration $\langle p, e, T, (\tilde{o}^h)_h, (o^i)_i, (o^j)_j, (\tilde{n}^h)_h, (n^i)_i, (d^j)_j, (z^j)_j\rangle$ such that (i) $\langle(\tilde{o}^h, \tilde{n}^h)_h, (o^i, n^i)_i, (o^j, d^j, z^j)_j\rangle$ solve (4.2), (3.2), and (3.7), respectively, for $\langle p, e, T\rangle$ and r^h defined as in (4.3) for all $h \in H$; (ii) $\langle e, T\rangle$ is a government budget proposal relative to price system p that is not unanimously rejected in favor of any other government budget proposal relative to price system p; and (iii)

$$\begin{aligned} \sum_h \tilde{o}^h &= \sum_i o^i + \sum_j o^j + \omega, \\ e - \xi &= \sum_j z^j, \\ n - \eta := n^i - \eta = \tilde{n}^h - \eta &= \sum_j (d^j + \sigma^j),\ \forall h \in H, i \in I. \end{aligned} \tag{4.5}$$

Lemma 1. Suppose u^h satisfies local nonsatiation for all $h \in H$ and $\langle p, e, T, (\tilde{o}^h)_h, (o^i)_i, (o^j)_j, (\tilde{n}^h)_h, (n^i)_i, (d^j)_j, (z^j)_j \rangle$ is a CCE of a private ownership economy $E_{pvt} = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, (\omega^h, \theta^{hi}, \theta^{hj}, \sigma^j)_{h,i,j}, \eta, \xi \rangle$ with $n - \eta \neq 0$.[22] Then, we have

$$\sum_h \tilde{p}_n^h - \sum_i p_n^i = p_d. \tag{4.6}$$

Proof. Since, u^h satisfies local nonsatiation for all $h \in H$, the consumer budget constraints hold as equalities. Summing up over consumer budget constraints, employing the definitions of Π^j and $\hat{\Pi}^i$ for all $j \in J$ and $i \in I$ and a government budget proposal relative to p, and using conditions (4.5) we obtain

$$\begin{aligned}
& p_o \sum_h \tilde{o}^h + \sum_h \tilde{p}_n^h n = \sum_i \Pi^i(p_o, p_n^i, e) - \sum_h T^h + \sum_j \Pi^j(p_o, p_d, p_z) + p_o \omega + \sum_h \tilde{p}_n^h \eta \\
& \Rightarrow p_o \sum_h \tilde{o}^h + \sum_h \tilde{p}_n^h [n - \eta] = \sum_i [p_o o^i + p_n^i [n - \eta]] - \sum_h T^h \\
& \qquad + \sum_j [p_o o^j + p_d [d^j + \sigma^j] + p_z z^j] + p_o \omega \\
& \Rightarrow [n - \eta][\sum_h \tilde{p}_n^h - \sum_i p_n^i - p_d] = p_z e - \sum_h T^h = 0 \\
& \Rightarrow \sum_h \tilde{p}_n^h - \sum_i p_n^i = p_d.
\end{aligned} \tag{4.7}$$

4.1. Every CCE is a Pareto Optimum.

We now prove the major result of this paper, *i.e.,* the optimality of a collective consumption equilibrium.

Theorem 6: *Suppose u^h satisfies local nonsatiation for all $h \in H$ and $\langle p, e, T, (\tilde{o}^h)_h, (o^i)_i, (o^j)_j, (\tilde{n}^h)_h, (n^i)_i, (d^j)_j, (z^j)_j \rangle$ with $n - \eta \neq 0$ is a CCE of a private ownership economy $E_{pvt} = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, (\omega^h, \theta^{hi}, \theta^{hj}, \sigma^j)_{h,i,j}, \eta, \xi \rangle$ derived from economy E. Then the CCE is a weak Pareto optimum of economy E.*

[22] The utility function $u(x)$ satisfies local-nonsatiation if for all $\overset{*}{u} \in \mathbf{R}$, we have $\geq_u (\overset{*}{u}) \subseteq Cl \ >_u (\overset{*}{u})$.

Proof: Suppose $\langle p, e, T, (\tilde{o}^h)_h, (o^i)_i, (o^j)_j, (\tilde{n}^h)_h, (n^i)_i, (d^j)_j, (z^j)_j \rangle$ is a CCE of E_{pvt} but it is not a weak Pareto optimum of E. Then there exists a feasible allocation of E, $\langle (\overset{*}{x}^h)_h, (\overset{*}{y}^i)_i, (\overset{*}{y}^j)_j \rangle$, such that $u^h(\overset{*}{o}^h, \overset{*}{e}^h, \overset{*}{n}^h) > u^h(\tilde{o}^h, \tilde{e}^h, \tilde{n}^h)$ for all h. Since,[23]

$$\begin{aligned} \sum_h \overset{*}{o}^h &= \sum_i \overset{*}{o}^i + \sum_j \overset{*}{o}^j + \omega, \\ \overset{*}{e}^h = \overset{*}{e} &= \sum_j \overset{*}{z}^j + \xi, \ \forall h, \\ \overset{*}{n}^h = \overset{*}{n} &= \sum_j \overset{*}{d}^j + \eta, \ \forall h, \end{aligned} \tag{4.8}$$

we have

$$\begin{aligned} p_o \sum_h \overset{*}{o}^h + p_z \overset{*}{e} + p_d[\overset{*}{n} - \eta] &= p_o \sum_i \overset{*}{o}^i + p_o \sum_j \overset{*}{o}^j + p_o\omega + p_z[\sum_j \overset{*}{z}^j + \xi] + p_d \sum_j \overset{*}{d}^j \\ &\le \sum_j \Pi^j(p_o, p_d, p_z) + p_o \sum_i \overset{*}{o}^i + \sum_i p_n^i[\overset{*}{n} - \eta] - \sum_i p_n^i[\overset{*}{n} - \eta] + p_o\omega + p_z\xi \\ &\le \sum_i \Pi^i(p_o, p_n^i, \overset{*}{e}) + \sum_j \Pi^j(p_o, p_d, p_z) - \sum_i p_n^i[\overset{*}{n} - \eta] + p_o\omega + p_z\xi. \end{aligned} \tag{4.9}$$

Hence, employing (4.6), we have

$$p_o \sum_h \overset{*}{o}^h - \sum_i \Pi^i(p_o, p_n^i, \overset{*}{e}) + \sum_h \tilde{p}_n^h \overset{*}{n} \le \sum_j \Pi^j(p_o, p_d, p_z) - p_z[\overset{*}{e} - \xi] + p_o\omega + \sum_h \tilde{p}_n^h \eta. \tag{4.10}$$

On the other hand, local non-satiation of consumer preferences implies that the aggregate budget constraint of the consumers holds as an equality.[24]

$$p_o \sum_h \tilde{o}^h - \sum_i \Pi^i(p_o, p_n^i, e) + \sum_h T^h + \sum_h \tilde{p}_n^h n = \sum_j \Pi^j(p_o, p_d, p_z) + p_o\omega + \sum_h \tilde{p}_n^h \eta. \tag{4.11}$$

Subtracting (4.10) from (4.11), we obtain

$$\sum_h T^h + p_o \sum_h [\tilde{o}^h - \overset{*}{o}^h] - \sum_i [\Pi^i(p_o, p_n^i, e) - \Pi^i(p_o, p_n^i, \overset{*}{e})] + \sum_h \tilde{p}_n^h [n - \overset{*}{n}] \ge p_z[\overset{*}{e} - \xi]. \tag{4.12}$$

For all h, choose

$$\overset{*}{T}^h = T^h + p_o[\tilde{o}^h - \overset{*}{o}^h] - \sum_i \theta^{hi}[\Pi^i(p_o, p_n^i, e) - \Pi^i(p_o, p_n^i, \overset{*}{e})] + \tilde{p}_n^h[n - \overset{*}{n}]. \tag{4.13}$$

---

23 For convenience, we have assumed $\sigma = 0$.

24 $n := \tilde{n}^h = n^i, \ \forall i \in I$ and $\forall h \in H$.

Then, it follows from (4.12) and (4.13) that

$$\sum_h \overset{*}{T}^h \geq p_z[\overset{*}{e} - \xi]. \tag{4.14}$$

Further, from (4.13) and (4.11), we have

$$\begin{aligned}\overset{*}{T}^h + p_o\overset{*}{\tilde{o}}^h - \sum_i \theta^{hi}\Pi^i(p_o, p_n^i, \overset{*}{e}) + \tilde{p}_n^h\overset{*}{\tilde{n}} &= T^h + p_o\tilde{o}^h - \sum_i \theta^{hi}\Pi^i(p_o, p_n^i, e) + \tilde{p}_n^h n \\ &= \sum_j \theta_j^h \Pi^j(p_o, p_d, p_z) + p_o\omega^h + \tilde{p}_n^h\eta.\end{aligned} \tag{4.15}$$

This implies that, for all $h \in H$, we have

$$p_o\overset{*}{\tilde{o}}^h + \tilde{p}_n^h\overset{*}{\tilde{n}} = \sum_i \theta^{hi}\Pi^i(p_o, p_n^i, \overset{*}{e}) \sum_j \theta_j^h \Pi^j(p_o, p_d, p_z) + p_o\omega^h + \tilde{p}_n^h\eta - \overset{*}{T}^h. \tag{4.16}$$

Thus, the bundle $\langle \overset{*}{\tilde{o}}^h, \overset{*}{\tilde{n}}^h \rangle$ is affordable with income $r^h(p, \overset{*}{T}^h, \overset{*}{e})$ for all $h \in H$. Now define A such that

$$\sum_h \overset{*}{T}^h = A + p_z[\overset{*}{e} - \xi]. \tag{4.17}$$

From (4.14), we have $A \geq 0$. Define, for all $h \in H$,

$$\hat{T}^h = \overset{*}{T}^h - \frac{A}{H}. \tag{4.18}$$

Then, for all $h \in H$, we have

$$r^h(p, \hat{T}^h, \overset{*}{e}) = r^h(p, \overset{*}{T}^h, \overset{*}{e}) + \frac{A}{H} \geq r^h(p, \overset{*}{T}^h, \overset{*}{e}), \tag{4.19}$$

and

$$V^h(p, \overset{*}{e}, r^h(p, \hat{T}^h, \overset{*}{e}) \geq V^h(p, \overset{*}{e}, r^h(p, \overset{*}{T}^h, \overset{*}{e}) \geq u^h(\overset{*}{\tilde{o}}^h, \overset{*}{\tilde{n}}^h, \overset{*}{e}) > u^h(\tilde{o}^h, \tilde{n}^h, e). \tag{4.20}$$

Also,

$$\sum_h \hat{T}^h = p_z[\overset{*}{e} - \xi]. \tag{4.21}$$

Thus, we have created a government budget proposal, $\langle \hat{T}, \overset{*}{e} \rangle$ relative to price system p such that the proposal $\langle T, e \rangle$ will be unanimously rejected in favor of $\langle \hat{T}, \overset{*}{e} \rangle$. This contradicts the fact that $\langle p, e, T, (\tilde{o}^h)_h, (o^i)_i, (o^j)_j, (\tilde{n}^h)_h, (n^i)_i, (d^j)_j, (z^j)_j \rangle$ is a CCE of E_{pvt}.■

4.2. Decentralization of a Pareto Optimum as a Restricted Collective Consumption Equilibrium.

Because of the technological nonconvexities that can arise for firms in I, we find that a given Pareto optimum of E is only a *restricted* collective consumption equilibrium, that is, it is a collective consumption equilibrium of a restricted economy where the set of government budget proposals, from which the consumers vote, is restricted to those for which the levels of the public goods are fixed (in this case, at the Pareto optimal levels). So consumers accept or reject based on the distribution of the contributions alone.

Definition. A restricted collective consumption equilibrium (RCCE) of $E_{pvt} = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, (\omega^h, \theta^{hi}, \theta^{hj}, \sigma^j)_{h,i,j}, \eta, \xi\rangle$ is a configuration $\langle p, e, T, (\tilde{o}^h)_h, (o^i)_i, (o^j)_j, (\tilde{n}^h)_h, (n^i)_i, (d^j)_j, (z^j)_j\rangle$ such that (i) $\langle (\tilde{o}^h, \tilde{n}^h)_h, (o^i, n^i)_i, (o^j, d^j, z^j)_j\rangle$ solve (4.2), (3.2), and (3.7), respectively, for $\langle p, e, T\rangle$ and r^h defined as in (4.3) for all $h \in H$; (ii) $\langle e, T\rangle$ is a government budget proposal relative to price system p that is not unanimously rejected in favor of any other government budget proposal $\langle \bar{e}, \bar{T}\rangle$ relative to price system p with $\bar{e} = e$; and (iii)

$$\begin{aligned} \sum_h \tilde{o}^h &= \sum_i o^i + \sum_j o^j + \omega, \\ e - \xi &= \sum_j z^j, \\ n - \eta := n^i - \eta = \tilde{n}^h - \eta &= \sum_j (d^j + \sigma^j), \ \forall h \in H, i \in I. \end{aligned} \tag{4.22}$$

Theorem 7: *Suppose $\langle (\overset{*}{x}{}^h)_h, (\overset{*}{y}{}^i)_i, (\overset{*}{y}{}^j)_j\rangle$ is a weak Pareto optimum of E and the following assumptions hold:*

(i) $\overset{*}{x}{}^h \in int\ X^h$, for all $h \in H$,[25]

(ii) u^h is quasi-concave, locally nonsatiated, and continuous, for all $h \in H$,

(iii) Y^j is convex, closed, and int $Y^j \neq \emptyset$ for all $j \in J$, and

(iv) Y^i is closed, int $Y^i \neq \emptyset$, and $P^i(e^i)$ is convex for all $e^i \in \mathbf{R}^M_+$ and for all $i \in I$,

Then for every distribution of initial resources (including collective ownership of ξ and η) and systems of shares $\langle (\omega^h)_h, (\theta^{hi})_{h,i}, (\theta^{hj})_{h,j}, (\sigma^j)_j\rangle$, there exists a price system $\overset{*}{p}$ and taxes $\overset{*}{T}$ such that the implied configuration $\langle \overset{*}{p}, \overset{*}{e}, \overset{*}{T}, (\overset{*}{\tilde{o}}{}^h)_h, (\overset{*}{o}{}^i)_i, (\overset{*}{o}{}^j)_j, (\overset{*}{\tilde{n}}{}^h)_h, (\overset{*}{n}{}^i)_i, (\overset{*}{d}{}^j)_j, (\overset{*}{z}{}^j)_j\rangle$ is a RCCE of $E_{pvt} = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, (\omega^h, \theta^{hi}, \theta^{hj}, \sigma^j)_{h,i,j}, \eta, \xi\rangle$, derived from economy E.

[25] An alternate assumption, called *irreducibility*, can be found in Ghosal and Polemarchakis [1999].

Proof: Assumptions (ii), (iii), and (iv) in the theorem ensure that Assumptions (a) to (d) of Lemma 2 hold. Hence, conclusions of Lemma 2 follow, and let us define $\overset{*}{p}_o = \overset{*}{\rho}_o$, $\overset{*}{p}_z = \overset{*}{\rho}_e$, $\overset{*}{p}{}^h_n = \overset{*}{\rho}{}^h_n$, for all $h \in H$, $\overset{*}{p}{}^i_n = \overset{*}{\rho}{}^i_n$, for all $i \in I$, and $\overset{*}{p}_d = \overset{*}{\rho}_n$.
Pick any distribution of endowments and profit shares $\langle(\omega^h)_h, (\theta^{hi})_{h,i}, (\theta^{hj})_{h,j}, (\sigma^j)_j\rangle$. Since $\overset{*}{\rho}{}^j \in N(Y^j, \overset{*}{y}{}^j)$ for all $j \in J$ and Y^j is convex, from Lemma A.8 it follows that $\Pi^j(\overset{*}{p}_o, \overset{*}{p}_d, \overset{*}{p}_z) = \overset{*}{p}_o \cdot \overset{*}{\delta}{}^j + \overset{*}{p}_d \cdot [\overset{*}{d}{}^j + \sigma^j] + \overset{*}{p}_z \cdot \overset{*}{z}{}^j$. Since $\overset{*}{\rho}{}^i \in N(Y^i, \overset{*}{y}{}^i)$ and $P^i(\overset{*}{e}{}^i)$ is convex for all $i \in I$, it follows from Lemmas A.8 and A.9 that $\hat{\Pi}^i(\overset{*}{p}_o, \overset{*}{p}{}^i_n, \overset{*}{e}) = \overset{*}{p}_o \cdot \overset{*}{\delta}{}^i + \overset{*}{p}{}^i_n \cdot [\overset{*}{n}{}^i - \eta]$.
For all $h \in H$, we define $\overset{*}{T}{}^h$, $\overset{*}{A}{}^h$, and $\overset{*}{r}{}^h$ as

$$
\begin{aligned}
-\overset{*}{T}{}^h &= \overset{*}{p}_o \overset{*}{o}{}^h + \overset{*}{p}{}^h_n \overset{*}{n} - \sum_i \theta^{hi} \hat{\Pi}^i(\overset{*}{p}_o, \overset{*}{p}{}^i_n, \overset{*}{e}) - \sum_j \theta^{hj} \Pi^j(\overset{*}{p}_o, \overset{*}{p}_d, \overset{*}{p}_z) - \overset{*}{p}_o \omega - \overset{*}{p}{}^h_n \eta \\
&=: \overset{*}{p}_o \overset{*}{o}{}^h + \overset{*}{p}{}^h_n \overset{*}{n} - \overset{*}{A}{}^h - \sum_i \theta^{hi} \hat{\Pi}^i(\overset{*}{p}_o, \overset{*}{p}{}^i_n, \overset{*}{e}), \text{ and} \qquad (4.23) \\
\overset{*}{r}{}^h &:= r^h(\overset{*}{p}{}^h, \overset{*}{e}, \overset{*}{T}{}^h) = -\overset{*}{T}{}^h + \overset{*}{A}{}^h + \sum_i \theta^{hi} \hat{\Pi}^i(\overset{*}{p}_o, \overset{*}{p}{}^i_n, \overset{*}{e}).
\end{aligned}
$$

For all $h \in H$ let $u^h(\overset{*}{x}{}^h) = \overset{*}{u}{}^h$. Since $-\overset{*}{\rho}{}^h \in N(\geq^h_u (\overset{*}{u}{}^h), \overset{*}{x}{}^h)$, X^h is convex, u^h satisfies local nonsatiation and is continuous and quasiconcave for all $h \in H$, we have $\geq^h_u (\overset{*}{u}{}^h)$ is convex, $\overset{*}{x}{}^h$ is a boundary point of $\geq^h_u (\overset{*}{u}{}^h)$, and $\overset{*}{\rho}{}^h \cdot \overset{*}{x}{}^h \leq \overset{*}{\rho}{}^h \cdot x^h$ for all $x^h \in \geq^h_u (\overset{*}{u}{}^h)$.[26] From Lemmas A.8 and A.9 it follows that, for all $h \in H$, $\geq^h_u (\overset{*}{u}{}^h, \overset{*}{e})$ is also convex, $-\langle \overset{*}{\rho}{}^h_o, \overset{*}{\rho}{}^h_n \rangle \in N(\geq^h_u (\overset{*}{u}{}^h, \overset{*}{e}), \langle \overset{*}{o}{}^h, \overset{*}{n}{}^h \rangle)$, and $\overset{*}{\rho}{}^h_o \cdot \overset{*}{o}{}^h + \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}{}^h \leq \overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h$ for all $\langle \tilde{o}^h, \tilde{n}^h \rangle \in \geq^h_u (\overset{*}{u}{}^h, \overset{*}{e})$. In addition, since $\overset{*}{x}{}^h \in int\ X^h$ for all $h \in H$, we have $\langle \overset{*}{o}{}^h, \overset{*}{n}{}^h \rangle \in int\{\langle \tilde{o}^h, \tilde{n}^h \rangle |\ \langle \tilde{o}^h, \tilde{n}^h, \overset{*}{e} \rangle \in X^h\}$. Hence, for all $h \in H$, we have $u^h(\overset{*}{o}{}^h, \overset{*}{n}{}^h, \overset{*}{e}) \geq u^h(\tilde{o}^h, \tilde{n}^h, \overset{*}{e})$ for all $\langle \tilde{o}^h, \tilde{n}^h \rangle \in \{\langle \tilde{o}^h, \tilde{n}^h \rangle |\ \langle \tilde{o}^h, \tilde{n}^h, \overset{*}{e} \rangle \in X^h\}$ such that $\overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h \leq \overset{*}{\rho}{}^h_o \cdot \overset{*}{o}{}^h + \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}{}^h = \overset{*}{r}{}^h$ (the last equality follows from (4.23)).[27] Thus, $V^h(\overset{*}{p}_o, \overset{*}{p}{}^h_n, \overset{*}{e}, \overset{*}{r}{}^h) = \overset{*}{u}{}^h$ for all $h \in H$.
Thus, the configuration $\langle \overset{*}{p}, \overset{*}{e}, \overset{*}{T}, (\overset{*}{o}{}^h)_h, (\overset{*}{\delta}{}^i)_i, (\overset{*}{\delta}{}^j)_j, (\overset{*}{n}{}^h)_h, (\overset{*}{n}{}^i)_i, (\overset{*}{d}{}^j)_j, (\overset{*}{z}{}^j)_j \rangle$ satisfies parts (i) and (iii) of the definition of a RCCE of E_{pvt}. We now show that part (ii) of this definition is also satisfied.
First note, local nonsatiation of u^h for all $h \in H$, feasibility of the Pareto allocation, and the fact that $\overset{*}{p}_d = \sum_h \overset{*}{p}{}^h_n - \sum_i \overset{*}{p}{}^i_n$ at the Pareto optimum (this follows from Lemma 2) imply, as a consequence of the Walras law, that $\sum_h \overset{*}{T}{}^h + \overset{*}{p}_z[\overset{*}{e} - \xi] = 0$. Suppose, there existed a government budget proposal $\langle \overset{*}{e}, T \rangle$ relative to price system $\overset{*}{p}$ such that

26 See Debreu [1959].
27 See Debreu [1959].

the government budget proposal $\langle \overset{*}{e}, \overset{*}{T} \rangle$ is unanimously rejected in favor of government budget proposal $\langle \overset{*}{e}, T \rangle$.

Suppose, for all $h \in H$, $\langle \tilde{o}^h, \tilde{n}^h \rangle$ solves (4.2) for price system $\overset{*}{p}$ and income $r^h = r^h(\overset{*}{p}, T^h, \overset{*}{e})$. Since, $\langle \overset{*}{o}^h, \overset{*}{n}^h \rangle$ solves (4.2) for price system $\overset{*}{p}$ and income $\overset{*}{r}^h = r^h(\overset{*}{p}, \overset{*}{T}^h, \overset{*}{e})$ and local nonsatiation is true, we have

$$
\begin{aligned}
\overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h &= \sum_i \theta^{hi} \hat{\Pi}^i(\overset{*}{p}_o, \overset{*}{p}{}^i_n, \overset{*}{e}) - \sum_j \theta^{hj} \Pi^j(\overset{*}{p}_o, \overset{*}{p}_d, \overset{*}{p}_z) - \overset{*}{p}_o \omega - \overset{*}{p}{}^h_n \eta - T^h \\
&=: B - T^h, \text{ and} \\
\overset{*}{\rho}{}^h_o \cdot \overset{*}{o}^h + \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}^h &= \sum_i \theta^{hi} \hat{\Pi}^i(\overset{*}{p}_o, \overset{*}{p}{}^i_n, \overset{*}{e}) - \sum_j \theta^{hj} \Pi^j(\overset{*}{p}_o, \overset{*}{p}_d, \overset{*}{p}_z) - \overset{*}{p}_o \omega - \overset{*}{p}{}^h_n \eta - \overset{*}{T}^h \\
&=: B - \overset{*}{T}^h.
\end{aligned}
\tag{4.24}
$$

Hence, for all $h \in H$, we have

$$\overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h + T^h = \overset{*}{\rho}{}^h_o \cdot \overset{*}{o}^h + \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}^h + \overset{*}{T}^h. \tag{4.25}$$

Summing up over all h and recalling that $\sum_h T^h = \overset{*}{p}_z \overset{*}{e} = \sum_h \overset{*}{T}^h$, we have

$$\sum_h \overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \sum_h \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h + \overset{*}{p}_z \overset{*}{e} = \sum_h \overset{*}{\rho}{}^h_o \cdot \overset{*}{o}^h + \sum_h \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}^h + \overset{*}{p}_z \overset{*}{e}. \tag{4.26}$$

This implies, from Lemma 2 and our definition of $\overset{*}{p}_z$, that

$$\sum_h \overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \sum_h \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h + [\sum_h \overset{*}{\rho}{}^h_e - \sum_i \overset{*}{\rho}{}^i_e] \overset{*}{e} = \sum_h \overset{*}{\rho}{}^h_o \cdot \overset{*}{o}^h + \sum_h \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}^h + [\sum_h \overset{*}{\rho}{}^h_e - \sum_i \overset{*}{\rho}{}^i_e] \overset{*}{e}. \tag{4.27}$$

Hence, we have

$$\sum_h \left[\overset{*}{\rho}{}^h_o \cdot \tilde{o}^h + \overset{*}{\rho}{}^h_n \cdot \tilde{n}^h + \overset{*}{\rho}{}^h_e \overset{*}{e} \right] = \sum_h \left[\overset{*}{\rho}{}^h_o \cdot \overset{*}{o}^h + \overset{*}{\rho}{}^h_n \cdot \overset{*}{n}^h + \overset{*}{\rho}{}^h_e \overset{*}{e} \right]. \tag{4.28}$$

Since, $\langle \tilde{o}^h, \tilde{n}^h, \overset{*}{e} \rangle \in >^h_u (\overset{*}{u}^h)$ and u^h is continuous, we have $\langle \tilde{o}^h, \tilde{n}^h, \overset{*}{e} \rangle \in int \ >^h_u (\overset{*}{u}^h)$ and there exists $\bar{x} := \langle \bar{\tilde{o}}^h, \bar{\tilde{n}}^h, \overset{*}{e} \rangle \in >^h_u (\overset{*}{u}^h)$ such that $\bar{x}^h \cdot \overset{*}{\rho}^h < \overset{*}{x}^h \cdot \overset{*}{\rho}^h$. This contradicts the fact that, for all $h \in H$, $\overset{*}{x}^h$ is cost minimizing in $\geq^h_u (\overset{*}{u}^h)$ at shadow price vector $\overset{*}{\rho}^h$.■

5. Conclusions.

In this paper, we reconcile Arrow/Starrett and Boyd and Conley general equilibrium models of externality. We argue that the Boyd and Conley externalities arise as a result of missing markets for certain nonproducible goods, which have public good properties for some agents. Boyd and Conley's assumption that these resources are bounded, alleviates the externality problem once Coasian markets are created for these goods. Arrow/Starrett externalities, on the other hand, have been argued as arising because of missing markets for certain producible goods. We show that these will continue to remain as sources of technological nonconvexities. To the extent these are prevalent, Hurwicz's impossibility result in finding *efficient* finite dimensional decentralized mechanisms applies. In this paper we propose an efficient decentralized mechanism motivated by the concept of a "public competitive equilibrium" proposed by Foley [1967, 1970]. Precisely because his mechanism allows decentralized choice based on both price and quantity signals, it permits a concept of an equilibrium for general (including nonconvex) economies that will always be efficient.

This first-best model also provides a benchmark to study second-best situations when the collective action is subject also to informational constraints in implementing the personalized taxes or transfers in its budget proposals. Further, we conjecture that it could also provide a framework of analysis for studying second-best settings for determining externality (including standard public goods) levels as well as redistribution policies based on an optimal mix of the benefit and ability to pay principles. Thus, these could be motivated by at least two dimensional informational asymmetries: the free rider problem and incentives of agents to incorrectly reveal their true abilities.[28]

Recently too, a public good purchase approach has been proposed by Bradford [2005] and Guesnerie [2005, 2006] for control of climate change. These papers promote the abatement of green house gases as a global public good that can be purchased by an agency of member countries who make contributions to finance this purchase. At a first level, these readings are suggestive of a collective consumption equilibrium in the context of global externalities. But the issues of the global economy impose more game-theoretic structure to this problem.

APPENDIX

Lemma 2, which is employed in proving Theorem 7, uses the Clarke's normal cone (the negative polar cone to the Clarke's tangent cone) to identify the cones of shadow

[28] For an example of second best policies based on two dimensional uncertainties, see Beaudry, Blackorby, and Szalay [2006].

prices associated with a Pareto optimal allocation. We present below some definitions and some properties of these cones, which were used to prove Lemma 2.[29]

Let $Y \subseteq \mathbf{R}^n$ and $y \in Cl\ Y$.[30] We present below the definition and properties in the form of Lemmas of the Clarke's Tangent and Normal Cones. These can be found in Clarke [1975, 1983, 1989], Rockafellar [1978], Khan and Vohra [1987], and Cornet [1989].

Definition: The Tangent Cone for Y relative to y is the set $T(Y,y) := \big\{x \in \mathbf{R}^n \big|\ \forall$ sequences $t_k \to 0$ and $y_k \to y$ with $y_k \in Cl\ Y,\ \exists$ a sequence $x_k \to x$ such that, for all large enough k, $y_k + t_k x_k \in Cl\ Y\big\}$.

Remark A.1: $T(Y,y)$ is a closed and convex cone with vertex 0^n.

Lemma A.1: The Interior of the Tangent Cone for Y relative to y is the set $int\ T(Y,y)$ $:= \big\{x \in \mathbf{R}^n \big| \exists\ \epsilon > 0,\ \eta > 0,\ \delta > 0$ such that, $\forall\ \lambda \in [0,\eta],\ (\{y'\} + \lambda Cl\ N_\epsilon(x)) \subseteq Y,\ \forall\ y' \in (Cl\ Y \cap\ Cl\ N_\delta(y))\big\}$.

Definition: The Negative Polar of a set $A \subseteq \mathbf{R}^n$ is the set $A^- := \big\{p \in \mathbf{R}^n \big| p \cdot x \leq 0\ \forall\ x \in A\big\}$.

Remark A.2: $A^- = (Cl\ A)^-$

Definition: The Normal Cone to Y relative to y is the set $N(Y,y) := T(Y,y)^-$.

Remark A.3: $N(Y,y) = (int\ T(Y,y))^-$ and $N(Y,y)$ is a closed and convex cone with vertex 0^n.

Lemma A.2: Let $Y = \big\{\hat{y} \in \mathbf{R}^n \big| f(\hat{y}) \leq 0\big\}$, where f is continuous. If f is differentiable at y and $f(y) = 0$, then $T(Y,y) = \big\{x \in \mathbf{R}^n \big| x \cdot \nabla f(y) \leq 0\big\}$, and $N(Y,y) = \big\{p \in \mathbf{R}^n \big| p = \lambda \nabla f(y), \lambda \geq 0\big\}$.

Lemma A.3: Let $Y \subseteq Z$, where Z is a closed subset of $\mathbf{R}^n$. Then $T(Y,y) \subseteq T(Z,y)$ and $N(Z,y) \subseteq N(Y,y)$.

Lemma A.4: Let $Y^i \in \mathbf{R}^n$, $int\ Y^i \neq \emptyset,\ i = 1,\ldots,m$, and $y \in \cap_i Cl\ Y^i$.

T Then $\cap_i int\ T(Y^i, y^i) = int\ T(\cap_i Y^i, y)$. If $\cap_i int\ T(Y^i, y^i) \neq \emptyset$, then $N(\cap_i Y^i, y) = \sum_i N(Y^i, y^i)$.

Lemma A.5: Let $K_1,\ldots,K_p$, be p open and non-empty convex cones with vertex 0^n in $\mathbf{R}^n$. Then $\bigcap_{i=1}^p K_i = \emptyset$ iff $\forall\ i = 1,\ldots,p,\ \exists\ q^i \in K_i^-$ with $q^i \neq 0^n$ for some $i = 1,\ldots,p$ such that $\sum_i q^i = 0^n$.

Lemma A.6: Let $Y := \prod_{i=1}^l Y^i$, and $y := \langle y^1,\ldots,y^l\rangle \in Y$, where $y^i \in Y^i \subseteq \mathbf{R}^n, i = 1,\ldots,l$. Then $int\ T(Y,y) = \prod_{i=1}^l int\ T(Y^i,y^i)$, and $N(Y,y) = \prod_{i=1}^l N(Y^i,y^i)$.

Lemma A.7: Suppose Y is convex and $int\ Y \neq \emptyset$. Then $int\ T(Y,y) \neq \emptyset$.

[29] This conceptualization of shadow prices for general cases and other alternative conceptualizations have been extensively used in the literature on nonconvex economies, motivated by issues such as increasing returns. Readings in this literature include Beato [1982], Beato and Mas-Colell [1985], Bonniseau and Cornet [1990], Brown and Heal [1979], Guesnerie (1975), Heal [1999], Khan and Vohra [1987], Quinzii [1992], *etc.*

[30] We denote the closure of a set $Y \in \mathbf{R}^n$ by $Cl\ Y$ and the interior of Y relative to $\mathbf{R}^n$ by $int\ Y$.

Lemma A.8: Suppose Y is convex and y belongs to the boundary of Y. Then for all $a \in N(Y, y)$, the hyperplane with normal a and constant $a \cdot y$ is a supporting hyperplane to Y at y.
Lemma A.9: Suppose we partition y as $y = \langle y_1, y_2 \rangle$, $a = \langle a_1, a_2 \rangle \in N(Y, y)$, and $A_1 := \{\bar{y}_1 | \ \exists \bar{y}_2 \text{ such that } \langle \bar{y}_1, \bar{y}_2 \rangle \in Y\}$. Then $a_1 \in N(A_1, y_1)$.

Ground Work for Lemma 2.

Given the economy $E = \langle (X^h, u^h)_h, (Y^i)_i, (Y^j)_j, \omega, \eta + \sigma, \xi \rangle$, we construct a new economy $\mathcal{E}$ with $K + (M + L)(H + I)$ commodities from the original economy E along the lines of Milleron. A typical vector in $\mathbf{R}^{K+(M+L)(H+I)}$, which will be denoted by N will be written as $\langle o, \tilde{e}^1, \ldots, \tilde{e}^H, \tilde{n}^1, \ldots, \tilde{n}^H, e^1, \ldots, e^I, n^1, \ldots, n^I \rangle$.

The consumption set of consumer h is $\mathcal{X}^h = \{\chi^h \in N_+ \mid \langle o, \tilde{e}^h, \tilde{n}^h \rangle \in X^h, \text{ and } \langle \tilde{e}^{h'}, \tilde{n}^{h'} \rangle = 0, \ \forall h' \neq h\}$. The utility function for consumer $h \in H$ in this economy is derived from u^h in an obvious way as $\mathcal{U}^h$. We define the production set of producer i in a similar way and denote it by $\mathcal{Y}^i$. A typical production bundle for i is denoted by $\gamma^i \in N$. The production set of $j \in J$ is defined as $\mathcal{Y}^j := \{\gamma^j \in N \mid \tilde{e}^h = e^i = z^j \text{ and } \tilde{n}^h = n^i = d^j, \ \forall i \in I \text{ and } h \in H, o = o^j, \text{ and } \langle o^j, z^j, d^j \rangle \in Y^j\}$. The vector of resources in this economy is $\Omega = \langle \omega, \xi, \ldots, \xi, \eta + \sigma, \ldots, \eta + \sigma, \xi, \ldots, \xi, \eta + \sigma, \ldots, \eta + \sigma \rangle \in N$.
Definition: A feasible state for $\mathcal{E} = \langle (\mathcal{X}^h, \mathcal{U}^h)_h, (\mathcal{Y}^i)_i, (\mathcal{Y}^j)_j, \Omega \rangle$ is a tuple $\langle (\chi^h)_h, (\gamma^i)_i, (\gamma^j)_j \rangle$ such that $\chi^h \in \mathcal{X}^h, \ \forall h \in H; \ \gamma^i \in \mathcal{Y}^i, \forall i \in I; \ \gamma^j \in \mathcal{Y}^j, \ \forall j \in J$; and

$$\sum_h \chi^h = \sum_i \gamma^i + \sum_j \gamma^j + \Omega. \tag{5.1}$$

Definition: A feasible state of $\mathcal{E}$ $\langle (\overset{*}{\chi}{}^h)_h, (\overset{*}{\gamma}{}^i)_i, (\overset{*}{\gamma}{}^j)_j \rangle$ is Pareto optimal if there does not exist another feasible state of $\mathcal{E}$ $\langle (\chi^h)_h, (\gamma^i)_i, (\gamma^j)_j \rangle$ such that $\mathcal{U}^h(\chi^h) \geq \mathcal{U}^h(\overset{*}{\chi}{}^h), \ \forall h$ and $\mathcal{U}^h(\chi^h) > \mathcal{U}^h(\overset{*}{\chi}{}^h)$ for some h.

For all h and $\overset{*}{\chi}{}^h \in \mathcal{X}^h$, we define

$$\begin{aligned} R^h(\overset{*}{\chi}{}^h) &:= \{\chi^h \in \mathcal{X}^h | \mathcal{U}^h(\chi^h) \geq \mathcal{U}^h(\overset{*}{\chi}{}^h)\} \text{ and} \\ P^h(\overset{*}{\chi}{}^h) &:= \{\chi^h \in \mathcal{X}^h | \mathcal{U}^h(\chi^h) > \mathcal{U}^h(\overset{*}{\chi}{}^h)\}. \end{aligned} \tag{5.2}$$

Lemma 2 (Support Prices at a Pareto Optimum). Let $\langle (\overset{*}{\chi}{}^h)_h, (\overset{*}{\gamma}{}^i)_i, (\overset{*}{\gamma}{}^j)_j \rangle$ be a Pareto optimal state of $\mathcal{E}$. Suppose $\mathcal{Y}^i$ and $\mathcal{Y}^j$ are closed for all i and j, $\mathcal{U}^h$ is continuous for all h, and the following hold:
(a) $int\ T(R^h(\overset{*}{\chi}{}^h), \overset{*}{\chi}{}^h)) \neq \emptyset, \forall h$,
(b) $int\ T(\mathcal{Y}^i, \overset{*}{\gamma}{}^i) \neq \emptyset, \ \forall i \in I$,
(c) $int\ T(\mathcal{Y}^j, \overset{*}{\gamma}{}^j) \neq \emptyset, \ \forall j \in J$,

(d) There exists h for whom $R^h(\overset{*}{\chi}^h) \subseteq Cl\ P^h(\overset{*}{\chi}^h)$ (existence of locally nonsatiated consumers).

Then for all $h \in H$, $i \in I$, and $j \in J$, there exists $-\overset{*}{\tilde{\rho}}^h \in N(R^h(\overset{*}{\chi}^h), \overset{*}{\chi}^h)$, $\overset{*}{\rho}^i \in N(\mathcal{Y}^i, \overset{*}{\gamma}^i)$, and $\overset{*}{\rho}^j \in N(\mathcal{Y}^j, \overset{*}{\gamma}^j)$, not all equal to 0, such that

1. $\overset{*}{\tilde{\rho}}^h_o = \overset{*}{\rho}^i_o = \overset{*}{\rho}^j_o,\ \forall h, i, j,$
2. $\overset{*}{\rho}^j_e = \sum_h \overset{*}{\tilde{\rho}}^h_e - \sum_i \overset{*}{\rho}^i_e =: \overset{*}{\rho}_e,\ \forall j$ and
3. $\overset{*}{\rho}^j_n = \sum_h \overset{*}{\tilde{\rho}}^h_n - \sum_i \overset{*}{\rho}^i_n =: \overset{*}{\rho}_n,\ \forall j.$

Proof: Let us represent any vector $u \in N$ by $u = \langle (\chi^h)_h, (\gamma^i)_i, (\gamma^j)_j \rangle$. For all $h \in H$ we define the following set in N: $(R^h(\overset{*}{\chi}^h)) := \{u \in N | \chi^h \in R^h(\overset{*}{\chi}^h)\}$. Similarly, we define $P^h(\overset{*}{\chi}^h))$, $(\hat{\mathcal{Y}}^i)$ and $(\hat{\mathcal{Y}}^j)$ for all h, i, and j. Let $W_k := \{u \in N | \sum_h \tilde{o}^h_k \le \sum_i o^i_k + \sum_j o^j_k + \omega_k\}, \forall\ k$, $\tilde{W}^h_e := \{u \in N | \tilde{e}^h - \xi \le \sum_j z^j\}, \forall h$, $W^i_e := \{u \in N | e^i - \xi \le \sum_j z^j\}, \forall i$, $\tilde{W}^h_n := \{u \in N | \tilde{n}^h - \eta \le \sum_j d^j + \sigma\}, \forall h$, $W^i_n := \{u \in N | n^i - \eta \le \sum_j d^j + \sigma\}, \forall i$.
Let $\overset{*}{u} := \langle (\overset{*}{\chi}^h)_h, (\overset{*}{\gamma}^i)_i, (\overset{*}{\gamma}^j)_j \rangle$. It follows from Lemma A.6 and the maintained assumptions (a) to (c) that $int\ T((R^h(\overset{*}{\chi}^h)), \overset{*}{u}) \neq \emptyset,\ \forall h, int\ T((\mathcal{Y}^i), \overset{*}{u}) \neq \emptyset,\ \forall i$, and $int\ T((\mathcal{Y}^j), \overset{*}{u}) \neq \emptyset,\ \forall j$. From Lemma A.2 above it follows that $int\ T(W_k, \overset{*}{u})$, $int\ T(\tilde{W}^h_e, \overset{*}{u}), int\ T(W^i_e, \overset{*}{u}), int\ T(\tilde{W}^h_n, \overset{*}{u})$, and $int\ T(W^i_n, \overset{*}{u})$ are not empty for all k, h, i, and j. Next, note that since $\overset{*}{u}$ corresponds to a Pareto optimum we have[31]

$$V := \bigcap_h (R^h(\overset{*}{\chi}^h)) \bigcap_{\text{h is LNS}} (P^h(\overset{*}{\chi}^h)) \bigcap_i (\mathcal{Y}^i) \bigcap_j (\mathcal{Y}^j) \bigcap_k W_k \bigcap_i \bigcap_{t=e,n} W^i_t \bigcap_h \bigcap_{t=e,n} \tilde{W}^h_t = \emptyset. \tag{5.3}$$

We show that under the maintained assumption (d), this implies that

$$\begin{aligned} \hat{V} := &\bigcap_h int\ T((R^h(\overset{*}{\chi}^h)), \overset{*}{u}) \bigcap_i int\ T((\mathcal{Y}^i), \overset{*}{u}) \bigcap_j int\ T((\mathcal{Y}^j), \overset{*}{u}) \\ &\bigcap_k int\ T(W_k, \overset{*}{u}) \bigcap_i \bigcap_{t=e,n} int\ T(W^i_t, \overset{*}{u}) \bigcap_h \bigcap_{t=e,n} int\ T(\tilde{W}^h_t, \overset{*}{u}) = \emptyset. \end{aligned} \tag{5.4}$$

Suppose not. Then using the definition of the interior of the tangent cone, we can show that $\exists v \in \hat{V}$ and $\langle \delta, \epsilon, \eta \rangle \gg 0$, such that

$$\overset{*}{u} + \lambda N_\epsilon(v) \subseteq \bigcap_h (R^h(\overset{*}{\chi}^h)) \bigcap_i (\mathcal{Y}^i) \bigcap_j (\mathcal{Y}^j) \bigcap_k W_k \bigcap_i \bigcap_{t=e,n} W^i_t \bigcap_h \bigcap_{t=e,n} \tilde{W}^h_t,\ \forall\ \lambda \in [0, \eta]. \tag{5.5}$$

Since there exists h' such that $R^{h'}(\overset{*}{\chi}^{h'}) \subseteq Cl\ P^{h'}(\overset{*}{\chi}^{h'})$, we have $\overset{*}{u} + \lambda N_\epsilon(v) \subseteq Cl\ P^{h'}(\overset{*}{\chi}^{h'})$, $\forall \lambda \in [0, \eta]$. Pick $a \in \overset{*}{u} + \lambda N_\epsilon(v)$ for a $\lambda \in [0, \eta]$. Then $a \in (Cl\ P^{h'}(\overset{*}{\chi}^{h'}))$. For any $\delta > 0$ such that $N_\delta(a) \subseteq \overset{*}{u} + \lambda N_\epsilon(v)$, since a is a limit point of $(Cl\ P^{h'}(\overset{*}{\chi}^{h'}))$, we have

[31] In the expression below, "LNS" stands for "locally nonsatiated".

$N_\delta(a) \cap (P^{h'}(\overset{*}{\chi}^{h'})) \neq \emptyset$. Hence $\exists\ b \in [N_\delta(a) \cap (P^{h'}(\overset{*}{\chi}^{h'}))] \subseteq \overset{*}{u} + \lambda N_\epsilon(v)$. Thus $b \in V$. This is a contradiction to $\overset{*}{u}$ being a Pareto optimal state.

By Lemma A.5, $\exists\ (-\overset{*}{\rho}^h) \in N((R^h(\overset{*}{\chi}^h)), \overset{*}{u})$, $(\overset{*}{\rho}^i) \in N((\mathcal{Y}^i), \overset{*}{u})$, $(\overset{*}{\rho}^j) \in N((\mathcal{Y}^j), \overset{*}{u})$, for all h, i, and j, $\psi_k \in N(W_k, \overset{*}{u})$, $\tilde{\psi}^h_{t=e,n} \in N(\tilde{W}^h_t, \overset{*}{u})$, $\psi^i_{t=e,n} \in N(W^i_t, \overset{*}{u})$, $\forall h$, and i; not all equal to 0, such that

$$\sum_h(\overset{*}{\rho}^h) + \sum_i(\overset{*}{\rho}^i) + \sum_j(\overset{*}{\rho}^j) + \sum_k \psi_k + \sum_{h,t=e,n} \tilde{\psi}^h_{t=e,n} + \sum_{i,t=e,n} \psi^i_{t=e,n} = 0. \tag{5.6}$$

Working through element-by-element of the left-hand side of (5.6) and employing Lemmas A.2 and A.6, we find that $\exists \lambda_k \geq 0\ \forall k$, $\lambda^i_{t=e,n} \geq 0$, $\forall i$, and $\tilde{\lambda}^h_{t=e,n} \geq 0$, $\forall h$ such that

(1') $\overset{*}{\rho}^h_k = \lambda_k$, $\forall k$, and h

(2') $\overset{*}{\rho}^i_k = \lambda_k$, $\forall k$, and i

(3') $\overset{*}{\rho}^j_k = \lambda_k$, $\forall k$, and j

(4') $\overset{*}{\rho}^h_{t=e,n} = \tilde{\lambda}^h_{t=e,n}$,

(5') $\overset{*}{\rho}^i_{t=e,n} = -\lambda^i_{t=e,n}$, and

(6') $\overset{*}{\rho}^j_{t=e,n} = \sum_h \tilde{\lambda}^h_{t=e,n} - \sum_i \lambda^i_{t=e,n}$.

Conclusions of Lemma 2 follow.■

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