APPROXIMATE CORES OF GAMES AND ECONOMIES WITH CLUBS

Alexander Kovalenkov

and

Myrna Holtz Wooders

No 634

WARRICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

Approximate cores of games and economies with clubs.[*]

Alexander Kovalenkov[†]
Department of Economics, Gardner Hall, University of North Carolina,
Chapel Hill, NC 27599-3305, U.S.A.

Myrna Holtz Wooders[‡]
Corresponding Author
Department of Economics, University of Warwick,
Coventry, CV4 7Al, United Kingdom
e-mail: M.Wooders@warwick.ac.uk
phone: (44) 2476 523796
fax: (44) 2476 523032.

January 1997; This revision, March 2002.

Running Head: Approximate cores, club economies
To appear in Journal of Economic Theory

[*]This paper combines two Autonomous University of Barcelona Working Papers, numbers WP 390.97 and 391.97. The authors are grateful to an anonymous referee and to Nizar Allouch for helpful comments.

[†]This research was undertaken with support from the European Union's Tacis ACE Programme 1995. At that time, this author was in the IDEA Ph.D. Program of the Autonomous University of Barcelona. Support by DGICYT, the IBM Fund Award, the Latané Fund, the University of North Carolina Research Council, and the Warwick Centre for Public Economics is gratefully acknowledged.

[‡]The support of the Direccio General d'Universitats of Catalonia, the Social Sciences and Humanities Research Council of Canada, and the Department of Economics of the Autonomous University of Barcelona is gratefully acknowledged.

Abstract

We introduce the framework of parameterized collections of games with and without sidepayments and provide three nonemptiness of approximate core theorems. The parameters bound (a) the number of approximate types of players and the size of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. Our theorems are based on a new notion of partition-balanced profiles and approximately partition-balanced profiles. The results are applied to a new model of an economy with clubs. In contrast to the extant literature, our approach allows both widespread externalities and uniform results.

Journal of Economic Literature Classification Numbers: C71, C78, D71.

Key words: cooperative games; clubs; approximate cores; effective small groups; parameterized collections of games, NTU games.

1 Introduction.

We introduce the notion of parameterized collections of games and show that, under apparently mild conditions, approximate cores of all sufficiently large games without side payments are nonempty. A collection of games is parameterized by (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. All games described by the same parameters are members of the same collection. The conditions required on a parameterized collection of games to ensure nonemptiness of approximate cores are merely that most players have many close substitutes and all or almost all gains to collective activities can be realized by groups of players bounded in size (small group effectiveness). For our final result, we also require per capita boundedness, which simply rules out arbitrarily large average payoff, and that transfers of payoff between players, although not necessarily at a one-to-one rate, are possible. The condition of small group effectiveness may appear to be restrictive but, in fact, in the context of a "pregame," if there are sufficiently many players of each type, then per capita boundedness and small group effectiveness are equivalent (Wooders [67]).[1]

As an application of our research, we develop a new model of an economy with clubs and obtain analogues of our nonemptiness results for games. Following Buchanan [8] and Shubik and Wooders [52], we allow individuals to belong to multiple clubs and clubs may be as large as the entire player set. In contrast to prior research in this area, our model allows utilities from forming a club to be directly affected by the size and composition of the economy containing the club. For example, there may be widespread externalities.

Turning to the motivation for our research, it is well understood that except in highly idealized situations cores of games may be empty and competitive equilibrium of economies may not exist. For example, within the context of an exchange economy, the conditions required for existence of equilibrium typically include convexity, implying infinite divisibility of commodities, and also nonsatiation. Even these two conditions may well not be satisfied; goods are usually sold in pre-specified units and there are some commodities that many individuals prefer not to consume. In economies with congestion, even if preferences and production technologies are convex, the core of the economy may be empty.[2] In the context of economies with coalition structures, such as economies with clubs and/or local public goods, the added difficulties of endogenous group formation compound the problem; even if, given club memberships, all conditions for existence of equilibrium and nonemptiness of the core are satisfied in each club, the core may be empty. One possible approach to the problem of existence of equilibrium is to restrict attention to models where

[1] Similar results hold for games without side payments [64].

[2] An example making this point appears in Shubik and Wooders [54].

equilibria exist, for example, economies with continuums of agents. But a model with a continuum of agents can only be an approximation to a finite economy. Another approach is to consider solution concepts for which existence is more robust, for example, approximate equilibria and cores. It seems reasonable to suppose that there are typically frictions that prevent attainment of an exact competitive equilibrium. At any time, most markets may have some unsatisfied demand or supply and most purchases might be made at prices that are only close to equilibrium prices. It also seems reasonable to suppose that there are typically costs of forming coalitions. These sorts of observations motivate the study of existence of approximate equilibria and nonemptiness of approximate cores.

Besides assumptions on the structure of the economies or games considered, solution theory also requires behavioral assumptions – the competitive equilibrium requires that individuals take prices as given (by some unknown source) and optimize while the core is based on the idea that if a group of individuals can be better off by forming a coalition and reallocating resources and activities within that coalition, then they will do so. These behavioral assumptions are problematic, those of the core perhaps no more so than those of the competitive equilibrium. An alternative to the behavioral assumption of the core that may be easier for economists to swallow is that entrepreneurs form coalitions whenever there exists an opportunity to profit from doing so; there is a long literature taking this approach or closely related approaches, for example, Pauly [42], Shapley and Shubik [49], [58] and Bennett and Wooders [5].[3] The literature on contestable markets, for example, Baumol, Panzer and Willig [4] takes a similar approach; roughly this literature suggests that the presence of entrepreneurs who are ever-ready to enter a market if there is an opportunity to profit ensures that prevailing prices are perfectly competitive. Thus, there is some motivation for both the core and the competitive equilibrium in the idea that if prices/payoffs are not competitive, profit maximizing firms will enter. From the viewpoint of the behavioral assumptions required, there are arguments in favor of both the core and the competitive equilibrium.

Since the seminal papers of Shubik [50], Debreu and Scarf [17] and Aumann [3], equivalence of the core and price-taking equilibrium and approximate equivalence has been shown in a variety of contexts, including in economies with coalition production (Hildenbrand [24] and Böhm [6] for example), in economies with public goods (Conley [9] and Vasil'ev, Weber and Wiesmeth [55] for example), economies with local public goods or clubs (several papers referenced herein – see [69] for further references) and in

[3]Pauly treats the case of essentially identical players and economies with quasi-transferable utilities. Wooders [58] treats multiple types; these results are applied in Bennett and Wooders [5] to problems of firm formation. For NTU games, it can be shown that under certain conditions, the core is equivalent to a "no-entry" equilibrium (a typescript with such results is available on request from the second author).

Shapley and Shubik [49] made an important connection between games with side payments and markets, defined as economies with quasi-linear preferences and concave utility functions.

economies with finite coalitions and widespread externalities (Hammond, Kaneko and Wooders [23] and Hammond [22] for example). Indeed, with quasi-linear utilities, large economies, including those with nonconvexities, indivisibilities, coalition production, public projects and clubs are equivalent to markets if and only if small groups are effective for the realization of all or almost all gains to coalition formation ([67]). All these results suggest that in diverse large economies, approximate cores are nonempty and close to competitive outcomes. These results also all depend on structures of specific economic models. The study of cores of general large games allows us to demonstrate that certain properties of large economies depend only on a few features of the model, most notably, small group effectiveness.[4]

The game theoretic environment we consider may be especially well suited to the core. Since the games treated have many players of each of a relatively few approximate types, informally, we might expect that some random sorting process and bargaining may lead to optimal outcomes. In the context of exchange economies, results in this spirit have been obtained by a number of authors, for example, Dagan, Serrano and Volij [15].

To position our model and results in the literature, recall that Shapley and Shubik [48] showed that large replica exchange economies with quasi-linear preferences have nonempty approximate cores. Under the assumption of per capita boundedness – finiteness of the supremum of average payoff – Wooders [59],[62] demonstrated nonemptiness of approximate cores of large games with and without side payments. Since then, there have been a number of advances in this literature, including Shubik and Wooders [54], Kaneko and Wooders [27], and Wooders and Zame [70]. The prior literature on approximate cores of large games all uses the framework of a pregame. A pregame consists of a compact metric space of player types, possibly finite, and a worth function ascribing a payoff possibilities set to every possible group of players. The worth function depends continuously on the types of players in a coalition. Note that the pregame framework treats collections of games that can all be described by a *single* worth function. This has hidden consequences; for example, as we will illustrate, the equivalence between small group effectiveness and per capita boundedness noted above depends on the structure of a pregame. Moreover, the pregame framework dictates that the payoff set of a coalition cannot depend on the total player set of the game in which it is embedded; widespread externalities are ruled out.[5]

To illustrate how parameterized collections can treat a broader class of situations than pregames, consider, for example, a sequence of economies where the nth economy has n identical players. Suppose that, due to widespread externalities, in the nth economy a group consisting of m players can realize a payoff of $m(1 + 1/n)$. The

[4] This has been the motivation of a continuing line of research on large games. See, for example, the discussion in [58], on-line on the second author's web page.

[5] Two exceptions are [62], which allows positive externalities, and [68]. Other exceptions due to the current authors are subsequent to this paper.

pregame framework rules out such sequences of games, where the payoff to a group of players depends on the total player set in which it is embedded. In contrast, parameterized collections of games incorporate games with widespread externalities and our results can be applied.[6] This example also illustrates that our club-theoretic results cannot be obtained in the pregame context.

In the remainder of this introduction, we first discuss our game-theoretic framework and results in more detail and then discuss economies with clubs. Related literature is discussed in the body of the paper.

1.1 The game-theoretic model and results.

We provide three theorems showing nonemptiness of approximate cores of arbitrary games. Given the specification of an approximate core – the particular approximate core notion and the parameters describing the closeness of the approximation – we obtain a lower bound η on the number of players so that any game in the class of games described by the specified parameters with at least η players has a nonempty approximate core. While our three theorems each use different notions of approximate cores, both the notions of approximate cores and the theorems build on each other. Our framework encompasses games derived from pregames with or without side payments and our results encompass, as special cases, a number of nonemptiness of approximate core results in the literature. In the concluding section of the current paper we remark on other applications of the notion of parametrized collections of games.

Our first result, for the ε-remainder core, requires a finite integer number T of types of players and a bound B on strictly effective group sizes.[7] Roughly, a payoff vector is in the *ε-remainder core* if it is in the core of a subgame containing all but a fraction ε of the players. Our result provides a lower bound, depending on T, B, and ε, on the number of players required to ensure nonemptiness of the ε-remainder core for *all* games with T types and bound B on effective group sizes. An important aspect of this result, like the result of Kaneko and Wooders [27], is that the conclusion is independent of the payoff sets for the games. The result is eminently applicable to models with bounded coalition sizes, such as marriage and matching games (cf., Kelso and Crawford [32] or Roth and Sotomayor [44]).

The *ε-core* of a game is the set of feasible payoff vectors that cannot be improved upon by any coalition of players by at least ε for each member of the coalition. A feasible payoff vector is in the *ε_1-remainder ε_2-core* if it induces a payoff vectors

[6]In spirit, the pregame framework is similar to the economic frameworks of Kannai [31] and Hildenbrand [24], for example, while our approach is more in the spirit of the economic models of Anderson [2] and Manelli [36], [37].

[7]We postpone discussion of the origins of approximate core concepts to their formal introduction, later in the paper.

in the ε_2-core of a subgame containing all but a fraction ε_1 of the players. Our nonemptiness theorem for the ε_1-remainder ε_2-core requires only that groups bounded in size are effective for the realization of *almost* all gains to cooperation. Instead of the assumption of a finite number of types, to show nonemptiness of the ε_1-remainder ε_2-core we require only that there be a partition of the set of players into a finite number of approximate types. Such an assumption would be satisfied by games derived from a pregame with a compact metric space of player types, for example.

Under two additional restrictions on the class of games, we obtain a nonemptiness result for ε-cores. The restrictions are that: (a) per capita payoffs are bounded; and (b) the games are strongly comprehensive in the sense that the boundaries of the total payoff set are bounded away from being "flat". Per capita boundedness dictates that average payoff is bounded. Strong comprehensiveness, called q-comprehensiveness with q bounded away from zero, dictates that it is possible to make transfers to the "remainder" players of the previous theorems. Since per capita payoffs are bounded, in games with sufficiently many players it is possible to transfer sufficiently large amounts of payoff to the remainder players so that they will cannot form an improving coalition. A corollary relaxes assumption (b).

1.2 Economies with clubs.

There are now numerous papers in the literature studying cores and equilibria of economies with local public goods, where a feasible state of the economy includes a partition of the set of agents into disjoint jurisdictions or clubs for the purposes of collective consumption of public goods within each club or jurisdiction.[8] There have been far fewer works on economies where an agent can belong to multiple clubs. In this paper we develop a model of an economy with clubs where: (a) a player may belong to multiple clubs – indeed, as many clubs as there are groups containing that player; (b) all players may differ from each other; (c) each club may provide a unique bundle of goods and/or services, including private goods, public goods subject to exclusion, and conviviality; and (d) the payoff set of a club may depend on the economy in which it is embedded – widespread externalities are permitted.

A club is a group of people who collectively consume and/or produce a bundle of goods and/or services for the members of the club. Often clubs have been treated as synonymous with groups/jurisdictions of players providing congestable and excludable public goods for their members where, in addition, the production possibilities and preferences may depend on the membership of the jurisdiction providing the public goods, 'Tiebout economies'. We observe, however, that clubs engage in a variety of activities. These activities may or may not require input of private goods. The goods

[8]Some early papers include, for example, Wooders [57] and Greenberg and Weber [21]. See Conley and Wooders [13] and Konishi, Le Breton, and Weber [33] for more recent references and [69] for a short survey.

provided by the club may include the enjoyment of the company of the other club members. In clubs of intellectuals, the exchange of ideas may be the aspect of the club that brings enjoyment to its members. Clubs may provide only private goods; for example, many academic departments have coffee clubs. Other clubs offer some goods and/or services to the general public. Some sorts of clubs offer private goods and/or services to their members in addition to public goods. There is frequently no requirement that members of the same club consume the same bundles of goods. Thus, in this paper for each club we assume that there is an abstract set of feasible club activities.[9]

It may be the case that some sorts of clubs are ruled out for legal, technical, or social reasons. For example, a marriage may be viewed as a club, and polyandrous marriages may be illegal. Thus, for each group of players in the economy there is an admissible club structure of that group. Admissible club structures are required to satisfy certain natural properties. In addition, for our final result, our model is required to satisfy the conditions that: (a) average utilities are bounded independently of the size of the economy; and (b) as the economy grows large, increasing returns to club size must become negligible.

Although the conditions on our model are remarkably non-restrictive, by application of our game-theoretic results we are able to show several forms of the result that approximate cores of large economies – with sufficiently many players – are nonempty. Our result applies simultaneously to all games in a parameterized collection.

1.3 Organization of the paper.

The paper is organized as follows. The next section introduces the basic definitions, including the notion of parametrized collections of games. Section 3 presents our three theorems on nonemptiness of approximate cores in the order presented above. Section 4 consists of our club model and results. Section 5 concludes the body of the paper. Appendix contains the proofs that are based on a new mathematical result on approximate balancedness of large profiles of player sets.

[9]The notion of public projects, introduced in Mas-Colell [39] and extended, in 1992, to local public projects in an unpublished paper due to J. Manning.

2 Definitions.

2.1 Cooperative games: description and notation.

Let $N = \{1, ..., n\}$ denote a *set of players.* A nonempty subset of N is called a *group.*[10] For any group S let $\mathbf{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbf{R}^N, x_S$ will denote its restriction to $\mathbf{R}^S$. To order vectors in $\mathbf{R}^S$ we use the symbols $>>$, $>$ and $\geq$ with their usual interpretations. The non-negative orthant of $\mathbf{R}^S$ is denoted by $\mathbf{R}^S_+$ and the strictly positive orthant by $\mathbf{R}^S_{++}$. We denote by $\vec{1}_S$ the vector of ones in $\mathbf{R}^S$, that is, $\vec{1}_S = (1, ..., 1) \in \mathbf{R}^S$. Each group S has a feasible set of payoff or utility vectors denoted by $V_S \subset \mathbf{R}^S$. By agreement, $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is nonempty, closed and bounded from above for any i. In addition, we will assume that

$$\max \left\{ x : x \in V_{\{i\}} \right\} = 0 \text{ for any } i \in N;$$

this is by no means restrictive since it can always be achieved by a normalization.

It is convenient to describe the feasible payoff vectors of a group as a subset of $\mathbf{R}^N$. For each group S let $V(S)$, called the *payoff set for* S, be defined by

$$V(S) := \left\{ x \in \mathbf{R}^N : x_S \in V_S \text{ and } x_i = 0 \text{ for } i \notin S \right\}.$$

A *game without side payments* (called also an *NTU game* or simply a *game*) is a pair (N, V) where the correspondence $V : 2^N \longrightarrow \mathbf{R}^N$ is such that $V(S) \subset \left\{ x \in \mathbf{R}^N : x_i = 0 \text{ for } i \notin S \right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is nonempty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbf{R}^N_+$ is bounded for all $S \subset N$, in the sense that there is a real number $K > 0$ such that if $x \in V(S) \cap \mathbf{R}^N_+$, then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1) + V(S_2) \subset V(S_1 \bigcup S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

We next introduce the uniform version of strong comprehensiveness assumed for our third approximate core result. Roughly, this notion dictates that payoff sets are

[10] In cooperative game theory the term "coalition" is typically used instead of "group"; this is consistent with an emphasis on cooperative behavior. To describe our model, we prefer the term "group" since what a group could do were its members to decide to cooperate is quite distinct from the assumption that the group actually forms a coalition. A referee, however, was concerned that the use of the two terms, "group" to describe simply a subset of players, and "coalition" for a group that forms an alliance, may be confusing so for the remainder of the paper we use only the term group.

both comprehensive and uniformly bounded away from having level segments in their boundaries. Consider a set $W \subset \mathbf{R}^S$. We say that W is *comprehensive* if $x \in W$ and $y \leq x$ implies $y \in W$. The set W is *strongly comprehensive* if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$.[11] Given (i) $x \in \mathbf{R}^S$, (ii) $i, j \in S$, (iii) $0 \leq q \leq 1$ and (iv) $\varepsilon \geq 0$, define a vector $x^q_{i,j}(\varepsilon) \in \mathbf{R}^S$, where

$$\begin{aligned} (x^q_{i,j}(\varepsilon))_i &= x_i - \varepsilon, \\ (x^q_{i,j}(\varepsilon))_j &= x_j + q\varepsilon, \text{ and} \\ (x^q_{i,j}(\varepsilon))_k &= x_k \text{ for } k \in S \backslash \{i, j\}. \end{aligned}$$

The set W is *q-comprehensive* if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \geq 0$.[12] For $q > 0$ this condition uniformly bounds the slopes of the Pareto frontier of payoff sets away from zero. Note that for $q = 0$, 0-comprehensiveness is simply comprehensiveness. Also note that if a set is q-comprehensive for some $q > 0$ then the set is q'-comprehensive for all q' with $0 \leq q' \leq q$.

Now let us consider $V(S) = \left\{x \in \mathbf{R}^N : x_S \in V_S \text{ and } x_i = 0 \text{ for } i \notin S\right\}$. Given q, $0 \leq q \leq 1$, let $W^q_S \subset \mathbf{R}^S$ be the smallest q-comprehensive set that includes the set V_S.[13] For $V(S)$ we define the set $c_q(V(S))$ in the following way:

$$c_q(V(S)) := \left\{x \in \mathbf{R}^N : x_S \in W^q_S \text{ and } x_i = 0 \text{ for } i \notin S\right\}.$$

Notice that for the relevant components – those assigned to the members of S – the set $c_q(V(S))$ is q-comprehensive, but not for other components. With some abuse of terminology, we will call this set the *q-comprehensive cover* of $V(S)$. When $q > 0$ we can think of the payoff set as having some degree of "side-paymentness" or as allowing transfers between players, not necessarily at a one-to-one rate, but at a rate q bounded away from zero (and not greater than one).

A *game with side payments* (also called a *TU game*) is a game (N, V) with 1-comprehensive payoff sets, that is $V(S) = c_1(V(S))$ for any $S \subset N$. This implies that for any $S \subset N$ there exists a real number $v(S) \geq 0$ such that $V_S = \left\{x \in \mathbf{R}^S : \sum_{i \in S} x_i \leq v(S)\right\}$. The numbers $v(S)$ for $S \subset N$ determine a function v mapping the subsets of N to $\mathbf{R}_+$. Then the TU game is represented as the pair (N, v).

[11] Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. This property has also been called "nonleveledness" and "quasi-transferable utility."

[12] The notion of q-comprehensiveness can be found in Kaneko and Wooders [30]. For the purposes of the current paper, q-comprehensiveness can be relaxed on portions of the payoffs sets not contained in the positive orthant.

[13] Notice that there exist q-comprehensive sets that contain V_S, specifically $\mathbf{R}^S$. The set W^q_S is the intersection of all q-comprehensive sets containing V_S.

2.2 Parameterized collections of games.

To introduce the notion of parameterized collections of games we will need the concept of Hausdorff distance. For every two nonempty subsets E and F of a metric space (M,d), define the *Hausdorff distance between E and F* (with respect to the metric d on M), denoted by $dist(E,F)$, as

$$dist(E,F) := \inf\{\varepsilon \in (0,\infty) : \; E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\},$$

where $B_\varepsilon(E) := \{x \in M : d(x,E) \leq \varepsilon\}$ denotes an ε-neighborhood of E.

Since payoff sets are unbounded below and payoffs that assign any player an amount significantly less than zero are not relevant for our results, we will use a modification of the concept of the Hausdorff distance so that the distance between any two payoff sets is the distance between the intersection of the sets and a subset of Euclidean space bounded below. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbf{R}^N$ defined by $M^* := \left\{x \in \mathbf{R}^N : x_i \geq -m^* \text{ for any } i \in N\right\}$. For every two nonempty subsets E and F of Euclidean space $\mathbf{R}^N$ let $H_\infty[E,F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x-y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbf{R}^N$.

The concepts defined below lead to the definition of parameterized collections of games. To motivate the concepts, each is related to analogous concepts in the pregame framework.

$\delta-$substitute partitions. In our approach we approximate games with many players, all of whom may be distinct, by games with finite sets of player types. Observe that for a compact metric space of player types, given any real number $\delta > 0$ there is a partition (not necessarily unique) of the space of player types into a finite number of subsets, each containing players who are "δ-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a finite number of approximate types.

Let (N,V) be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. Formally, given a set $W \subset \mathbf{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1,..,T\}$ of N, a permutation τ of N is *$type-preserving$* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1,..,T\}$ of N with the property that, for any type-preserving permutation τ and any group S,

$$H_\infty\left[V(S), \sigma_\tau^{-1}(V(\tau(S)))\right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

(δ,T)*- type games.* The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game (N, V) is a (δ, T)-*type game* if there is a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N. The set $N[t]$ is interpreted as an *approximate type.* Players in the same element of a δ-substitute partition are δ*-substitutes.* When $\delta = 0$, they are *exact substitutes.*

profiles. Another notion that arises in the study of large games is that of the profile of a player set, a vector listing the number of players of each type in a game. This notion is also employed in the definition of a parameterized collection of games, but profiles are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let (N, V) be a game and let $\{N[t] : t = 1, .., T\}$ be a partition of N into δ-substitutes. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$ and a *subprofile* s of a profile f is a profile satisfying the condition that $s \leq f$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

β*−effective* B*−bounded groups.* In all studies of approximate cores of large games, some conditions are required to limit gains to collective activities, such as per capita boundedness and/or small group effectiveness, as in [59], [62],[66],[67], or the more restrictive condition of boundedness of individual marginal contributions in boundedness of marginal contributions to groups, as in Wooders and Zame [70]. Small groups are effective if all or *almost all* gains to collective activities can be realized by groups bounded in size of membership. The following notion formulates the idea of small effective groups in the context of parameterized collections of games.

Informally, groups of players containing no more than B members are β-effective if, by restricting groups to having fewer than B members, the loss to each player is no more than β. Let (N, V) be a game. Let $\beta \geq 0$ be a given non-negative real number and let B be a given positive integer. For each group $S \subset N$, define a corresponding set $V(S; B) \subset \mathbf{R}^N$ in the following way:

$$V(S; B) := \bigcup \left[\sum_k V(S^k) : \left\{ S^k \right\} \text{ is a partition of } S, \ \left| S^k \right| \leq B \right].$$

The set $V(S;B)$ is the payoff set of the group S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S;B) = V(S)$ for $|S| \leq B$. We might think of $c_q(V(S;B))$ as the payoff set to the group S when groups are restricted to have no more than B members and transfers are allowed between groups in the partition. If the game (N,V) has q-comprehensive payoff sets then $c_q(V(S;B)) \subset V(S)$ for any $S \subset N$. The game (N,V) with q-comprehensive payoff sets has β-effective *B-bounded groups* if for every group $S \subset N$

$$H_\infty \left[V(S), c_q(V(S;B))\right] \leq \beta. \tag{1}$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

<u>*parameterized collections of games $G^q((\delta,T),(\beta,B))$.*</u> With the above definitions in hand, we can now define parameterized collections of games. Let T and B be positive integers, let δ and β be nonnegative real numbers, and let q be a real number, $0 \leq q \leq 1$. Let $G^q((\delta,T),(\beta,B))$ be the collection of all (δ,T)-type games that have q-comprehensive payoff sets and β-effective B-bounded groups.

Our results hold for all parameters δ and β that are sufficiently small, that is, $2(\delta+\beta) < m^*$, where m^* is a positive real number used in the definition of the Hausdorff distance. (Since m^* can be chosen to be arbitrarily large, this requirement is nonrestrictive.)

3 Nonemptiness of approximate cores of games.

For TU games the concept of the core is attributed to Gilles [20] and unpublished research of Lloyd Shapley. The ε-core was introduced for a TU game derived from an exchange economy in Shapley and Shubik [48].[14] The ε-core of an NTU game was introduced in Weber [56],[15] Ichiishi and Schäffer [26] and independently in Wooders [60].

<u>*the core and the ε-core.*</u> Let (N,V) be a game. A payoff vector x is *ε-undominated* if, for all $S \subset N$ and $y \in V(S)$, it is not the case that $y_S >> x_S + \vec{1}_S\varepsilon$. The payoff vector x is *feasible* if $x \in V(N)$. The *ε-core* of a game (N,V) consists of all feasible and ε-undominated payoff vectors. When $\varepsilon = 0$, the ε-core is the *core.*

[14] Related constructs now appear in a number of papers for NTU economies, cf., Kannai [31].

[15] As Weber [56] demonstrates, for games with a continuum of players, extensions of the concepts of balancedness (from finite games) do not necessarily ensure nonemptiness of the core of the game. Thus, even with a continuum player set and with a balancedness condition, further conditions are required.

3.1 The ε-remainder core.

The concept of the ε-remainder core is based on the idea that all requirements of the core should at least be satisfied for almost all players with the remainder of players representing a small fraction of "unemployed" or "underemployed" players. The ε-remainder core was suggested in Shubik [51] for an example involving bridge games and introduced in Wooders [61][16] to demonstrate nonemptiness of "weak" approximate cores and to obtain the results published in Wooders [62].[17] The ε-remainder core subsequently appeared in Shubik and Wooders [54], Kaneko and Wooders [27], and other papers. This approximate core notion is frequently used as a stepping stone to other notions of approximate cores. There are game-theoretic situations, however, in which the notion of the ε-remainder core may naturally arise – for example, the demand games of Selten [46] or multi-sided matching games with bounds on the numbers of players in an effective group

the ε-remainder core. Let (N, V) be a game. A payoff vector x belongs to the *ε-remainder core* if x is feasible and if, for some group $S \subset N$, $\frac{|N|-|S|}{|N|} \leq \varepsilon$ and x_S belongs to the core of the subgame (S, V).

Note that the following theorem requires no restrictions on the degree of comprehensiveness – the usual notion of comprehensiveness suffices.

Theorem 1. (**Nonemptiness of the ε-remainder core**). Let T and B be positive integers and let q be a nonnegative real number satisfying $q \leq 1$. For any $\varepsilon > 0$, there exists an integer $\eta_1(\varepsilon, T, B)$ such that if
(a) $(N, V) \in G^q((0, T), (0, B))$ and
(b) $|N| \geq \eta_1(\varepsilon, T, B)$
then the ε-remainder core of (N, V) is nonempty.

While the assumptions of Theorem 1 are strong – a fixed number T of exact player types and strictly effective groups of size less than or equal to B, they provide a strong conclusion. The Theorem states that for any $\varepsilon > 0$ there exists a lower bound $\eta_1(\varepsilon, T, B)$ on the number of the players such that all games satisfying the

[16]The strategy of the proofs of this paper and [59], [62] is similar to that of the current paper, except in the prior papers the results are restricted to games with a fixed distribution of player types. Thus, to arrive at the final result of these papers, the same sort of intermediate steps as in the current paper are used. From the Lemmas in Wooders ([61],[62]) analogues of the first two nonemptiness results of the current paper, but for sequences of games with a fixed distribution of player types, follow. Our current results are much stronger since the framework of parameterized collections is broader and we do not impose any relationships between individual games in a parameterized collection.

[17]We thank an anonymous referee for encouraging this detailed discussion of the history of the ε-remainder core. We note that this concept has appeared under different names, such as the weak ε-core; the term ε-remainder core is chosen since it seems the most informative.

assumptions with more than $\eta_1(\varepsilon, T, B)$ players have nonempty ε-remainder cores. Since the bound depends only on ε, T, and B, the bound is uniform across all the games characterized by the parameters; there is no restriction to replica games. Our result extends the result of Kaneko and Wooders [27] from replication sequences to arbitrary large games. As in Kaneko and Wooders [27] *the result is independent of the characteristic function of the games; the same bound holds for all games in the collection parameterized by T and B.*

To illustrate the application of Theorem 1, consider the collection of *all* games with at most two types of players and with two-person effective groups – marriage games, buyer-seller games, coalition production economies where two and only two workers are required for a productive group, and so on. Note that this collection of games cannot be described by a pregame – the collection is too large and cannot be accommodated by one space of player types and one worth function. Our result shows that given $\varepsilon > 0$, provided the number of players in the game is sufficiently large, *any* game in the collection has a nonempty ε-remainder core.

To give some intuition into Theorem 1 and the following Theorems, consider a particular game where all players are identical and only two-player groups are effective. It is immediate that the core of the game is nonempty if there is an even number of players. In fact, any even replication of a given game has a nonempty core. This doesn't depend on the payoff sets of the games; instead, it reflects the fact that any set with an even number of players can be partitioned into two-person "optimal" groups. When the total number of players is odd, one player can be assigned his individually rational payoff; thus, given $\varepsilon > 0$, for all sufficiently large total player sets, the ε-remainder core is nonempty.[18] In Appendix we provide a general version of the result that when effective group sizes are bounded and there is a finite number of types of players, large player sets can be partitioned so that most players are in "optimal" groups. With this result in hand, Theorem 1 is immediate. Theorems 2 and 3 follow by approximation arguments.

3.2 The ε_1-remainder ε_2-core.

The requirements of the ε-core were first relaxed to allow an exceptional set of player, containing no more than the fraction ε of the total player set in Wooders [61] and in Shubik and Wooders [54]. We provide a small refinement of the ε-remainder core paper by allowing the fraction of players in the exceptional set to differ from ε. For this less restrictive definition of the approximate core we can treat a significantly more general class of games than those of Theorem 1, in particular, we can allow

[18] This intuition, for the case of identical players, already appears in Pauly [43] and, especially, Shubik [51]. (Pauly's assumption in the two-type case, relating marginal contributions of players of different types to the worths of coalitions implies that there is no loss in efficiency in mixing both types. A paper demonstrating this result is available on request from the second author.)

approximate types ($\delta > 0$) and almost effective groups ($\beta > 0$). For example, the class of models covered by our next Theorem includes replica models of economies with private goods as in Debreu and Scarf [17] and models of local public good economies satisfying per capita boundedness, as in [63].

the ε_1-remainder ε_2-core. Let (N, V) be a game. A payoff vector x belongs to the ε_1-remainder ε_2-*core* if x is feasible and if, for some group $S \subset N$, $\frac{|N|-|S|}{|N|} \leq \varepsilon_1$ and x_S belongs to the ε_2-core of the subgame (S, V).

The following result extends the nonemptiness results of Wooders [59], [62], [65], Shubik and Wooders [54], and Wooders and Zame [70], [71] from pregames to parameterized collections of games.[19] For the same values of the parameters T and B the bound on the sizes of games in the following theorem can be chosen to equal the bound in the preceding theorem. Note that there are no restrictions on the value of q except that it be greater than or equal to zero – strong comprehensiveness is not required.

Theorem 2. (**Nonemptiness of the ε_1-remainder ε_2-core**). Let T and B be positive integers and let q be a nonnegative real number satisfying $q \leq 1$. For any $\varepsilon_1 > 0$ and $\varepsilon_2 \geq 0$, there exists an integer $\eta_1(\varepsilon_1, T, B)$ such that if
(a) $(N, V) \in G^q((\delta, T), (\beta, B))$ with $\delta \geq 0$, $\beta \geq 0$, $(\delta + \beta) \leq \varepsilon_2$ and
(b) $|N| \geq \eta_1(\varepsilon_1, T, B)$
then the ε_1-remainder ε_2-core of (N, V) is nonempty.

To describe the intuition behind Theorem 2, we begin with the same sort of example as discussed above where only two-player groups are effective. Suppose now that only two-player groups are effective but that the players are not identical, they are only all δ-substitutes for each other. Suppose that the total number of players is even. Consider a feasible payoff vector x that assigns the largest possible amount of payoff to each player, subject to the constraint that x has the equal-treatment property. Since all players are δ-substitutes, no two players can improve on x by more than δ for each player. Thus, x is in the δ-core. If there is an odd number of players and if $\varepsilon_1 > \frac{1}{|N|}$ then the ε_1-remainder δ-core is nonempty; one player can be assigned his individually rational payoff.

Now let us suppose that two-person groups are only β-effective. When $|N|$ is even, selecting x as in the preceding paragraph will lead to a payoff vector in the $(\delta + \beta)$-core. When $|N|$ is odd, the ε_1-remainder $(\delta + \beta)$-core is nonempty for any ε_1 satisfying $\varepsilon_1 > \frac{1}{|N|}$; again, one player can be assigned his individually rational payoff.

[19] Further nonemptiness results for approximate cores of NTU pregames have been obtained by Wooders and Zame in unpublished research. A copy of a typescript is available on request from the second author.

In effect, a parameterized collection of games $G^q((\delta, T), (\beta, B))$, by its definition, can be approximated by the collection $G^q((0, T), (0, B))$.

Observe that by definition the ε-remainder 0-core coincides with the ε-remainder core. Therefore, Theorem 2 is a strict generalization of Theorem 1 (Theorem 1 is a subcase for $\delta = \beta = 0$). But both Theorem 1 and Theorem 2 are based on the idea that some small proportion of the players can be ignored.

3.3 The ε-core.

Our third Theorem provides conditions for the nonemptiness of the ε-core of large games. The proof is based on the idea of compensating the "remainder" players from the previous theorems, as in [62] and a number of subsequent papers. This compensation is possible under q-comprehensiveness with $q > 0$ and with one more condition, typically called per capita boundedness.

per capita boundedness. Let C be a positive real number. A game (N, V) has a *per capita payoff bound* of C if, for all groups $S \subset N$,

$$\sum_{a \in S} x_a \leq C\,|S| \text{ for any } x \in V(S).$$

Theorem 3. (**Nonemptiness of the ε-core**). Let T and B be positive integers. Let C and q be positive real numbers with $q \leq 1$. For each $\varepsilon > 0$ if

(a) $(N, V) \in G^q((\delta, T), (\beta, B))$ with $\delta \geq 0$, $\beta \geq 0$, $\delta + \beta < \varepsilon$ and

(b) (N, V) has per capita payoff bound C,

then there exists an integer $\eta_2(\varepsilon - (\delta + \beta), T, B, C, q)$ such that the ε-core of (N, V) is nonempty whenever $|N| \geq \eta_2(\varepsilon - (\delta + \beta), T, B, C, q)$.

To describe the intuition behind Theorem 3, return again to the situation where all players are identical, only two-player groups are effective and $|N|$ is odd. Suppose also that this is a game with side payments and any two-person group can earn 1; thus, the payoff vector $(\frac{1}{2}, ..., \frac{1}{2}, 0)$ is feasible. Then for any ε satisfying $\varepsilon\,|N| > \frac{1}{2}$ the payoff vector $(\frac{1}{2} - \varepsilon, ..., \frac{1}{2} - \varepsilon, y)$ is in the ε-core where $y = (|N| - 1)\varepsilon$. Informally, we can make transfers of ε from each of $|N| - 1$ players in optimal groups to the reminder player so that *every* player has a payoff of at least $(\frac{1}{2} - \varepsilon)$. It follows that no group of players can significantly improve (by more than ε in this case) upon such a payoff. The feature that per capita payoffs are bounded puts an upper bound on the required transfer to the remainder players. In games without side payments, the feature that q is greater than zero means that a small amount of payoff can be transferred from players in optimal groups to remainder players (but not necessarily at a one-to-one rate) until the remainder players are as well off as those players in optimal groups, with the consequence that no group of players can significantly improve upon the resulting payoff vector.

Now return to the situation where two-player groups are β-effective and all players are δ-substitutes for each other, $\beta+\delta < \varepsilon$. As we've observed, if $|N|$ is odd, the $(\delta+\beta)$-core may be empty. When per capita payoffs are bounded and there is some means of making transfers ($q > 0$) then, for sufficiently large $|N|$, it's possible to begin with some payoff vector in the $(\varepsilon - \delta - \beta)$-remainder $(\delta + \beta)$-core, and make transfers to the remainder players so that, *no* group could improve on the resulting payoff by ε for each player.

We note that the assumption of per capita boundedness is actually stronger than required. For our proofs (as the proof of [62]) we may use only the fact that the set of equal-treatment payoff vectors satisfy the per capita boundedness condition.

An example in Appendix shows indispensability of the condition in Theorem 3 that q is greater than zero, thus clarifying the need for this condition in Theorem 3 in contrast to our previous theorems. Indispensability of the other conditions of Theorem 3 is demonstrated by the examples in Subsection 3.6. The following Corollary shows that Theorem 3 can be applied to obtain nonemptiness of approximate cores of games that are "close" to q-comprehensiveness games (with $q > 0$).[20] The proof of this result is left to the reader.

Corollary. (**Nonemptiness with near q-comprehensiveness**). Let (N, W) be a game. Suppose that for some and $\varepsilon > 0$ there exist a game $(N, V) \in G^q((\delta, T), (\beta, B))$ with $0 < q \leq 1$, $\delta \geq 0$, $\beta \geq 0$, $(\delta + \beta) < \varepsilon$ such that:

(a) (N, V) has per capita payoff bound C,

(b) $H_\infty[W(S), V(S)] \leq \frac{\gamma}{2}$ for all $S \subset N$, where $\gamma < \varepsilon - (\delta + \beta)$, and

(c) $|N| \geq \eta_2(\varepsilon - (\delta + \beta + \gamma), T, B, C, q)$.

Then the ε-core of (N, W) is nonempty.

We remark that in [62], to obtain nonemptiness of approximate cores, q-comprehensiveness is not required. To obtain her equal-treatment result, however, Wooders employs approximating games satisfying strong comprehensiveness. A similar idea is at work in our Corollary. If there is another game (N, V) with payoff sets close – within $\frac{\gamma}{2}$ – to those of (N, W) and (N, V) satisfies the conditions of Theorem 3, it is not necessary that the game (N, W) itself satisfies those conditions. In these circumstances, we can find a payoff vector that is in the $(\varepsilon - \gamma)$-core of (N, V) and adjust this payoff vector so that it is in the ε-core of (N, W). In essence, this is similar to the sort of approximation technique used for our Theorems 2 and 3.

[20] Any comprehensive payoff set can be approximated arbitrarily closely by a q-comprehensive payoff set, for q small ([62], Appendix).

3.4 A simple example of a production economy.

The following example illustrates the application of our results to a familiar sort of situation – one with firms and workers. The example also illustrates the application of our results to situations where there is a compact metric space of player types, essentially a special case.

Example 1. Suppose a pregame has two sorts of players, firms and workers.[21] The set of possible types of workers is given by the points in the interval $[0, 1)$ and the set of possible types of firms is given by the points in the interval $[1, 2]$. To derive a game from the information given above, let N be any finite player set and let ξ be an *attribute function*, that is, a function from N into $[0, 2]$. In interpretation, if $\xi(i) \in [0, 1)$ then i is a worker and if $\xi(i) \in [1, 2]$ then i is a firm. Firms can profitably hire up to three workers and the payoff to a firm i and a set of workers $W(i) \subset N$, containing no more than 3 members, is given by $v(\{i\} \bigcup W(i)) = \xi(i) + \sum_{j \in W(i)} \xi(j)$. Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any group $S \subset N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into groups containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game.

We leave it to the reader to verify that for any positive integer m every game derived from the pregame is a member of the class $G^1((\frac{1}{m}, 2m), (0, 4))$ and has a per capita bound of 2. Theorem 3 states that given $\frac{1}{m} < \varepsilon$, if $|N| \geq \eta_2(\varepsilon - \frac{1}{m}, 2m, 4, 2, 1)$ then the game (N, v) has a nonempty ε-core. In fact, Theorem 3 states this conclusion for an *arbitrary* game (N, V) described by the same parameter values, $T = 2m$, $B = 4$, $\delta + \beta = \frac{1}{m}$, $C = 2$ and $q = 1$.

For this example, an exact bound on $|N|$ can be calculated. However it was not an intension of the present paper to provide precise bounds; we refer the reader to another of our papers [35] for computation of bounds.

3.5 Per capita boundedness and small group effectiveness.

Our final result, Theorem 3, requires both per capita boundedness and small group effectiveness. As noted previously, in the context of pregames with side payments, when arbitrarily small percentages of players of any particular type is ruled out, then these two conditions are equivalent. But in important economic contexts, neither condition implies the other. The next example illustrates a voting games satisfying

[21] We refer the reader to Wooders [58], [59] for a definition of a pregame (with side payments) with a finite number of types of players and Wooders and Zame [70] or Wooders [65] for a definition of a pregame with a compact metric space of player types. For games without side payments, see Kaneko and Wooders [30] or Wooders [64], for example.

per capita boundedness. There is only one player type in each game so the "thickness" condition of the equivalence result of [67] is satisfied. But small group effectiveness does not hold and Theorem 3 does not apply.

Example 2. *Voting games.* Consider a sequence of games $(N^m, v^m)_{m=1}^{\infty}$ with side payments and where the m^{th} game has $3m$ players. Suppose that there are widespread positive externalities so that in the m^{th} game, any group S consisting of at least $2m$ players can get up to $2m$ units of payoff to divide among its members, that is, $v^m(S) = 2m$. Assume that if $|S| < 2m$, then $v^m(S) = 0$.

We can think of the games as a sequence of voting games where a winning group must contain $\frac{2}{3}$ of the population, for example, impeachment of a President of the United States or ratification of a treaty in some parliaments.

Observe that each game in the sequence has one exact player type and a per capita bound of 1. That is, $q = 1, T = 1, C = 1$, and $\delta = 0$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m.

To see that the $\frac{1}{7}$-core is empty, observe that for any feasible payoff vector there are m players that are assigned, in total, no more than $\frac{2m}{3m}m = \frac{2}{3}m$. There are another m players that get in total no more than $\frac{2m}{2m}m = m$. These $2m$ players can form a group and receive $2m$ in total. This group can improve upon the given payoff vector for each of its members by $\frac{1}{6}$, since $(2m - \frac{5}{3}m)\frac{1}{2m} = \frac{1}{6}$.

The following example, of matching games with widespread positive externalities, illustrates economic situations where, because of small group effectiveness, ε-remainder cores are nonempty for positive ε, but per capita boundedness does not hold and Theorem 3 does not apply.

Example 3. *A matching game with widespread positive externalities.*[22] The economic situation we've in mind is one where any two players can carry out some job but their reward from the job depends on the size of the economy in which they live. (It would be easy to modify the example to become a two or many-sided matching game.) Consider a sequence of games with side payments $(N^m, v^m)_{m=1}^{\infty}$ where the m^{th} game has $2m+1$ players. Assume that any player alone can get only 0 units or less, that is $v^m(\{i\}) = 0$ for all $i \in N$. Also assume that any two-player group can get up to $2m$ units of payoff to divide; $v^m(S) = 2m$ if $|S| = 2$. An arbitrary group can gain only what it can obtain in partitions where no member of the partition contains more than two players.

The games $(N^m, v^m)_{m=1}^{\infty}$ are members of the collection of games with one exact player type and strictly effective small groups of two. That is, $q = 1$, $T = 1$, $B = 2$, and $\delta = \beta = 0$. Given $\varepsilon > 0$ the ε-remainder core of the game

[22] A similar example in Wooders and Zame [70].

(N^m, v^m) is nonempty for all m satisfying $\frac{1}{2m+1} < \varepsilon$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m; the remainder player cannot be compensated by the sum of taxes on the players in two-person groups.

To see that the $\frac{1}{7}$-core is empty, observe that for any feasible payoff vector there is a player whose payoff is no more than $\frac{2m^2}{2m+1}$. There is another player whose payoff must be no more than $\frac{2m^2}{2m} = m$. These two players may form a group and realize $2m$. Thus they gain $m - \frac{2m^2}{2m+1} = \frac{m}{2m+1} \geq \frac{m}{3m} = \frac{1}{3}$. Obviously, together this two-player group can improve upon the given payoff by $\frac{1}{6}$ for each member of the group.

3.6 Remarks.

Remark 1. *q-comprehensiveness or convexity?* It is possible to obtain a result similar to Theorem 3 using convexity of payoff sets and "thickness" instead of q-comprehensiveness (see [34]). Strong comprehensiveness, however, can be naturally satisfied by games derived from economies. Moreover, "1-strongly comprehensive games" are games with side payments, so we can incorporate this important special case. Furthermore, in models of economies with local public goods or with clubs, convexity may be difficult to satisfy. Although examples show that none of the assumptions can be omitted, our Corollary relaxes q-comprehensiveness.

Remark 2. *Exact bounds.* It may be possible to compute the bounds on the size of the total player sets given in Theorems 1, 2, and 3 in terms of the parameters describing the games. A simple bound is obtained in [35], although under somewhat different assumptions. Also, the proofs of that paper, relative to those of this paper, are quite complex.

Remark 3. *Absolute or relative sizes?* It is possible to obtain similar results with bounds on *relative* sizes of effective groups. In a finite game with a given number of players, assumptions on absolute sizes and on relative sizes of effective groups are equivalent. We have chosen to develop our results using bounds on absolute sizes of near-effective groups since this seems to reflect typical economic and social situations. Examples include: marriage and matching models (see Kelso and Crawford [32] and Roth and Sotomayor [44]); models of economies with shared goods and crowding (see Conley and Wooders [12]) for a survey); and private goods exchange economies (see Mas-Colell [38] and Kaneko and Wooders [29] for example). In fact, assumptions on proportions of economic agents typically occur only when there is a continuum of players, cf. Ostroy [41]).

Remark 4. *Limiting gains to group formation.* In the pregame framework several different conditions limiting returns to group formation have been used. For situations with a fixed distribution of a finite number of player types, Wooders [59], [62] and Shubik and Wooders [54] require per capita boundedness. To treat compact metric spaces of player types, Wooders and Zame [70], [71] require boundedness of marginal contributions to groups while Wooders [65], [66] requires the less restrictive condition of small group effectiveness. As noted in the introduction, in the context of games derived from pregames, small group effectiveness and per capita boundedness are equivalent. In Subsection 3.5, we have shown that in the broader framework of parameterized collections of games both β-effective B-bounded groups and per capita boundedness are required.

Remark 5. *Relationships between the Theorems.* The notions of the ε-core and the ε-remainder core are conceptually independent of each other, while the ε_1-remainder ε_2-core combines features of both. One may wonder if nonemptiness of the ε_1-remainder ε_2-core (Theorem 2) can be deduced from the nonemptiness of the ε-core and the ε-remainder core (Theorems 1 and 3). The assumptions required for Theorems 1 and 3, however, are *strictly stronger* than those for Theorem 2; thus, while a less general nonemptiness result for the ε_1-remainder core ε_2-core can be deduced from the other two results, Theorem 2, requiring weaker assumptions, cannot be. Indeed, none of the Theorems can be deduced from the others.

4 Economies with clubs.

We define admissible club structures in terms of natural properties and take as given the set of all admissible club structures for each group of players. Generalizing Mas-Colell's [39] notion of public projects to club activities, there is no necessary linear structure on the set of club activities. Indeed, our results could be obtained even without any linear structure on the space of private commodities. We remark that it would be possible to separate crowding types of players (those observable characteristics that affect the utilities of others, or, in other words, their external characteristics) from taste types, as in Conley and Wooders [10], [11], and have players' roles as club members depend on their crowding types. In those papers, however, the separation of crowding type and taste type has an important role; the authors show that prices for public goods – or club membership prices – need only depend on observable characteristics of players and not on their preferences. The current paper treats only the core; at this point separation of taste and crowding type would have no essential role.

<u>agents</u>. There are T "types" of agents. Let $\rho = (\rho_1, ..., \rho_T)$ be a given profile, called the *population profile.* The set of agents is given by

$$N_\rho = \{(t, q) : q = 1, ..., \rho_t \text{ and } t = 1, ..., T\},$$

and (t, q) is called the q^{th} *agent of type* t. It will later be required that all agents of the same type may play the same role in club structures. For example, in a traditional marriage model, all females could have the role of "wife". Define $N_\rho[t] := \{(t, q) : q = 1, ..., \rho_t \}$. For our first Proposition members of $N_\rho[t]$ will be exact substitutes for each other and for our next two Propositions, approximate substitutes.

commodities. The economy has L private goods. A vector of private goods is denoted by $y = (y_1, ..., y_\ell, ..., y_L) \in \mathbf{R}^L_+$.

clubs. A *club* is a nonempty subset of players. For each club $S \subset N_\rho$, a *club structure of* S, denoted by $\mathcal{S}$, is a set of clubs whose union coincides with S. The set of *admissible club structures for* S, denoted by $\mathcal{C}(S)$, is assumed to be nonempty for any $S \neq \emptyset$. This assumption ensures that a club of one player has a unique admissible club structure - a singleton set. The sets $\mathcal{C}(S)$ are also required to satisfy the following two properties:

1. If S and S' are nonempty disjoint subsets of players and $\mathcal{S}$ and $\mathcal{S}'$ are admissible club structures of S and S' respectively, then $\{C : C \in \mathcal{S} \bigcup \mathcal{S}'\}$ is an admissible club structure of $S \bigcup S'$ (unions of admissible club structures of disjoint groups are admissible club structures of the unions of the groups).

2. Let S and S' be subsets of players with the same profiles, let $\mathcal{S}$ be an admissible club structure of S and let φ be a type-preserving 1-1 mapping from S onto S' (that is, if $(t, q) \in S$ then $\varphi((t, q)) = (t, q')$ for some $q' = 1, ..., \rho_t$). Then

$$\mathcal{S}' = \{C \subset S' : \varphi^{-1}(C) \in \mathcal{S}\}$$

is an admissible club structure of S' (admissible club structures depend only on profiles, that is, all players of the same type have the same roles in clubs).

The first property is necessary to ensure that the game derived from the economy is superadditive. It corresponds to economic situations where one option open to a group is to form smaller groups. Since the singletons are always admissible club structures for clubs of one player, this property implies that the partition of any set $S \subset N_\rho$, $S \neq \emptyset$, into singletons is an admissible club structure for S. The second property corresponds to the idea that the opportunities open to a group depend on the profile of the group.

club activities. For each club C there is a given set of *club activities* $\mathcal{A}(C)$. An element α of $\mathcal{A}(C)$ requires input $x(C, \alpha) \in \mathbf{R}^L$ of private goods. For any two clubs C and C' with the same profile we require that if $\alpha \in \mathcal{A}(C)$, then $\alpha \in \mathcal{A}(C')$ and $x(C, \alpha) = x(C', \alpha)$. For 1-player clubs $\{(t, q)\}$, we assume that there is an activity α_0 with $x(\{(t, q)\}, \alpha_0) = 0$, that is, there is an activity requiring no use of inputs.

preferences and endowments. Only private goods are endowed. Let $\omega^{tq} \in \mathbf{R}^L_+$ be the *endowment of the $(t,q)^{th}$ participant of private goods.*

Given $S \subset N_\rho$, $(t,q) \in S$, and a club structure $\mathcal{S}$ of S, *the consumption set of the* $(t,q)^{th}$ player (relative to $\mathcal{S}$) is given by

$$\Phi^{tq}(\mathcal{S}) := X^{tq}(\mathcal{S}) \times \prod_{C \in \mathcal{S}} \mathcal{A}(C),$$

where $X^{tq}(\mathcal{S}) \subset \mathbf{R}^L$ is the *private goods consumption set relative to* $\mathcal{S}$, assumed to be closed. Note that the private goods part of the consumption set of a player, $X^{tq}(\mathcal{S})$, may depend on the club structure; one function of a club may be to provide certain private goods to its members. The entire consumption set of the $(t,q)^{th}$ player is given by

$$\Phi^{tq} := \bigcup_{S \subset N_\rho : (t,q) \in S} \; \bigcup_{\mathcal{S} \in \mathcal{C}(S)} \Phi^{tq}(\mathcal{S}).$$

We assume that the $(t,q)^{th}$ player can subsist in isolation. That is

$$(\omega^{tq}, \alpha_0) \in \Phi^{tq}(\{(t,q)\}).$$

It is also assumed that for each (t,q), each $S \subset N_\rho, (t,q) \in S$, and each club structure $\mathcal{S}$ of S, the preferences of the $(t,q)^{th}$ agent are represented by a continuous utility function $u^{tq}(\cdot;\mathcal{S})$ defined on $\Phi^{tq}(\mathcal{S})$.

states of the economy. Let S be a nonempty subset of N_ρ and let $\mathcal{S}$ be a club structure of S. A *feasible state of the economy S relative to $\mathcal{S}$*, or simply *a state for $\mathcal{S}$*, is a pair $(y^{\mathcal{S}}, \alpha^{\mathcal{S}})$ where:

(a) $y^{\mathcal{S}} = \{y^{tq}\}_{(t,q) \in S}$ with $y^{tq} \in X^{tq}(\mathcal{S})$ for $(t,q) \in S$;

(b) $\alpha^{\mathcal{S}} = \{\alpha^C\}_{C \in \mathcal{S}}$ with $\alpha^C \in \mathcal{A}(C)$ for $C \in \mathcal{S}$; and

(c) the allocation of private goods is feasible, that is,

$$\sum_{C \in \mathcal{S}} x(C, \alpha^C) + \sum_{(t,q) \in S} y^{tq} = \sum_{(t,q) \in S} \omega^{tq}.$$

feasible payoffs vectors. A payoff vector $U = (\bar{u}^{tq})_{(t,q) \in N_\rho}$ is *feasible for a group S* if $\overline{u}^{tq} = 0$ for all $(t,q) \in N_\rho \backslash S$ and there is club structure $\mathcal{S}$ of S and a feasible state of the economy for S relative to $\mathcal{S}, (y^{\mathcal{S}}, \alpha^{\mathcal{S}})$, such that $\overline{u}^{tq} = u^{tq}(y^{tq}, \alpha^{\mathcal{S}}; \mathcal{S})$ for each $(t,q) \in S$.

the game induced by the economy. For each group $S \subset N_\rho$, define

$$V(S) = \{(\widehat{u}^{tq})_{(t,q) \in N_\rho} : \text{there is a payoff vector } (\bar{u}^{tq})_{(t,q) \in S} \text{ that is feasible for } S \text{ and } \widehat{u}^{tq} \le \overline{u}^{tq} \text{ for all } (t,q) \in S;\ \widehat{u}^{tq} = 0 \text{ for all } (t,q) \notin S\}.$$

It is immediate that the player set N_ρ and function V determine a game (N_ρ, V) with comprehensive payoff sets.

ε-domination. Let $\mathcal{N}_\rho$ be a club structure of the total player set N_ρ and let $(y^{\mathcal{N}_\rho}, \alpha^{\mathcal{N}_\rho})$ be a feasible state of the economy N_ρ relative to $\mathcal{N}_\rho$. A group S can *ε-dominate* the state $(y^{\mathcal{N}_\rho}, \alpha^{\mathcal{N}_\rho})$ if there is a club structure $\mathcal{S} = \{S_1, ..., S_K\}$ of S and a feasible state $(y'^{\mathcal{S}}, \alpha'^{\mathcal{S}})$ for the economy S such that for all consumers $(t, q) \in S$ it holds that

$$u^{tq}(y'^{tq}, \alpha'^{\mathcal{S}}; \mathcal{S}) > u^{tq}(y^{tq}, \alpha^{\mathcal{S}}; \mathcal{N}_\rho) + \varepsilon.$$

the core of the economy and ε-cores. A state $(y^{\mathcal{N}_\rho}, \alpha^{\mathcal{N}_\rho})$ of the economy N_ρ relative to $\mathcal{N}_\rho$ is in the *core of the economy* if it cannot be 0-dominated by any group S. It is clear that if $(y^{\mathcal{N}_\rho}, \alpha^{\mathcal{N}_\rho})$ is a state in the core of the economy then the utility vector induced by that state is in the core of the induced game. Similarly, if $(\bar{u}^{tq})_{(t,q)\in N_\rho}$ is in the core of the game induced by an economy then there is a state in the core of the economy $(y^{\mathcal{N}_\rho}, \alpha^{\mathcal{N}_\rho})$ such that the utility vector induced by that state is $(\bar{u}^{tq})_{(t,q)\in N_\rho}$. Notions of the ε-remainder core, the ε_1-remainder ε_2-core, and the ε-core of the economy are defined in the obvious way.

4.1 Nonemptiness of approximate cores.

To obtain our results we require few restrictions on the economy. Our first Proposition requires exact player types and strictly effective small groups. It applies naturally to the sorts of matching models considered in Roth and Sotomayer [44] or Crawford [14], for example.

(A.0) For each t and all $q, q' \in \{1, ..., \rho_t\}$, $u^{tq}(\cdot) = u^{tq'}(\cdot)$ and $\omega^{tq} = \omega^{tq'}$. In addition, in the game induced by the economy the players (t, q) and (t, q') are exact substitutes. (All players of the same type are identical in terms of their endowments, preferences and crowding types – their effects on others.)

(A.1) There is a bound B such that for any population profile ρ, any group $S \subset N_\rho$, and any club structure $\mathcal{S}$ of S, if $U = (\bar{u}^{tq} : (t, q) \in S)$ is a feasible payoff vector for the club structure $\mathcal{S}$ then there is a partition of S into groups, say $\{S^1, ..., S^K\}$ and club structures of these groups, $\{\mathcal{S}^1, ..., \mathcal{S}^K\}$ such that for each k, $\left|S^k\right| \leq B$ and $U^k := (\overline{u}^{tq} : (t, q) \in S^k)$ is a feasible payoff vector for $\mathcal{S}^k$.

Our approach requires that for each group S, the set of individually rational and feasible payoffs is compact, (A.2) below. This can be ensured by assuming, for example, in addition to (A.1), that: (i) there is a bound on the number of clubs to which an agent can belong; (ii) for any club in any admissible club structure, the set of club activities $\mathcal{A}(C)$ is compact and (iii) the consumption of private goods sets

$X^{tq}(\mathcal{S})$ are bounded below for any admissible club structure $\mathcal{S}$ of S. More general economic environments will also ensure compactness but for the purposes of this example, we prefer to keep the assumptions relatively simple.

(A.2) For each subset of players $S \subset N_\rho$ the set $V(S) \cap \mathbf{R}^N_+$ is compact.[23]

The following result is an immediate application of Theorem 1.

Proposition 1. (**Nonemptiness of the ε-remainder core**). Let T and B be positive integers. Assume (A.0)-(A.2) hold. Given $\varepsilon > 0$, there exists an integer $\eta_1(\varepsilon, T, B)$ such that if ρ, the profile of the economy, satisfies the property that $\|\rho\| \geq \eta_1(\varepsilon, T, B)$ then the ε-remainder core of the economy is nonempty.

Proposition 1 is most natural if there is some composite commodity – one private good or, given prices of private goods, money – or if private goods are indivisible so that all gains from trade in private goods can be realized by trade within groups of bounded sizes. If we require only nonemptiness of the ε_1-remainder ε_2-core we can weaken the restrictions on the economy – players of the same type need only be approximate substitutes and small groups need only be nearly effective. For brevity, these assumptions will not be made on the primitives of the economy. Thus, instead of (A.0) and (A.1), for the following two Propositions, we will assume (A.0$'$) and (A.1$'$):

(A.0$'$) For some $\delta \geq 0$ the players in the set $N_\rho[t] = \{(t, q) : q = 1, ..., \rho_t\}$ are δ-substitutes for each other in the game induced by the economy.

(A.1$'$) There are $\beta \geq 0$ and an integer B so that the game derived from the economy has β-effective B-bounded groups.

Then the following result, for the ε_1-reminder ε_2-core, follows from Theorem 2.

Proposition 2. (**Nonemptiness of the ε_1-remainder ε_2-core**). Let T and B be positive integers. Given $\varepsilon_2 \geq 0$, assume (A.0$'$), (A.1$'$) and (A.2) hold and $(\delta + \beta) \leq \varepsilon_2$. Given $\varepsilon_1 > 0$, there exists an integer $\eta_1(\varepsilon_1, T, B)$ such that if ρ, the profile of the economy, satisfies the property that $\|\rho\| \geq \eta_1(\varepsilon_1, T, B)$ then the ε_1-remainder ε_2-core of the economy is nonempty.

To motivate the above result, one might suppose that, in general, group, firm or club sizes are not bounded nor is there only a bounded number of player types. Even

[23] Note that for those results which require q-comprehensiveness for $q > 0$, the sets $V(S)$ cannot be compactly generated. But had we defined q-comprehensiveness only for each set $V(S) \cap M^*$, then instead of assuming (A.2) we could assume instead that $V(S)$ is compactly generated. In fact, other than for further notational complexity, our theorems would remain unchanged by requiring q-comprehensiveness only on the sets $V(S) \cap M^*$.

if group sizes are strictly bounded, all players may differ. One may view Proposition 2 as providing justification for the study of the situations described by Proposition 1.

For our next result, to ensure q-comprehensiveness for some $q > 0$, we assume that the L^{th} commodity is a sort of "quasi-money" for which everyone has a separable preference. Let y^{tq}_{-L}, and $X^{tq}_{-L}(\mathcal{S})$ denote the restriction of $y^{tq} \in \mathbf{R}^L$ and $X^{tq}(\mathcal{S})$ respectively to their first $(L-1)$ coordinates. We assume that, for some real number $q^* \in (0,1]$, the marginal utility of the L^{th} commodity, for all sufficiently large amounts of the commodity, is greater than or equal to q^*.

(A.3) (q^*-comprehensiveness)**:** For good L and all players $(t,q) \in N_\rho$, for any state of the economy $(y^{\mathcal{S}}, \alpha^{\mathcal{S}})$ it holds that :

(a) $X^{tq}(\mathcal{S}) = X^{tq}_{-L}(\mathcal{S}) \times \mathbf{R}$ (the consumption set is separable and the projection of the L^{th} coordinate is $\mathbf{R}$),[24]

(b) $u^{tq}(y^{tq}, \alpha^{\mathcal{S}}; \mathcal{S}) = u^{tq}_{-L}(y^{tq}_{-L}, \alpha^{\mathcal{S}}; \mathcal{S}) + u^{tq}_L(y^{tq}_L, \alpha^{\mathcal{S}}; \mathcal{S})$ for some functions $u^{tq}_{-L}(\cdot;\cdot)$ and $u^{tq}_L(\cdot;\cdot)$ (utility is separable),

(c) for a real number q^*, $0 < q^* \leq 1$, for all players (t,q) the marginal utility of the $(t,q)^{th}$ player for the L^{th} good is between q^* and 1.

(A.4) (per capita boundedness): There is a constant C such that the condition of per capita boundedness is satisfied by the games derived from the economies.

Assumption (A.3) ensures that starting from any allocation, through redistribution of good L the wealth can always be transferred to the "remainder players" at a rate that is allowed to depend on the allocation but is restricted to be between q^* and 1. This assumption implies q^*-comprehensiveness of the game derived from the economy.[25]

A simple example of (A.3) is that utility functions are linear in the L^{th} commodity with some positive slope. Specifically, for some positive constant b^{tq} for each type t, $q^* \leq b^{tq} \leq 1$,

$$u^{tq}(x^{tq}, \alpha^{\mathcal{S}}; \mathcal{S}) = u^{tq}_{-L}(x^{tq}_{-L}, \alpha^{\mathcal{S}}; \mathcal{S}) + b^{tq} x^{tq}_L.$$

If b^{tq} were equal to one for each player then the economy would have "transferable utility."[26] Notice however that our conditions allow much more general forms of the

[24] The condition that the L^{th} commodity can be consumed in arbitrarily large negative amounts significantly simplifies the example. This condition could be relaxed but at some cost in terms of complexity and the length of this paper.

[25] Such an assumption, in the literature on private goods economies with indivisibilities, goes back at least to Broome [7]. It was introduced in the club/local public good literature in several papers due to Wooders, for example [63].

[26] It would be more appropriate, in some respects, to reserve the term "games with side payments" to the situations permitted by (A.3) since, according to the common usage of the words, (A.3) permits side payments while "transferable utility" requires that the side payments can be made at a one-to-one rate.

utility of the L^{th} commodity, for example,

$$\begin{aligned} u^{tq}(x^{tq}, \alpha^{\mathcal{S}}; \mathcal{S}) &= u^{tq}_{-L}(x^{tq}_{-L}, \alpha^{\mathcal{S}}; \mathcal{S}) + (b^{tq}_1 + b^{tq}_2)x^{tq}_L \text{ for } x^{tq}_L \leq 1, \\ &= u^{tq}_{-L}(x^{tq}_{-L}, \alpha^{\mathcal{S}}; \mathcal{S}) + b^{tq}_1 x^{tq}_L + b^{tq}_2 (x^{tq}_L)^{\frac{1}{2}} \text{ for } x^{tq}_L \geq 1 \end{aligned}$$

for any values b^{tq}_1 and b^{tq}_2, where $q^* \leq b^{tq}_1 \leq 1$ and $0 \leq b^{tq}_2 \leq 1 - b^{tq}_1$ for each $(t, q) \in N_\rho$.

The most restrictive part of condition (A.3) may be that there is no lower bound of the amount of the L^{th} commodity that can be consumed. It is possible to weaken (A.3) to the sort of assumption introduced in Hammond, Kaneko and Wooders [23], that the endowment is preferred to any bundle containing only indivisible (in our case, club) goods but, for our purposes here, we prefer to avoid this additional complexity.[27]

The next result follows from Theorem 3.

Proposition 3. (Nonemptiness of the ε-core). Let T and B be positive integers. Let a per capita bound C and q^* be positive real numbers. Given $\varepsilon > 0$, assume that (A.0′), (A.1′), (A.2)-(A.4) hold and $(\delta + \beta) < \varepsilon$. Then there exists an integer $\eta_2(\varepsilon - (\delta + \beta), T, B, C, q^*)$ such that if ρ, the profile of the economy, satisfies $\|\rho\| \geq \eta_2(\varepsilon - (\delta + \beta), T, B, C, q^*)$ then the ε-core of the economy is nonempty.

4.2 Further applications.

The class of economies defined above is very broad. The results can be applied to extend results already in the literature on economies with group structures, such as those with local public goods (called club economies by some authors), cf., Shubik and Wooders [52].[28]

For example, there is a number of papers showing core-equilibrium equivalence in finite economies with local public goods and one private good and satisfying strict effectiveness of small groups, cf., Conley and Wooders [13] and references therein. In these economies, from the results of Wooders [62] and Shubik and Wooders [54], existence of approximate equilibrium where an exceptional set of agents is ignored is immediate. (Just take the largest subgame having a nonempty core and consider the equilibria for that subeconomy; ignore the remainder of the consumers.) Our results allow the immediate extension of these results to results for all sufficiently large economies – no restriction to replication sequences is required.

[27] Were we to weaken (A.3) we could also modify our notion of q-comprehensiveness so that it is required only on the sets $V(S) \cap M^*$ for some appropriate set M^*. See also footnote 20.

[28] The results of this paper are easily available in Shubik and Wooders [53], on-line on the second author's web page.

5 Conclusions.

Except in certain idealized situations, cores of games are typically empty. This has the consequences that important classes of economies typically have empty cores and a competitive equilibrium does not exist. Examples include economies with indivisibilities and other nonconvexities, economies with public goods subject to crowding, and production economies with non-constant returns to scale. The standard justification for convexity, assumed in Arrow-Debreu-McKenzie models of exchange economies, is that the economies are "large," rendering nonconvexities negligible – the convexifying effect of large numbers. Similarly results on nonemptiness of approximate cores rely on large numbers of players and the *balancing effect* of large numbers. An important aspect of our results in this paper is that they are for arbitrary games and the bounds depend on the parameters describing the games; the compact metric space of player types assumed in previous work is a special case. Moreover, our approach allows both widespread externalities and uniform results.

It appears that the framework of parametrized collections of games and our approach will have a number of uses. In ongoing research this framework is used to demonstrate further market-like properties of arbitrary games[29]: approximate cores are nearly symmetric (treat similar players similarly); arbitrary games are approximately market games and; arbitrary games satisfy a "law of scarcity," dictating that an increase in the abundance of players of a given type does not increase the core payoff vectors to members of that type. In addition, some initial results have been obtained on convergence of cores and approximate cores. A particularly promising direction appears to be the application of ideas of lottery equilibrium in games of Garratt and Qin [19] to parameterized collections of games. Another possible application is to games with asymmetric information, as in Allen [1], for example, and Forges, Heifetz and Minelli [18].[30]

6 Appendix.

6.1 Mathematical foundations: Partition-balanced profiles.

This subsection formalizes a key idea about profiles that underlies the nonemptiness of approximate cores of large games. Throughout this subsection, let the number of types of players be fixed at T. Thus, every profile f has T components and $f \in \mathbf{Z}^T$. Our key definitions follow.

[29]See Shapley and Shubik [48], [49] for seminal results of this nature and Wooders [66], [67] for more recent references to related results in the context of pregames and economies with clubs/local public goods.

[30]We are grateful to Francoise Forges for pointing out this possible application.

B–partition-balanced profiles. A profile f is *B-partition-balanced* if *any* game $(N, V) \in G^q((0,T),(0,B))$ where the profile of N is f (that is, $|N[i]| = f_i$ for any $i = 1, .., T$) has a nonempty core.

replicas of a profile. Given a profile f and a positive integer r, the profile rf is called the r^{th} *replica* of f.

The Lemma 1 below is a very important step. It states that, for any profile f, there is a replica of that profile that is B-partition-balanced. The smallest such replication number is called the *depth* of the profile.

Lemma 1. (**Kaneko and Wooders, 1982, Theorem 3.2 - The partition-balancing effect of replication**). Let B be a positive integer and let f be *any* profile. Then there is an integer $r(f,B)$, the depth of f, such that, for any positive integer k, the profile $kr(f,B)f$ is B-partition-balanced.

We refer the reader to Kaneko and Wooders [27] for a proof.[31] For a recent discussion and an interesting application of this sort of result to dynamic matching processes, see Myerson [40]. The following concept of ε-B-partition-balanced profiles completes our construction.

ε-B-partition-balanced profiles. Given a positive integer B and a non-negative real number ε, $0 \leq \varepsilon \leq 1$, a profile f is *ε-B-partition-balanced* if there is a subprofile f' of f such that $\frac{\|f'\|}{\|f\|} \geq 1 - \varepsilon$ and f' is B–partition-balanced.

The next result is key: given $\varepsilon > 0$ and B, any sufficiently large profile is ε-B-partition-balanced. Note that this result is *uniform across all large profiles.*

Fundamental Proposition. (**The partition-balancing effect of large numbers**). Given a positive integer B and a positive real number $\varepsilon, 0 < \varepsilon \leq 1$, there is a positive integer $k(\varepsilon, B)$ such that *any* profile f with $\|f\| \geq k(\varepsilon, B)$ is ε-B-partition-balanced.

A technical lemma is required. Denote by $\|\cdot\|$ the sum-metric in $\mathbf{R}^T$, that is, for $x, y \in \mathbf{R}^T$, $\|x - y\| := \sum_i |x_i - y_i|$. Let us consider the simplex in $\mathbf{R}^T_+$: $\triangle_+ := \left\{\lambda \in \mathbf{R}^T_+ : \sum_{i=1}^T \lambda_i = 1\right\}$. For any positive integer η, let us define the following finite set in the simplex:

$$\triangle_\eta := \left\{x \in \triangle_+ : \eta x \in Z^T_+\right\}.$$

Finally, let us define $\eta(\varepsilon) = \min\left\{\eta \in Z_+ : \eta \geq \frac{T}{\varepsilon}\right\}$. Now we can state Lemma 2.

[31] The reader may also derive the proof from [62], Lemmas 3 and 5.

Lemma 2. For each $\varepsilon > 0$ and for any $f \in \mathbf{R}^T_+$ there is a vector $g \in \mathbf{R}^T_+$ satisfying $g \leq f$, $\|f\| - \|g\| = \|f - g\| \leq \varepsilon \|f\|$ and $\frac{g}{\|g\|} \in \triangle_{\eta(\varepsilon)}$.

Proof of Lemma 2: Let us first prove that $(1+\varepsilon)\triangle_+ \subset \triangle_{\eta(\varepsilon)} + \varepsilon\ \triangle_+$. Consider any $a = (a_1, .., a_T) \in (1+\varepsilon)\triangle_+$. (That is $\sum_{i=1}^T a_i = 1 + \varepsilon$ and $a_i \geq 0$ for each $i = 1, .., T$.) Now for any i let us consider $a_i = \frac{l_i}{\eta(\varepsilon)} + p_i$, where l_i is a nonnegative integer and $0 \leq p_i < \frac{1}{\eta(\varepsilon)}$. Then $\sum_{i=1}^T p_i < \frac{T}{\eta(\varepsilon)} \leq \varepsilon$ and hence $\sum_{i=1}^T l_i > \eta(\varepsilon)$. Now consider any vector of nonnegative integers $(l_1^*, .., l_T^*)$ such that $l_i^* \leq l_i$ for any i and $\sum_{i=1}^T l_i^* = \eta(\varepsilon)$. By construction $(\frac{l_1^*}{\eta(\varepsilon)}, .., \frac{l_T^*}{\eta(\varepsilon)}) \in \triangle_{\eta(\varepsilon)}$ and $(a_1 - \frac{l_1^*}{\eta(\varepsilon)}, .., a_T - \frac{l_T^*}{\eta(\varepsilon)}) \in \varepsilon\triangle_+$. Therefore

$$a = (a_1, .., a_T) = (\frac{l_1^*}{\eta(\varepsilon)}, .., \frac{l_T^*}{\eta(\varepsilon)}) + (a_1 - \frac{l_1^*}{\eta(\varepsilon)}, .., a_T - \frac{l_T^*}{\eta(\varepsilon)}) \subset \triangle_{\eta(\varepsilon)} + \varepsilon\triangle_+.$$

Now given a profile f, observe that $\frac{f}{\|f\|} \in \triangle_+ \subset (\triangle_{\eta(\varepsilon)} + \varepsilon\ \triangle_+)\frac{1}{1+\varepsilon}$. Therefore there exists $h \in \frac{1}{1+\varepsilon}\triangle_{\eta(\varepsilon)}$ such that $\frac{f}{\|f\|} \in (\{h\} + \frac{\varepsilon}{1+\varepsilon}\triangle_+)$. Now, define $g := h\|f\|$. Then $g \leq f$ and, by construction, $\frac{g}{\|g\|} = (1+\varepsilon)h \in \triangle_{\eta(\varepsilon)}$. Moreover $\frac{\|f\|-\|g\|}{\|f\|} = 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon$. ■

Proof of Fundamental Proposition: Given a positive integer η, we first define an integer that will play an important role in the proof. Arbitrarily select $x \in \triangle_\eta$ and define $y(x) := \eta x \in Z^T_+$. Since $y(x)$ is a profile, by Lemma 1 there is an integer $r(y(x), B)$ such that for any integer k the profile $kr(y(x), B)y(x)$ is B-partition-balanced. There exists such an integer $r(y(x), B)$ for each $x \in \triangle_\eta$. Since $\triangle_\eta$ contains only a finite number of points, there is a finite integer $M(\eta, B)$ such that $\frac{M(\eta,B)}{r(y(x),B)}$ is an integer for any $x \in \triangle_\eta$.

By Lemma 2, given $\frac{\varepsilon}{2} > 0$, there exists a positive integer $\eta' := \eta(\frac{\varepsilon}{2})$, such that for any $f \in \mathbf{R}^T_+$ there exists a vector $g \in \mathbf{R}^T_+$ satisfying

$$\begin{aligned} &g \leq f, \\ &\|f\| - \|g\| = \|f - g\| \leq \tfrac{\varepsilon}{2}\|f\| \text{ and} \\ &\tfrac{g}{\|g\|} \in \triangle_{\eta'}. \end{aligned}$$

Arbitrarily select $f \in \mathbf{R}^T_+$ and let $g \in \mathbf{R}^T_+$ be a vector satisfying the above conditions. Define $y^* := \eta'\frac{g}{\|g\|}$. Since $\frac{g}{\|g\|} \in \triangle_{\eta'}$, it holds that $y^* \in Z^T_+$. Therefore y^* is a profile. Moreover, by the choice of $M(\eta', B)$, the $kM(\eta', B)^{th}$-replica of the profile y^* is B-partition-balanced for any positive integer k.

Observe that there is an integer k^0, possibly equal to zero, such that

$$k^0 M(\eta', B)y^* \leq g < (k^0 + 1)M(\eta', B)y^*.$$

Define

$$f' := k^0 M(\eta', B)y^* = k^0 M(\eta', B)\eta'\frac{g}{\|g\|}.$$

Obviously, f' is a profile ($f' \in Z_+^T$) and $f' \leq g \leq f$. Suppose that $k^0 > 0$. (This is the only relevant case since we are concerned only with "sufficiently large" profiles – that is, those profiles for which k_0 is greater than zero. See below.) Then

$$\frac{f'}{\|f'\|} = \frac{g}{\|g\|} \text{ and } \|g\| - \|f'\| = \|g - f'\| \leq M(\eta', B)\eta'.$$

Moreover, the profile f' is B-partition-balanced since it is a replica of the profile y^*.

Now, define $k(\varepsilon, B) := M(\eta', B)\eta' \frac{2}{\varepsilon}$. If $\|f\| \geq k(\varepsilon, B)$, then

$$\begin{aligned} &k^0 > 0,\ f' \leq g \leq f\ , \\ &\|f - g\| \leq \tfrac{\varepsilon}{2} \|f\|\ , \text{ and} \\ &\|g - f'\| \leq M(\eta', B)\eta' \leq \tfrac{\varepsilon}{2} \|f\|\ . \end{aligned}$$

Therefore $\|f\| - \|f'\| = \|f - f'\| \leq \varepsilon \|f\|$. Thus f' is a subprofile of f, $\frac{\|f'\|}{\|f\|} \geq 1 - \varepsilon$, and f' is B-partition-balanced. ■

6.2 Proofs of the Theorems.

Proof of Theorem 1: Fix the number of types T and consider the bound $k(\varepsilon, B)$ from the Fundamental Proposition. Let $\eta_1(\varepsilon, B, T) := k(\varepsilon, B)$. Let (N, V) be a game with $|N| \geq k(\varepsilon, B)$. Denote the profile of N by f. By the Fundamental Proposition, f is ε-B-partition-balanced. That is, there is a B-partition-balanced subprofile f' of f such that $\frac{\|f'\|}{\|f\|} \geq 1 - \varepsilon$. Now select some $S \subset N$ such that $|S\,[i]| = f'_i$ for any $i = 1, .., T$. Then $\frac{|N| - |S|}{|N|} \leq \varepsilon$ by choice of S and the subgame (S, V) has a nonempty core. Thus the ε-remainder core of (N, V) is nonempty. ■

Proof of Theorem 2: For any $S \subset N$ define $V'(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S)))$, where the intersection is taken over all type-preserving permutations τ of the player set N. Then $(N, V') \in G^q((0, T), (\beta, B))$. Moreover, from the definition of $V'(S)$ it follows that $V'(S) \subset V(S)$. (Informally, taking the intersection over all type-preserving permutations makes all players of each approximate type no more productive than the least productive members of that type.) From the definition of δ-substitutes, it follows that $H_\infty[V'(S), V(S)] \leq \delta$ for any $S \subset N$.

Now for any $S \subset N$, define $V''(S) := c_q(V'(S; B))$. Then, by construction,

$$(N, V'') \in G^q((0, T), (0, B)).$$

Moreover,

$$V''(S) \subset V'(S) \subset V(S)$$

and

$$H_\infty[V''(S), V(S)] \leq H_\infty[V''(S), V'(S)] + H_\infty[V'(S), V(S)] \leq \beta + \delta.$$

By Theorem 1, if $|N| \geq \eta_1(\varepsilon, T, B)$ then the ε_1-remainder core of the game (N, V'') is nonempty. That is, there exists $S \subset N$ such that $\frac{|N|-|S|}{|N|} \leq \varepsilon_1$ and such that (S, V) has a nonempty core. Let x be a payoff vector in the core of the game (S, V''). Since $V''(S) \subset V(S)$, the payoff vector x is feasible for (S, V). Since $H_\infty[V''(S), V(S)] \leq \beta + \delta$ and x is in the core of (S, V''), x is $(\beta + \delta)$-undominated for the game (S, V). Since $(\beta + \delta) \leq x_2$ and $\frac{|N|-|S|}{|N|} \leq \varepsilon_1$, x is in the ε_1-remainder ε_2-core of (N, V). ■

Proof of Theorem 3: As in the proof of Theorem 2 first construct the game $(N, V'') \in G^q((0, T), (0, B))$. As noted in the proof of Theorem 2, $V''(S) \subset V(S)$ and $H_\infty[V''(S), V(S)] \leq \beta + \delta$ for any $S \subset N$. In addition, the game (N, V'') has a per capita bound of C. To simplify notations let us define $\varepsilon_1 = \varepsilon - (\beta + \delta) > 0$. Recall that we required $2(\beta + \delta) < m^*$. Assume first that $2\varepsilon_1 \leq m^*$. Thus $\varepsilon = (\varepsilon_1 + \beta + \delta) < m^*$.

Applying Theorem 1 for $\varepsilon^0 := \frac{q}{BC}\varepsilon_1$ to the game $(N, V^{''})$ we found that for $|N| \geq \eta_1(\varepsilon^0, B, T)$ there is some subset of players $S \subset N$ with $\frac{|N|-|S|}{|N|} \leq \varepsilon^0$ such that the game (S, V'') has a nonempty core. Let x be a payoff vector in the core of (S, V''). We now construct a payoff vector $y \in \mathbf{R}^N$ for the game (N, V''). For $i \in S$, define $y_i := x_i - \varepsilon_1$ and for $i \notin S$, define $y_i := BC - \varepsilon_1$. Observe that y is in the ε_1-core of the game (S, V'').

We next need to show that $y \in V(N)$. Since $\frac{|N|-|S|}{|N|} \leq \varepsilon^0 = \frac{q}{BC}\varepsilon_1$, it holds that

$$q\varepsilon_1 |N| \geq BC(|N| - |S|).$$

Since $q \leq 1$, it follows that

$$q\varepsilon_1 |S| \geq (BC - \varepsilon_1)(|N| - |S|).$$

Informally, this means that we can take ε_1 away from each player in S, transfer this amount to the players in $N \backslash S$ at the rate q, and increase the payoff to each player in $N \backslash S$ to $BC - \varepsilon_1$. Therefore since $x \in V''(S) \subset V(S)$, by superadditivity and by q-comprehensiveness of payoff sets it holds that $y \in V(N)$.

We now prove that the payoff vector y is ε_1-undominated in the game (N, V''). (We then proceed to show that y is ε-undominated in the game (N, V)). The strategy of the proof is to show that if y is ε_1-dominated in the game (N, V'') then it can be ε_1-dominated by some group (to be called) $A \subset S$. We thus obtain a contradiction. The proof proceeds through two steps.

The first step is to construct the group A. Suppose that y is ε_1-dominated in the game (N, V'') by some group W. Specifically, suppose there exists a payoff vector z such that

$$z \in V''(W) = c_q(V'(W; B)) \text{ and}$$

$$z_W >> y_W + \vec{1}_W \varepsilon_1.$$

Since $z \in V''(W)$ there exists some partition $\left\{W^k\right\}$ of W, $\left|W^k\right| \leq B$ and some payoff vector $z' \in \sum_k V'(W^k)$ such that z can be obtained from z' by making "transfers" at the rate q between players in W. Let

$$A := \bigcup \left\{W^k : W^k \subset S\right\} \text{ and let } A^L := \bigcup \left\{W^k : W^k \backslash S \neq \emptyset\right\};$$

that is, A consists of those members of subsets in $\{W^k\}$ that are contained in S and A^L consists of those members of subsets of $\{W^k\}$ that contain at least one player from $N \backslash S$.

The second step is to show that the set A is nonempty and the player set $\cup_{W^k \in A}$ can ε_1-dominate the payoff vector y. Since y is in the ε_1-core of the subgame (S, V'') it is clear that the group W must contain at least one member of $N \backslash S$; if not, a contradiction is immediate. Therefore the set A^L must be nonempty. Observe that for any $W^k \subset A^L$ and $x' \in V'(W^k)$, since $\left|W^k\right| \leq B$ it holds that $\sum_{i \in W^k} x'_i \leq BC$. From the assumption that $0 \in V(\{i\})$, the construction of y and the fact that $z_W >> y_W + \vec{1}_W \varepsilon_1$, it follows that $z_i >> 0$ for all $i \in W^k$. Since there exists, however, $i \in W^k \backslash S$ such that $z_i >> y_i + \varepsilon_1 = BC$, we have $\sum_{i \in W^{k'}} z_i > BC \left|W^{k'} \cap N \backslash S\right| > BC$. From per capita boundedness, $\sup_{x'' \in V(W^{k'})} \sum_{i \in W^{k'}} x''_i \leq BC$. This is a contradiction. Thus, z can be feasible in $V''(W)$ only by some transfers from the players in the set $\cup_{W^k \in A} W^k$ to the players in the set $\cup_{W^k \in A^L} W^k$. This implies that the set A is nonempty. Moreover the group A is not a net beneficiary of transfers. This implies that there is a payoff vector $z'' \in V''(\cup_{W^k \in A} W^k)$ such that for all players $i \in \cup_{W^k \in A} W^k$,

$$z''_i \geq z_i > y_i + \varepsilon_1.$$

Since $A \subset S$, this is a contradiction to the construction of y as a payoff vector in the ε_1-core of the game (S, V''). We conclude that y is ε_1-undominated in the game (N, V'').

Since the payoff vector y is ε_1-undominated in the game (N, V''), for $|N| \geq \eta_1(\frac{q}{BC}\varepsilon_1, B, T)$ the payoff vector y is in the ε_1-core of the game (N, V''). This implies that y is feasible and $(\varepsilon_1 + \beta + \delta)$-undominated in the initial game (N, V), providing that $|N| \geq \eta_1(\frac{q}{BC}\varepsilon_1, B, T)$. Let $\eta_2(\varepsilon_1, B, T, C, q) := \eta_1(\frac{q}{BC}\varepsilon_1, B, T)$. Thus, we proved that for $|N| \geq \eta_2(\varepsilon - (\beta + \delta), B, T, C, q)$ the ε-core of the game (N, V) is nonempty.

For $\varepsilon_1 > \frac{m^*}{2}$ let us define $\eta_2(\varepsilon - (\beta + \delta), B, T, C, q) := \eta_2(\frac{m^*}{2}, B, T, C, q)$. Then for $|N| \geq \eta_2(\varepsilon - (\beta + \delta), B, T, C, q)$ again the ε-core of the game (N, V) is nonempty. ■

6.3 Final Example.

This example shows indispensability of the condition in Theorem 3 that q is greater than zero, thus clarifying the need for this condition in Theorem 3 in contrast to Theorem 1 and Theorem 2.

Example A. *The positivity of q.* Consider a sequence of games without side payments $(N^m, V^m)_{m=1}^{\infty}$ where the m^{th} game has $2m+1$ players. Suppose that any player alone can earn only 0 units or less. Suppose that any two-player group can distribute a total payoff of 2 units in any agreed-upon way, while there is no transferability of payoff between groups. Suppose only one- and two-player groups are effective. Then the game is described by the following parameters: $q = 0, T = 1, B = 2, \delta = \beta = 0$. Moreover, the game has per capita bound $C = 1$. Thus the game satisfies strict small group effectiveness and per capita boundedness. However, the $\frac{1}{3}$-core of the game is empty for arbitrarily large values of m. (At any feasible payoff vector, at least one player gets 0 units and some other player no more than 1 unit. These two players can form a group and gain $\frac{1}{2}$ each.) ■

References

[1] B. Allen, Incentives in market games with asymmetric information: Approximate NTU cores in large economies," Chapter 11 in *Social Choice, Welfare, and Ethics: Proceedings of the Eighth International Symposium in Economic Theory and Econometrics* (W.A. Barnett, H. Moulin, M. Salles, and N.J. Schofield Eds.), Cambridge: Cambridge University Press, 1994.

[2] R.M. Anderson, An elementary core equivalence theorem, *Econometrica* 46 (1978), 1438-1487.

[3] R.J. Aumann, Markets with a continuum of traders, *Econometrica* 32 (1964), 39-50.

[4] W.J. Baumol, J.C. Panzer and R.D. Willig, *Contestable Markets and the Theory of Industry Structure*, San Diego: Harcourt, Brace, Javanovitch, 1982.

[5] E. Bennett and M. Wooders, Income distribution and firm formation, *Journal of Comparative Economics* 3 (1979), 304-317.

[6] V. Böhm, The limit of the core of an economy with production, *International Economic Review* 15 (1974), 143-148.

[7] J. Broome, Approximate equilibrium in economies with indivisible commodities, *Journal of Economic Theory* 5 (1972), 224-249.

[8] J. Buchanan, An economic theory of clubs, *Economica* 33 (1965), 1-14.

[9] J. Conley, Convergence theorems on the core of a Public Goods Economy: Sufficient Conditions, *Journal of Economic Theory* 62 (1994), 161-85.

[10] J. Conley and M.Wooders, Taste homogeneity of optimal jurisdictions in a Tiebout economy with crowding types and endogenous educational investment choices, *Richerche Economiche* 50 (1996), 367-387.

[11] J. Conley and M. Wooders, Equivalence of the core and competitive equilibrium in a Tiebout economy with crowding types, *Journal of Urban Economics* (1997), 421-440.

[12] J. Conley and M. Wooders, Anonymous pricing in Tiebout economies and economies with clubs,in *Topics in Public Finance*, (D. Pines, E. Sadka, and I. Zilcha Eds.), Cambridge: Cambridge University Press, 1998.

[13] J. Conley and M. Wooders, Tiebout economies with differential genetic types and endogenously chosen crowding characteristics, *Journal of Economic Theory* 98 (2001), 261-294.

[14] V.P. Crawford, Comparative Statics in Matching Markets, *Journal of Economic Theory* 54 (1991), 389-400.

[15] N. Dagan, R. Serrano and O. Volij, Bargaining, Coalitions and Competition, *Economic Theory* 15 (2000), 279-296.

[16] G. Debreu, New concepts and techniques for equilibrium analysis, *International Economic Review* 3 (1962), 257-273.

[17] G. Debreu and H. Scarf, A limit theorem on the core of an economy,*International Economic Review* 4 (1963), 235-246.

[18] F. Forges, A. Heifetz and E. Minelli, Incentive compatible core and competitive equilibria in differential information economies, *Economic Theory* 18 (2001), 349-365.

[19] R. Garratt and C.-Z. Qin Cores and competitive equilibria with indivisibilities and lotteries, *Journal of Economic Theory* 68 (1996), 531-543.

[20] D.B. Gilles, Some Theorems on n-person Games, Ph.D. Dissertation, Department of Mathematics, Princeton University, Princeton, 1953.

[21] J. Greenberg and S. Weber, Strong Tiebout equilibrium with restricted preference domain, *Journal of Economic Theory* 38 (1986), 101-117.

[22] P.J. Hammond, On f-core equivalence with general widespread externalities, *Journal of Mathematical Economics* 32 (1999), 177-184.

[23] P.J. Hammond, M. Kaneko, and M. Wooders (1989) Continuum economies with finite coalitions; Core, equilibrium and widespread externalities, *Journal of Economic Theory* 49 (1989), 113-134.

[24] W. Hildenbrand, *Core and Equilibria of a Large economy*, Princeton: Princeton University Press, 1974.

[25] W. Hildenbrand and A.P. Kirman, *Introduction to Equilibrium Analysis*, Amsterdam- Oxford: North Holland, 1976.

[26] T. Ichiishi and J. Schäffer, The τ-core of a game without side payments,Carnegie Mellon University Graduate School of Industrial Administration Working Paper No. 9-79-80, 1979.

[27] M. Kaneko and M. Wooders, Cores of partitioning games, *Mathematical Social Sciences* 3 (1982), 313-327.

[28] M. Kaneko and M. Wooders, The core of a game with a continuum of players and finite coalitions: The model and some results, *Mathematical Social Sciences* 12 (1986), 105-137.

[29] M. Kaneko and M. Wooders, The core of a continuum economy with widespread externalities and finite coalitions: From finite to continuum economics, *Journal of Economic Theory* 49 (1989), 135-168.

[30] M. Kaneko and M. Wooders, The nonemptiness of the f-core of a game without side payments, *International Journal of Game Theory* 25 (1996), 245-258.

[31] Y. Kannai, Continuity properties of the core of a market, *Econometrica* 38 (1970), 791-815.

[32] A.S. Kelso and V.P. Crawford, Job matching, coalition formation, and gross substitutes, *Econometrica* 50 (1982), 1483-1504.

[33] H. Konishi, M. Le Breton, and S. Weber, Equilibrium in a finite local public goods economy, *Journal of Economic Theory* 79 (1998), 224-244.

[34] A. Kovalenkov and M. Wooders, Epsilon cores of games and economies with limited side payments, Autonomous University of Barcelona WP 392.97, revised, *Games and Economic Behavior* 36 (2001), 193-218.

[35] A. Kovalenkov and M. Wooders, An exact bound on ε for nonemptiness of ε-cores of games,Autonomous University of Barcelona WP 393.97 and 394.97, revised, *Mathematics of Operations Research* 26 (2001), 654-678.

[36] A.M. Manelli, Core convergence without monotone preferences and free disposal, *Journal of Economic Theory* 55 (1991), 400-415.

[37] A.M. Manelli, Monotonic preferences and core equivalence, *Econometrica* 59 (1991), 123-138.

[38] A. Mas-Colell, A refinement of the core equivalence theorem, *Economic Letters* 3 (1979), 307-310.

[39] A. Mas-Colell, Efficiency and decentralization in the pure theory of public goods, *Quarterly Journal of Economics* 94 (1980), 625-641.

[40] R. Myerson, *Game Theory*, Cambridge- London: Harvard University Press, 1991.

[41] J. Ostroy, A reformulation of the marginal productivity theory of distribution, *Econometrica* 52 (1984), 599-630.

[42] M. Pauly, Clubs, commonality and the core: An integration of game theory and the theory of public goods, *Economica* (1967), 314-324.

[43] M. Pauly, Cores and clubs, *Public Choice* 9 (1970), 53-65.

[44] A. Roth and M. Sotomayor, *Two-sided Matching; A Study in Game-theoretic Modeling and Analysis*, Cambridge: Cambridge University Press, 1990.

[45] H.E. Scarf, The core of an n-person game, *Econometrica* 35 (1967), 50-67.

[46] R. Selten, A non-cooperative model of characteristic function bargaining, in "*Essays in Game Theory and Mathematical Economics in Honor of O.Morgenstern*," (V. Böhm and M. Nachtkamp Eds.), Manheim- Wein- Zürich: Bibliographisches Institut, 1981.

[47] L.S. Shapley, On balanced sets and cores, *Naval Research Logistics Quarterly* 14 (1967), 453-460.

[48] L.S. Shapley and M. Shubik, Quasi-cores in a monetary economy with nonconvex preferences, *Econometrica* 34 (1966), 805-827.

[49] L.S. Shapley and M. Shubik, On market games, *Journal of Economic Theory* 1 (1969), 9-25.

[50] M. Shubik, Edgeworth Market Games, in "*Contributions to the Theory of Games IV, Annals of Mathematical Studies 40*," (F.R. Luce, and A.W. Tucker, eds.), Princeton: Princeton University Press, 1959.

[51] M. Shubik, The 'Bridge Game' economy: An example of indivisibilities, *Journal of Political Economy* 79 (1971), 909-912.

[52] M. Shubik and M. Wooders, Near-markets and market-games, Cowles Foundation Discussion Paper No. 657, 1982.

[53] M. Shubik and M. Wooders, Clubs, near-markets and market-games, in "*Topics in Mathematical Economics and Game Theory; Essays in Honor of Robert J. Aumann,*" (M. Wooders ed.), Fields Institute Communication Volume, Providence: American Mathematical Society, 1999.

[54] M. Shubik and M. Wooders, Approximate cores of replica games and economies: Part I. replica games, externalities, and approximate cores, *Mathematical Social Sciences* 6 (1983), 27-48.

[55] V. Vasil'ev, S. Weber and H. Wiesmeth, Core equivalence with congested public goods, *Economic Theory* 6 (1995), 373-387.

[56] S. Weber, On ϵ-cores of balanced games, *International Journal of Game Theory* 8 (1979), 241-250.

[57] M. Wooders, Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good, *Journal of Economic Theory* 18 (1978), 328-348.

[58] M. Wooders, A characterization of approximate equilibria and cores in a class of coalition economies, Stony Brook Department of Economics Working Paper No. 184, 1977, Revised 1979, on-line at http://www.warwick.ac.uk/fac/soc/Economics/wooders/.

[59] M. Wooders, Asymptotic cores and asymptotic balancedness of large replica games, Stony Brook Department of Economics Working Paper No. 215, 1979, revised 1980, on-line at http://www.warwick.ac.uk/fac/soc/Economics/wooders/.

[60] M. Wooders, "The ε-core of an N-person game," State University of New York at Stony Brook Working Paper No. 216, September 1979.

[61] M. Wooders, "The epsilon core of a large game," Cowles Foundation Discussion Paper 612, 1981.

[62] M. Wooders, The epsilon core of a large replica game, *Journal of Mathematical Economics* 11 (1983), 277-300.

[63] M. Wooders, Stability of jurisdiction structures in economies with a local public good, *Mathematical Social Sciences* 15 (1988), 24-29.

[64] M. Wooders, The efficaciousness of small groups and the approximate core property in games without side payments, University of Bonn Sonderforschungsbereich 303 Discussion Paper No. B-17, 1991.

[65] M. Wooders, Inessentiality of large groups and the approximate core property; An equivalence theorem, *Economic Theory* 2 (1992), 129-147.

[66] M. Wooders, Large games and economies with effective small groups, in "*Game Theoretic Methods in General Equilibrium Analysis*," (J.-F. Mertens and S. Sorin Eds.), Dordrecht- Boston- London: Kluwer Academic Publishers, 1994.

[67] M. Wooders, Equivalence of games and markets, *Econometrica* 62 (1994), 1141-1160.

[68] Wooders, M. H. (1994c), "Approximating games and economies by markets," University of Toronto Working Paper No. 9415.

[69] M. Wooders, Multijurisdictional economies, the Tiebout Hypothesis, and sorting, *Proceedings of the National Academy of Sciences* 96 (1999), 10585-10587, on-line at http://www.pnas.org/cgi/content/full/96/19/10585.

[70] M. Wooders, and W.R. Zame, Approximate cores of large games, *Econometrica* 52 (1984), 1327-1350.

[71] M. Wooders and W.R. Zame, NTU values of large games, IMSSS Technical Report No. 503, 1987.