Computation of Power Indices

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COMPUTATION OF POWER INDICES

by

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1.1 Introduction: Voting Power and Power Indices

Many organizations have systems of governance by voting that are designed to give different amounts of influence over decision making to different members. For example the joint stock company gives each shareholder a number of votes in proportion to his or her ownership of ordinary stock; the shareholder body is designed to be a democratic decision-making group with each share having equal influence but with individual shareholders having different numbers of shares to reflect their relative capital contributions. Many international economic organizations have been designed on a similar principle, each country being entitled to a number of votes based on its financial contribution, the most prominent examples being the Bretton Woods institutions: the International Monetary Fund and World Bank. Federal political bodies which use the principle of weighted voting where the weights reflect populations rather than contributions include the European Union Council of Ministers and the US Presidential Electoral College, where the individual states' votes are cast as blocs of different sizes.

As general voting systems, considered in the abstract without reference to their different contexts, these are all formally similar and can be classed as weighted voting games. They contain considerable analytical interest because, when we consider their practical implications, by studying all theoretically possible voting outcomes, and how individual members' votes relate to them, then it turns out that the resulting distribution of power is often different from what the designers intended. On the other hand, it is almost always assumed, by writers analysing the distribution of votes, that the power of a member is the same as his or her share of the votes. For example, it is often the case in discussions of the IMF, that a member with five percent of the votes is described

1

as possessing five percent of the voting *power*, or that the United States with almost 18 percent of the votes, thereby has 18 percent of the voting *power*. Yet the proportion of decisions that may - at least theoretically - be taken by vote in which the member who has five percent may be pivotal in determining the outcome may not actually be five percent at all, and the votes of the United States may in fact be capable of being decisive in more or less than 18 percent of cases. Therefore it is untrue to claim that their respective shares of the total voting power are 5% and 18%.

A simple example that illustrates the point clearly is that of a company with three shareholders, two having 49 percent of the shares each and the third with 2 percent. It is not useful to describe these figures as shares of the power each has in running the company because if the decision rule requires a simple majority of more than 50 percent of the votes, then any two shareholders are required to support a motion for it to pass. Any shareholder can win by combining with one other and therefore the one with 2 percent has exactly the same power as one with 49 percent. Therefore by considering all possible voting outcomes it becomes clear that each shareholder has equal power despite the disparity in their votes. Many such examples can be constructed or found in the real world, in which the distribution of power among members of a weighted voting body - a member's power being his or her ability to join coalitions of others which do not have the required majority and make them winning - is not at all the same as the distribution of votes.

Another, well known, example is the original Council of the European Economic Community. Between 1958 and 1972 it had six member countries and used a system of qualified majority voting that allocated 4 votes each to France, West Germany and Italy, 2 votes each to

Belgium and the Netherlands and one vote to Luxembourg. From these figures one might assume that the smaller countries would have a disproportionately large amount of power. For example, Luxembourg, with 5.88 percent of the votes and less than 0.2 percent of the population, had 25 percent as many votes as West Germany with only 0.57% of its population; Luxembourg had one vote for 310,000 people while West Germany had one vote for every 13,572,500, suggesting that Luxemburgers were 43.78 times more powerful than Germans. In fact, however, since the number of votes required for a decision was fixed at 12, Luxembourg's one vote could never make any difference: it was impossible for it to add its vote to those of any losing group of other countries with precisely 11 votes and therefore its formal voting power was precisely zero. This is an extreme case, but a real one, which illustrates the analytical importance of looking at the possible outcomes of a weighted majority vote, as well as the nominal voting strengths, in considering voting power. The same point arises also in the context of the corporation when we study the power of large stockholders. Obviously if there is a majority shareholder he or she has all the voting power and none of the other shareholders has any voting power at all. However, it is well known that if the largest shareholder has a very substantial minority holding his or her vote will often be decisive in a proxy ballot or fully attended company meeting, even to the extent that he or she could be said to have working control of the corporation, if his or her voting power were sufficiently large. For example, it is almost universally accepted by writers who have studied corporate ownership and control that a single 20 percent shareholder faced with many small shareholders is very powerful indeed. This power is certainly not reflected in the number of its

shares and may in fact be very close to that of a majority shareholder. Although necessarily strictly less than 100 percent, it may be extremely close to it.

The question has been studied by the use of power indices as measures of the ability of members to influence voting outcomes. As a branch of co-operative game theory the field of power indices may be thought to date from the publication of the seminal paper by Shapley and Shubik in 1954, although if it is construed more generally than part of game theory, voting power analysis is much older. However it has failed to achieve wide acceptance due to ambiguity due to different indices yielding different results for the same data. This has meant that the field has remained at the frontier for almost fifty years. See Shapley and Shubik (1954), Banzhaf (1965), Coleman (1971), Dubey and Shapley (1979), Lucas (1983), Straffin (1994), Owen (1995), Felsenthal and Machover (1998). Good surveys are provided by Felsenthal and Machover (1998), Lucas (1983) and Straffin (1994).

This paper is mostly concerned with the *computation* of the so called *classical* power indices, proposed by Shapley and Shubik (1954) and by Banzhaf (1965) – the latter was in fact originally proposed by Penrose in 1946 in a paper that was for long overlooked[1] - both of which have been widely applied. (I will give some consideration also to the indices proposed by Coleman which have certain similarities to the Banzhaf indices but also some important differences.) The difficulty of computing the indices, especially when the number of members is large, has been a major factor limiting the use of the technique as a means of studying real institutions. Both indices

are based on a common idea that a member's power rests on how often he or she can add his or her votes to those of a losing coalition so that it wins, but they differ in the way that such coalitions are counted. In consequence, where both indices have been used to analyse the same voting body, they have been found to give different results. This has meant that in the absence of any objective evidence on the *actual* distribution of power, it has not been possible either to test the power indices approach or to establish the respective utility of the indices. This question is not addressed in the present paper; however, see Leech (2002b) for a discussion of it and some evidence.

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[1] See Felsenthal and Machover (1998) for the history of the measurement of voting power.

1.2 Power Indices: Notation and Definitions

I consider a weighted majority game of voting in a legislature with n members or players represented by a set N = {1, 2, . . . , n} whose voting weights are $w_1, w_2, \ldots, w_n$. The combined voting weight of all members of a coalition represented by a subset T, T $\subseteq$ N, will be denoted by the function w(T), where $w(T) = \sum_{i \in T} w_i.$

The decision rule is defined in terms of a quota, q, by which a coalition of players represented by subset T is winning if w(T) = q and losing if w(T) < q. (It is customary to require q > w(N)/2 to ensure a unique decision and hence that the voting game is a proper game, although this is not actually essential to the definition of power and the use of power indices – see Coleman (1971).) The weights and the quota are real numbers in general; although in many applications and textbook examples they are integers, this is not essential to the theory, although it may be relevant to certain methods of computation.

I will sometimes use the notation {q; $w_1, w_2, \ldots, w_n$} to represent the voting body. Note there is one decision rule with one set of weights. In some applications it is necessary to generalise this: for example the system of qualified majority voting in the EU council agreed at Nice is formally a triple-majority rule in which three conditions must be satisfied in terms of weighted votes, population and number of member countries. Also in some legislatures different types of decisions require different majorities – some may require a simple majority, some a supermajority (for example the IMF and World Bank boards of governors require a 50% majority for some decisions and a 85% majority for others) – so that power varies according to the kind of

issue being decided. I am not going into these matters here, since they are complications that are non-essential from the point of view of computation.

A power index is an n-vector whose elements measure the respective ability of each player to determine the outcome of a general vote. The index for each player is defined in terms of the relative number of times that player can influence the decision by transferring his or her voting weight to a coalition which is losing without him but wins with him. This is referred to as a swing. Formally a swing for player i can be defined as a pair of subsets, $(T_i, T_i + \{i\})$ such that T_i is losing, but $T_i + \{i\}$ is winning. In terms of voting weight, T_i is a swing if $q - w_i = w(T_i) < q$.

The power index for player i is defined as the relative frequency of swings for i with respect to a coalition model where, in some sense, each possible coalition is treated equally. If coalitions are regarded as being formed randomly (and equally likely in some sense) then the index is a probability. The two indices, however, employ different probability models and are mathematically distinct.

The Shapley-Shubik index is the probability that i swings (or is "pivotal" in the terminology of Shapley and Shubik) if all *orderings* of players are equally likely. Thus, given a particular swing for a member, the index is the number of orderings of both the members of the coalition T_i and the players not in T_i relative to the number of orderings of the set of all players N: every reordering is counted separately. The index is the probability of a swing for the player within this probability model.

For a given swing for player i, the number of orderings of the members of the subset T_i and its complement (apart from player i), $N-T_i-\{i\}$, is $t!(n-t-1)!$ where t is the number of members of T_i and n is the total number of players, members of N. The total number of swings for i defined in this way for this coalition model is $\sum_{T_i} t!(n-t-1)!$. The index, ϕ_i , is this number as a proportion of the number of orderings of all players in N,

$$\phi_i = \sum_{T_i} \frac{t!(n-t-1)!}{n!} \, . \qquad (1.)$$

If all orderings are equiprobable, it is the probability of a swing.

The Banzhaf index, on the other hand, treats all *coalitions* T_i as equiprobable, players being arranged in no particular order. A member's power index is then the number of swings expressed as a fraction of either the total number of coalitions (measuring the probability of a swing), or of the total number of swings for all players (measuring the player's relative capacity to swing).

The number of swings for i is $\eta_i = \sum_{T_i} 1$. The two versions of the index are defined by expressing this number over different denominators. The Non-Normalized Banzhaf index (or Banzhaf Swing Probability), β_i', uses the number of coalitions which do not include i , 2^{n-1}, the number of subsets of $N-\{i\}$, as denominator, and therefore it can be written as:

$$\beta_i' = \sum_{T_i} 1/2^{n-1} \, . = \eta_i/2^{n-1} \qquad (2.)$$

The Normalized Banzhaf Index, β_i, uses the total number of swings for all players as the denominator to measure relative voting power among players:

$$\beta_i = \eta_i / \sum \eta_i, \tag{3.}$$

or alternatively, $\beta_i = \beta_i' / \sum \beta_i'$ (Therefore in the discussion of computation of the Banzhaf index it is only necessary to consider the details of computing the swing probability version, (2), or find the numbers of swings. Both approaches are followed.)

Both the normalized indices sum to unity over players: $\sum \phi_i = 1$ and $\sum \beta_i = 1$.

The Coleman indices require, in addition, the number of winning coalitions, $\omega = \sum_S 1$, for all $S \subset N$ where $w(S) = q$. This imposes an extra computing requirement but enables us to calculate the three indices:

Power of the voting body to Act: $\qquad A = \dfrac{w}{2^n} \tag{4.}$

Power of member i to Prevent Action: $\qquad PPA_i = \dfrac{\eta_i}{\omega} \tag{5.}$

Power of member i to Initiate Action: $\qquad PIA_i = \dfrac{\eta_i}{2^n - \omega} \tag{6.}$

Neither (5) nor (6) have meaningful normalized versions, since they are both linear transformations of β_i' and therefore normalizing would reduce them both to β_i. It is unnecessary to calculate the Coleman indices when the game is one with a simple majority since then (5) and (6) are equal to (2).

1.3 The Uses of Power Indices

A power index provides a method of finding the *a priori* distribution of voting power in a given voting body, that is, how much influence each member has, either absolutely or relatively to the other players. It is important to understand the limitations of the approach. It does not provide a description of the disposition of *actual* power. That would depend on the preferences of the members and the range of issues over which decisions are taken, which would introduce the need to give different coalitions differently in the measure of power. *A priori* voting power is an element in actual power and therefore a useful analysis, based on idealized assumptions, that is informative but limited, rather in the same way as in many other areas of economics. The model of perfect competition does not provide a full account of any real industry but is nevertheless a useful analytical tool. Another example is a price index which uses fixed weights: it is a truism that this gives is an inaccurate measure of inflation, but since this is well understood and no other index is necessarily better, that does not prevent its widespread use in the real world. There are many other similar examples. *A priori* voting power analysis is useful because it tells us about the power implications of the legislature itself in a pure sense.

Power indices have an important use in the design of voting systems. When a constitution or rule is being framed, involving weighted voting, it is necessary to determine the quota and the distribution of weights. The practice of doing this is not well developed (as is evident from the problems surrounding the Nice Intergovernmental Council) and *a priori* power indices are a potentially useful technique to help in this. The framers of the rule can state explicitly what the

powers of the members ought to be – for example the members of the EU might agree that the weights should be such as to give equal power to all citizens of each member state - and then the weights and quota can be chosen so that the power indices correspond. Thus the computational problem is inverted: the power indices are the givens and the weights are the outputs to be determined. This is a much more formidable computational problem than finding the power indices, and our understanding of it is not yet well developed. This is sometimes referred to in the context of power indices as "the inverse problem".

The case for using power indices to design decision making systems is much less open to question than their use in measuring actual power, because the issue is normative rather than positive. In designing a voting system with given properties, the question to be answered is what the weights and quota *should* be, rather than what the power implications of a particular institutional arrangement *are*. In this case it is not appropriate to take account of preferences and the likelihood of different coalitions being formed since such informal information would not normally be built into a formal rule or constitution. A voting system for an expanding European Union would need to treat each member state formally as a sovereign and independent actor free to determine its own policies and alliances. Therefore *a priori* voting power, based on treating each voting coalition as equally likely, would be a natural approach. I discuss some aspects of this computation problem below.

2.1 Computation Methods for Finite Games

Several methods are available to compute the indices, and they are described in turn in the following sections. The different algorithms vary a lot in their approach and in their computing requirements. None is universally ideal and for each its advantages and limitations are considered. Two dimensions of algorithmic complexity are of supreme importance: running time (time complexity) and memory requirements (space complexity). Other dimensions of the complexity of an algorithm include its ease of implementation, its input requirements, and so on. It is also necessary to consider the suitability of the algorithm for a particular use, in particular the "inverse problem".

The direct application of the definitions, embodied in equations (1) to (6), that I have called the Direct Enumeration method described in the next section. It is the simplest algorithm but not suitable for games with a large number of players (greater than about 30). For larger games other methods have had to be devised. These are described in the sections below as follows.

Section	Method
2.2	Direct Enumeration.
2.3	Monte Carlo simulation due to Mann and Shapley (1960).
2.4	Generating Functions, due to Mann and Shapley (1962).

2.5 Multilinear Extensions Approximation methods, due to Owen (1972, 1975a).

2.6 Modified MLE Approximation, proposed in Leech (2002a).

Dennis Leech

2.2 Direct Enumeration

The simplest method is to compute the indices directly from their definitions. This requires the use of an algorithm to find each subset of players exactly once (via a search which finds each corner of a hypercube exactly once, known as a Hamilton walk; for example that in Nijenhuis and Wilf (1983) that I have used.).

For each (proper) subset it finds all swings and updates expressions (1) and (2) repeatedly. That is, for each $S \subset N$, it evaluates $w(S) = \sum_{j \in S} w_j,$ which requires n operations, summing over all n players taking account of whether each is a member of S or not.

For each i = 1, n it tests for a swing and updates as follows.

If $i \in$ N-S and $q - w_i = w(S) < q$ then

   set $\phi_i = \phi_i + s!(n-s-1)!/n!$ (for the Shapley-Shubik index), and

   set $\eta_i = \eta_i + 1$ and also $\eta = \eta + 1$ (for the Banzhaf index). (Alternatively $\beta_i' = \beta_i' + 2^{1-n}$.)

If $w(S) = q$ then

   set $\omega = \omega + 1$ (for the Coleman indices).

Then move to the next subset S and repeat. When all subsets have been searched the indices (1) to (6) are found.

The number of subsets of N is 2^n. ? ? erefore,? ? valuating the indices for each player has time complexity of order 2^n since computation time doubles each time n increases by 1. Despite this severe limitation, however, experience with it shows that it is a practical method to analyse QMV in the European Council with up to 27 members. (It has been used in Felsenthal and Machover (2001), Leech (forthcoming) and others.). However it is not at all feasible for something like the States game in the US presidential Electoral College which has 51 members.

One advantage of this direct approach is that it can be applied not only to evaluating power indices for simple games but it can also easily be adapted to find Shapley values, Banzhaf values (and other value concepts for cooperative games which associate a characteristic function to each coalition).

The advantages of Direct Enumeration are its simplicity, its generality in that it can be used for any problem, that its results are exact and its modest storage needs. The disadvantage is its exponential time complexity that severely limits its application to small-n games.

2.3 Monte Carlo Simulation

The earliest method proposed for computing power indices for large games was Monte Carlo simulation, described in Mann and Shapley (1960). The impetus for this was their need to compute Shapley-Shubik indices for the US presidential electoral college states game (which then had n = 50).

The method is simply to estimate t? ? ? ? ? ? ? ? ? from the basic definitions (1) and (2) from a random sample of coalitions. Suppose a coalition, T, is selected by randomly sampling the players. Define a random variable for each player, X, equal to 1 if it is a swing for player i, and 0 if it is not. Then,

$$E(X) = \beta_i' \text{ and } E(\frac{t!(n-1-t)!}{n!} X) = \phi_i, \text{ , where t is the number of players in the coalition,}$$

and the corresponding variances are

$$\text{Var}(X) = \beta_i' (1-\beta_i') \text{ and } \text{Var}(\frac{t!(n-1-t)!}{n!} X) = \phi_i(1-\phi_i).$$

Now take a sequence of m independent drawings, $X_1, \ldots, X_m$, with corresponding coalition sizes, $t_1, \ldots, t_m$.

The power indices are estimated by

$$\hat{\phi}_i = \frac{1}{m} \sum_{j=1}^{m} \frac{t_j!(n-1-t_j)!}{n!} X_j , \quad \text{and} \quad \hat{\beta}_i' = \frac{1}{m} \sum_{j=1}^{m} X_j ,$$

with variances

$$\text{Var}(\hat{\phi}_i) = \frac{\phi_i(1-\phi_i)}{m} \quad \text{and} \quad \text{Var}(\hat{\beta}_i') = \frac{\beta_i'(1-\beta_i')}{m}, \quad \text{for all } i = 1, \ldots, n.$$

The variances approach zero as m increases without limit.

This description is basic and ignores important details of how the samples are to be taken, which have a great impact on the method's effectiveness. (It is what Mann and Shapley call the Type 0 method.) Efficiency improvements are possible allowing better accuracy with given computing time. There is a large literature on simulation methods which I am not going to discuss here.

The advantages of this approach are its simplicity, its modest data storage requirements. Its disadvantage is that it is not exact. However the approximation errors can be substantially reduced with the use of greater computing power to increase the sample sizes used. Also the much greater use that is made of simulation methods generally now suggests it is a better method today than it was in 1960.

2.4 Generating Functions

The method of generating functions was first proposed by Mann and Shapley in 1962 from a suggestion by David G. Cantor. They used it to find the Shapley-Shubik indices exactly for the problem they had previously solved approximately by Monte Carlo simulation, the States game with n=50. Despite this it has not been widely used in applied studies subsequently.

The method of generating functions is an exact procedure avoiding the problem of exponential time complexity that limits the direct enumeration method and therefore it can be applied to larger games. However it is not without limitations. Its storage requirements can be very substantial, both in terms of integer sizes and array dimensions. (Thus this method trades off one form of complexity for another.) It also has the limitation (not shared by any other of the algorithms) that it can only be applied to games with integer weights and quota.

Accounts of the method can be found in Brams and Affuso (1976), Lucas (1983) and Lambert (1988). Lambert also gave a listing of a computer program and applied it to the US Electoral College. There are several implementations of it on the on-line web sites referred to below. The study by Brams and Affuso was the first to apply it to the Banzhaf and Coleman indices. A recent contribution is that by Bilbao et al. (2000).

<u>Computing the Banzhaf and Coleman Indices using Generating Functions</u>

Consider first the Banzhaf and Coleman indices. We must find the number of swings for each member and the number of winning coalitions. This requires first finding the number of coalitions, S, $S \subseteq N$, of each size of $w(S)$.

The number of possible occurrences of coalitions of each size can be found by using the generating function:

$$f(x) = \prod_{j=1}^{n}\left(1 + x^{w_j}\right) \tag{7.}$$

where x is a variable that has no significance of its own in the sense of measuring something, but whose role is to define the coefficients, which have an important meaning.

Equation (7) is a polynomial of degree $w(N)$, which may be written in general as,

$$f(x) = \sum_{j=0}^{w(N)} a_j x^{j} \tag{8.}$$

whose coefficients a_j are equal to the number of coalitions with total weight equal to j. That is, a_j is equal to the number of subsets S, such that $w(S) = j$, for $j = 0, 1, 2, \ldots, w(N)$.

Obviously $a_0 = 1$ always, because there is only one subset with $w(S) = 0$, the empty set. Henceforth, I will use the notation w to represent w(N) for brevity. Now the question arises of how to evaluate the coefficients in (8) from (7).

The coefficients a_j can be found numerically by building up the generating function (7) by successive multiplications. This gives rise to a simple rule.

Thus,
$$f(x) = \prod_{j=1}^{n} \left(1 + x^{w_j}\right),$$

$$= \left[1 + x^{w_1}\right] \prod_{j=2}^{n} \left(1 + x^{w_j}\right),$$

$$= \left[1 + x^{w_1} + x^{w_2} + x^{w_1 + w_2}\right] \prod_{j=3}^{n} (1 + x^{w_j}), \quad \text{etc.}$$

Let the successive polynomials inside the square brackets be denoted, in general,

$$1 + a_1^{(r)}x + a_2^{(r)}x^2 + \ldots + a_w^{(r)}x^w$$

at the rth stage, where $r = 1, 2, \ldots, n$. Obviously, $a_j^{(n)} = a_j$ for all j. Then $f(x)$ can be built up recursively as follows.

$$f(x) = \left[1 + a_1^{(1)}x + \ldots + a_w^{(1)}x^w\right] \prod_{j=2}^{n} \left(1 + x^{w_j}\right)$$

$$= \left[1 + a_1^{(2)}x + \ldots + a_w^{(2)}x^w\right] \prod_{j=3}^{n} \left(1 + x^{w_j}\right)$$

$$\ldots$$

$$= 1 + a_1^{(n)}x + \ldots + a_w^{(n)}x^w.$$

Computation of Power Indices

Therefore, these coefficients can be found from this by the following recursion. Let $a_j^{(0)} = 0$ for all $j \neq 0$ (both positive and negative values of j) and with $a_0^{(0)} = 1$. The coefficients are updated according to the rule:

$$a_j^{(r)} = a_j^{(r-1)} + a_{j-w_r}^{(r-1)} \tag{9.}$$

for $j = w_r, \ldots, s_r$, and $a_j^{(r)} = a_j^{(r-1)}$ otherwise, where $s_r = w(\{1, 2, 3, \ldots, r\})$. After n iterations, this gives the required coefficients, a_j, equal to the number of coalitions with weight j.

There is no need to store two arrays containing both the $a_j^{(r)}$ and $a_j^{(r-1)}$ values; one array, a, can be used if the updating rule is applied in reverse order. This can be a major consideration when w is very large. The storage requirements are less if one is computing the Banzhaf indices only, and not the Coleman indices, because only the swings are needed, and therefore the dimension of a is at most q. However the Coleman indices require the number of winning coalitions, $\omega = \sum_{j=q}^{w} a_j$, that necessitates finding a_1 to a_w.

The coefficients a_j can be used to find the number of swings for each member, i, which depends also on the member's weight, w_i. This can be found by finding, for each j, such that $q - w_i < j < q$, the number, c_j, of coalitions not containing i, then summing over j.

The coefficients c can be obtained by dividing the generating function (8) by the factor $(1 + x^{w_i})$. Thus, writing

$$f(x) = \left(1 + c_1 x + c_2 x + ... + c_v x^v\right)\left(1 + x^{w_i}\right) = 1 + a_1 x + ... a_w x^w,$$

where $v = w - w_i$, gives the rule:

$$c_j = a_j - c_{j-w_i} \qquad \text{for } j = 1, 2, ..., v, \qquad (10.)$$

where a coefficient with a negative subscript is zero.

Then the number of swings for member i, η_i, is

$$\eta_i = \sum_{j=q-w_i}^{q-1} c_j \ .$$

This process is then repeated for each member i and the indices are calculated in the usual way.

<u>Example</u>   Voting Body $\{9; 1, 1, 2, 3, 4, 6\}$,

The generating function is:

$$f(x) = (1 + x)(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^6)$$

This is built up recursively as follows.

Step #	Function
0	1
	……1+ x
2	$1 + 2x + x^2$
3	$1 + 2x + 2x^2 + 2x^3 + x^4$
4	$1 + 2x + 2x^2 + 3x^3 + 3x^4 + 2x^5 + 2x^6 + x^7$
5	$1 + 2x + 2x^2 + 3x^3 + 4x^4 + 4x^5 + 4x^6 + 4x^7 + 3x^8 + 2x^9 + 2x^{10} + x^{11}$
6	$1 + 2x + 2x^2 + 3x^3 + 4x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 5x^9 + 6x^{10}$ $+ 5x^{11} + 4x^{12} + 4x^{13} + 3x^{14} + 2x^{15} + 2x^{16} + x^{17}$

In this example n = 6, q = 9, w = 17.

The rule (9) is then:

$$a_j^{(r)} = a_j^{(r-1)} + a_{j-w_r}^{(r-1)} \quad , \qquad\qquad r = 1, ..., 6, \ \ j = 1, ..., 17$$

whose operation can be seen from the following table

24

Table 1. The evolution of the array $a=a^{(n)}$

$r =$	0	1	2	3	4	5	6
$w_r =$		1	1	2	3	4	6
$j = 0$	1	1	1	1	1	1	1
1	0	1	2	2	2	2	2
2	0	0	1	2	2	2	2
3	0	0	0	2	3	3	3
4	0	0	0	1	3	4	4
5	0	0	0	0	2	4	4
6	0	0	0	0	2	4	5
7	0	0	0	0	1	4	6
8	0	0	0	0	0	3	5
9	0	0	0	0	0	2	5
10	0	0	0	0	0	2	6
11	0	0	0	0	0	1	5
12	0	0	0	0	0	0	4
13	0	0	0	0	0	0	4
14	0	0	0	0	0	0	3
15	0	0	0	0	0	0	2
16	0	0	0	0	0	0	2
17	0	0	0	0	0	0	1

The number of winning coalitions $w = 5+6+5+4+4+3+2+2+1 = 32$.

The number of swings for each member, i, is h_i. This is found from the number of coalitions with weight j which do not include i, c_j, obtained from (10): $c_j = a_j - c_{j-w_i}$

These values are in Table 2.

Table 2. Number of swings for each i and j.

The table shows the c coefficients. Swings are in **bold**

		i =	1	2	3	4	5	6
j	a_j	w_i =	1	1	2	3	4	6
0	1		1	1	1	1	1	1
1	2		1	1	2	2	2	2
2	2		1	1	1	2	2	2
3	3		2	2	1	2	3	**3**
4	4		2	2	3	2	3	**4**
5	4		2	2	3	2	**2**	**4**
6	5		3	3	2	**3**	**3**	**4**
7	6		3	3	**3**	**4**	**3**	**4**
8	5		**2**	**2**	**3**	**3**	**2**	**3**
9	5		3	3	2	2	3	2
10	6		3	3	3	2	3	2
11	5		2	2	3	2	2	1
12	4		2	2	1	2	1	0
13	4		2	2	1	2	1	0
14	3		1	1	2	1	0	0
15	2		1	1	1	0	0	0
16	2		1	1	0	0	0	0
17	1		0	0	0	0	0	0
η_i		$\sum \eta_i = 52$	2	2	6	10	10	22
β_i'			0.0625	0.0625	0.1875	0.3125	0.3125	0.6875
β_i			0.003846	0.03846	0.11538	0.19231	0.19231	0.42308

$\omega = 32$; PTA = 32/64 = 0.5; PPA_i = PIA_i = β_i'.

Computing the Shapley-Shubik Index using Generating Functions

The index is computed using the generating function,

$$f(x) = \prod_{j=1}^{n} \left(1 + x^{w_j} y\right) \qquad (11.)$$

which has two arguments, x and y, to allow for the fact that this index requires the size of each coalition in terms both of number of players and number of votes.

This function (11) can be written in general as,

$$f(x,y) = \sum_{j=0}^{w} \sum_{k=0}^{n} d_{jk} x^j y^k$$

where d_{jk} is the number of coalitions with k members who have combined voting weight j. The (w+1)(n+1) matrix D (= $D^{(n)}$) is computed iteratively by the rule, for $D^{(r)}$,

$$d_{jk}^{(r)} = d_{jk}^{(r-1)} + d_{j-w_r, k-1}^{(r-1)}$$

where, as before, a negative subscript implies that the coefficient is zero.

The index for player i is found by calculating the number of swings with k members with votes equal to j, c_{jk}, using the rule, derived in the same way as before,

$$c_{jk} = d_{jk} - c_{j-w_i, k-1} ., \qquad (12.)$$

Then the index (1) is obtained from the expression,

$$\phi_i \;=\; \sum_{k=0}^{n-1} \frac{k!(n-1-k)!}{n!} \sum_{j=q-w_i}^{q-1} c_{jk}$$

(13.)

Then (12) and (13) are repeated for each player.

Example   {4; 1, 2, 3} n=3, q=4.

Table 3. The Build up of the Array $D^{(n)} = D$

	k=	r=1				r=2				r=3			
		0	1	2	3	0	1	2	3	0	1	2	3
	0	1	0	0	0	1	0	0	0	1	0	0	0
	1	0	1	0	0	0	1	0	0	0	1	0	0
	2	0	0	0	0	0	1	0	0	0	1	0	0
j=	3	0	0	0	0	0	0	1	0	0	1	1	0
	4	0	0	0	0	0	0	0	0	0	0	1	0
	5	0	0	0	0	0	0	0	0	0	0	1	0
	6	0	0	0	0	0	0	0	0	0	0	0	1

The indices are obtained from D by first finding the C matrices using (12) which are shown in

Table 4.

From the table, and expression (13), these are:

$$\phi_1 = \frac{1!1!}{3!}(1) = \frac{1}{6}$$

$$\phi_2 = \frac{1!1!}{3!}(0+1) = \frac{1}{6}$$

$$\phi_3 = \frac{1!1!}{3!}(1+1+0) + \frac{2!0!}{3!}(0+0+1) = \frac{4}{6}$$

Table 4 The C matrices for each player
Swings in **bold**

j=	k =	i=1			i=2			i=3		
		0	1	2	0	1	2	0	1	2
0		1	0	0	1	0	0	1	0	0
1		0	0	0	0	1	0	**0**	**1**	**0**
2		0	1	0	**0**	**0**	**0**	0	1	0
3		**0**	**1**	**0**	**0**	**1**	**0**	0	0	1
4		0	0	0	0	0	1			
5		0	0	1						

The method of Generating Functions, whether the version for the Banzhaf (and Coleman) or for the Shapley-Shubik indices, is efficient in terms of computing time. Its time complexity is linear in n and therefore it is a feasible method for games where the number of players is larger than can be handled by direct enumeration. However it is very demanding in storage which may

limit its use for large games and perhaps it should be regarded as an appropriate method for small and medium sized games (as Lucas (1983) suggested).

It makes substantial storage demands in two ways. First it requires a large array to store the frequencies of the coalitions of different sizes. For the Banzhaf indices a one-dimensional array, a, of size q (and for the Coleman indices this becomes w+1), and for the Shapley-Shubik indices a two-dimensional array, D, of size (q+1)(n+1). This can be a significant determinant of complexity when the weights are large integers. For example the IMF board of governors (studied in Leech (2002c)) has n=178, w = 2,118,076 and q = 1,059,039 for ordinary decisions or q=1,800,365 for decisions requiring a special majority of 85% of the voting weight.

Secondly, some of the integers can become very large. The number of coalitions of size j is of the order of 2^n - that is, $O(2^n)$ - and the number of swings is of $O(2^{n-1})$. This means that when n is large some of the integers, including the elements of the arrays, can be of this order of magnitude. For the IMF this means the swings are integers of the order of 2^{177}. This far exceeds the maximum size of integer that 32- bit computers can handle with total accuracy. However neither of these problems is insurmountable in this particular case, given modern computing power.

The storage requirement are likely to affect the algorithm's suitability for the "inverse problem" of finding the appropriate weights given the power indices. This procedure requires iteratively updating the weights and recomputing the indices each time and comparing them with their target values to a chosen accuracy. This in turn means that the weights must be in units that

32

make this possible. Therefore if they are constrained to being integers, and we seek maximum accuracy, some of them must be very large. This problem will increase the more the inequality between the desired powers.

2.5 Multilinear Extensions Approximation Algorithms

Owen introduced methods for large games based on the multilinear extension of the game and approximation methods which use the central limit theorem to get approximations to expressions (1) and (2). The key references are Owen (1972, 1975a). I will describe the approach to the Shapley-Shubik and Banzhaf indices; its application to find the Coleman indices is straightforward.

Expression (1) for the Shapley-Shubik index can be rewritten by noting that the term inside the summation is a beta function:

$$B(t+1, \text{ n-t}) \ = \ \frac{t!\,(n-t-1)!}{n!} \ = \ \int_0^1 x^t(1-x)^{n-t-1}\,dx \qquad\qquad (14.)$$

The integrand on the RHS of (14), $x^t(1\text{-}x)^{n\text{-}t\text{-}1}$, can be regarded as the probability that the (random) subset T_i appears, when x is the probability that any member joins T_i , assumed constant and independent for all players $j, j \in N - \{i\}$.

Summing this expression over all swings gives the probability of a swing for i. Let us call this probability $f_i(x)$:

$$f_i(x) = \sum_{T_i} \ x^t(1\text{-}x)^{n\text{-}t\text{-}1}. \qquad\qquad (15.)$$

Integrating x out of (15) gives the Shapley-Shubik index, because, substituting (14) into (1) gives:

$$\phi_i = \sum_{T_i} \ \int_0^1 x^t(1-x)^{n-t-1}\,dx \ = \ \int_0^1 \ [\sum_{T_i} \ x^t(1\text{-}x)^{n\text{-}t\text{-}1} \ ]\ dx$$

$$= \int_0^1 f_i(x)\, dx \,. \tag{16.}$$

Expression (15) is equal to the Banzhaf index when x=1/2.

Expression (15) is the multilinear extension of the game and it can be used to compute the indices for small n games. But it means evaluating a function whose size doubles every time a new player is added. Thus this method, too, like direct enumeration, suffers from exponential time complexity. Widgren (1994) reports that the exact multilinear extensions approach did not prove computationally feasible for a game with 19 members and Owen's MLE approximation method had to be used instead.

We can evaluate ϕ_i approximately using a suitable approximation for the probability $f_i(x)$. In large games with many small weights, and no very large weights, this can be done with reasonable accuracy using suitable probabilistic voting assumptions and the normal distribution.

The probability of a swing $f_i(x)$ can be approximated using the following probabilistic-voting model. Assuming each player j ? i votes the same way as i with probability x, independently of the others, defines a random variable, v_j with the following dichotomous distribution:

$$\Pr(v_j=w_j) = x, \quad \Pr(v_j=0) = 1 -x, \quad \Pr(v_j \ ? \ w_j \text{ and } v_j \ ? \ 0) = 0.$$

The random variable v_j can be interpreted as the number of votes cast by player j, at random, on the same side as those of player i. Its first two moments are:

$$E(v_j) = xw_j, \qquad Var(v_j) = x(1-x)w_j^2, \text{ all j.}$$

The total number of votes cast by players j in the same way as that of player i is a random variable $v_i(x) = \sum_{j \in N-\{i\}} v_j$. It is useful to define a sum-of-squares function: let this be $h(T) = \sum_{i \in T} w_i^2$.

Then $v_i(x)$ has an approximate normal distribution with moments:

$$E(v_i(x)) = xw(N-\{i\}) = \mu_i(x), \text{ say, and}$$

$$Var(v_i(x)) = x(1-x)\, h(N-\{i\}) = \sigma_i(x)^2.$$

Then the required probability,

$$f_i(x) = Pr[q - w_i = v_i(x) < q], \qquad\qquad (17.)$$

can be obtained approximately using the normal distribution function, $\Phi(.)$ by evaluating the expression:

$$f_i(x) = \Phi\left(\frac{q - \mu_i(x)}{\sigma_i(x)}\right) - \Phi\left(\frac{q - \mu_i(x) - w_i}{\sigma_i(x)}\right). \qquad\qquad (18.)$$

The Shapley-Shubik index in (16) is approximated by numerically integrating out x in (18) using a quadrature routine (such as Patterson (1968) which I have used). The Banzhaf index is obtained by setting $x = 0.5$ in (18), since then (15), for which (18) is an approximation, reduces to (2).

These methods have linear complexity. The calculations for the Shapley-Shubik index and the Non-Normalised Banzhaf index for a player depend on the number of players n only in the data input and calculation of the statistics w(N) and h(N) (which need only be done once since they are common to all players) because neither (18) nor its numerical integral (16) depend on n.

The Normalised Banzhaf indices require the normalizing constant which necessitates that all n indices are found.

These methods for both indices have been used in a number of studies, for example Owen (1975a, 1975b), Leech (1988, 1992), Widgren (1994) and others. But their accuracy depends on the validity of the normal approximation. In some real world weighted voting bodies the approximation is not good and consequent computation errors may be substantial because of a failure of the central limit theorem due to concentration of the voting weights in the hands of a few. Examples of this have recently been reported by Leech (2002a) and Widgren (2000).

2.6 Modified MLE Approximation for Large Finite Games

For games where n is too large for exact methods to be feasible, and where the distribution of weights is highly skewed, we can combine the essential features of both the Direct Enumeration and Multilinear Extensions Approximation approaches. The general procedure is as follows.

The players are ordered by their weight representing their respective number of votes, so that $w_i = w_{i+1}$ for all i. The players are divided into two subsets: major players with the largest weight, $M = \{1, 2, .. , m\}$ and minor players N - M. The value of m here is chosen for computational convenience, along a tradeoff between accuracy and efficiency. A general rule would be to choose m as large as possible while computing time is not too great.

The algorithm searches all subsets of M. Given a particular subset, $S \subseteq M$, it then evaluates the approximate conditional swing probability for each player making Owen's standard assumptions about random voting by <u>minor</u> players only, conditional on S. The probability of the swing is then obtained as the product of the probability of the formation of S, by random voting by <u>major</u> players, and that of the conditional swing. The index is obtained by summing these joint probabilities over all the subsets. There are two cases to consider: (1) player i is a major player, $i \in M$; (2) i is a minor player, $i \in N - M$.

(1) Major Players

It is necessary to search over all subsets of M which do not include player i; any subset of which i is a member cannnot define a swing and the swing probability associated with it is definitionally zero. For each such subset consider the probability of its forming and the probability of its being a swing for i.

Suppose S is a subset of M - {i}. We let the swing probability be $f_i(x)$ as before. This can be written as:

$$f_i(x) = \Pr(\text{swing for i}) = \sum_S \Pr(S)\Pr(\text{swing for i}|S)$$

Defining the conditional probability of a swing given S as the function $g_i(S, x)$, and the probability of selecting S randomly by the function $p(s, m-1, x)$, we can write:

$$f_i(x) = \sum_S p(s, m-1, x)g_i(S, x).$$

The first factor inside the summation is:

$$p(s, m-1, x) = x^s(1-x)^{m-s-1}.$$

To find the second factor, define the random variable:

$$v_i(x) = \sum_{j \in N-M} v_j,$$

where v_j is as before, to represent the random number of votes cast by the minor players.

So, $\qquad E(v_i(x)) = xw(N-M) = \mu_i(x),$

and $\qquad Var(v_i(x)) = x(1-x)h(N-M) = \sigma_i(x)^2.$

Using these moments and the normal approximation to the distribution of $v_i(x)$, we can obtain the

required probability as:

$$g_i(S, x) = \Pr[q - w(S) - w_i = v_i(x) < q - w(S)]$$

$$= \Phi\left(\frac{q - w(S) - \mu_i(x)}{\sigma_i(x)}\right) - \Phi\left(\frac{q - w(S) - w_i - \mu_i(x)}{\sigma_i(x)}\right). \quad (19.)$$

Therefore, we can write

$$f_i(x) = \sum_{S \subseteq M-\{i\}} x^s (1-x)^{m-s-1} g_i(S, x). \quad (20.)$$

The required index is then:

$$\phi_i = \int_0^1 f_i(x)\, dx = \int_0^1 \left[\sum_{S \subseteq M-\{i\}} x^s(1-x)^{m-s-1} g_i(S, x)\right]dx$$

$$= \sum_{S \subseteq M-\{i\}} \int_0^1 x^s(1-x)^{m-s-1} g_i(S, x)dx \quad (21.)$$

which can be found by searching over all subsets of M-{i}, integrating out x by numerical

quadrature at each subset then summing. The Banzhaf index β'_i is obtained from (20) on setting

x=0.5 instead of integrating it out, then summing to give $\beta'_i = f_i(0.5)$.

The summation in expression (20) above is over all subsets of M-{i}, but it is clear that

operationally we can search over all subsets of M since any set which includes i has a zero

probability of a swing for i. Writing $g_i(S,x) = 0$ for all S where $i \in S$, and as expression (19)

where $i \notin S$ then we can rewrite (20) and (21) as:

$$f_i(x) = \sum_{S \subseteq M} x^s(1-x)^{m-s-1} g_i(S, x), \quad (22.)$$

$$\phi_i \;=\; \sum_{S \subseteq M} \int_0^1 x^s(1-x)^{m-s-1} g_i(S,\,x)dx, \qquad\qquad (23.)$$

and the Banzhaf index

$$\beta_i{}' = \sum_{S \subseteq M} 0.5^{m-1} g_i(S,0.5) = f_i(0.5). \qquad\qquad (24.)$$

It is therefore possible to compute the indices for both major and minor players in a single search over the subsets of M.

(2) Minor Players

Now the computation of the indices for the smaller players, $i \in N-M$, is described. The subset S can now be considered to be any subset of M. Since we are now treating the votes of all m major players as random (not just m-1 of them), the probability of the subset S is:

$$Pr(S) = p(s,\,m,\,x) = x^s(1-x)^{m-s}.$$

The behavior of the minor players other than i is described by a random variable

$y_i(x) = \displaystyle\sum_{j \in N-M-\{i\}} v_j$ which has an approximate normal distribution with moments:

$$\mu_i(x) = xw(N\text{-}M\text{-}\{i\})$$

and
$$\sigma_i(x)^2 = x(1-x)h(N\text{-}M\text{-}\{i\}).$$

Hence we can evaluate the conditional swing probability $g_i(S, x)$ which now can be written as

$$g_i(S,\,x) = Pr[q - w(S) - w_i = y_i(x) < q - w(S)],$$

approximately by the normal probability in expression (19) after making the required notational substitutions.

Writing

$$f_i(x) = \sum_{S \subseteq M} p(s, m, x) \, g_i(S, x), \tag{25.}$$

the Shapley-Shubik index is found again by quadrature, then summing,

$$\phi_i = \sum_{S \subseteq M} \int_0^1 p(s, m, x) \, g_i(S, x) \, dx \tag{26.}$$

and the Banzhaf index by setting x=0.5, then summing,

$$\beta_i' = \sum_{S \subseteq M} 0.5^m \, g_i(S, 0.5) = f_i(0.5). \tag{27.}$$

within the same subset search as before, $S \subseteq M$.

These algorithms require a search over all subsets S of M, in order to find (20), therefore the calculations have to be repeated 2^m times. Expression (19) does not depend on either m or n once the statistics w(N-M) and h(N-M) have been evaluated, requiring O(n) operations, and these are common to all players. Therefore the indices have complexity exponential in m and linear in n.

These algorithms have proved to be very successful for large games. Leech (2002c) has n=178 and in Leech (2001, 2002b) there are numerous cases of company voting games with n>400. These applications have required only a moderate choice of m. The obvious rule for choosing the value of m is that it should be large enough to ensure accuracy without being too large to prevent all subsets of M to be enumerated in a reasonable computing time.

3 Methods for Oceanic Games

An oceanic game is a limiting case of a large game in which the number of players is allowed to become infinite. Here there are two types of players: a finite number of "atomic" players whose weights are finite, and an infinite number of "non-atomic" players whose weights are infinitesimally small. The Shapley-Shubik indices for such games were first analysed in Shapley and Shapiro (1978) and later the Banzhaf indices were discussed in Dubey and Shapley (1979). I describe the computation of such games here for completeness. I have reported indices for such games in several papers on shareholder control of companies, for example Leech (2001, 2002b).

In an oceanic game there are a finite number, m, of major players with fixed voting weights, and a very large number, n-m, (in the limit an "ocean" of minor players) with very small weights. Then as n goes to infinity it can be shown that the Shapley-Shubik index for major player i converges on the value:

$$\phi_i = \sum_{S \subseteq M_i} \int_a^b u^s (1-u)^{m-s-1} du \qquad\qquad i = 1, ..., m \qquad (28.)$$

where $M = \{1, 2, . . , m\}$ is the set of major players, $M_i = M - \{i\}$, and

$a = \text{median}(0, (q-w(S))/(1-w(M)), 1)$, $b = \text{median}(0, (q-w(S)-w_i)/(1-w(M)),1)$. Expression (28) is not difficult to evaluate, requiring only a minor extension of the modified MLE approximation algorithm. If m is small we can perform a Hamilton search over the m-dimensional hypercube and evaluate the integral for each S, and sum.

If m is larger this may not be feasible and therefore it becomes necessary to partition the finite players into two groups for computational convenience and use an approach based on expression (21) with suitable changes to the limits of integration. For example, in the corporate control problems analysed in Leech (2001 and 2002b) for 444 British companies, the oceanic games defined had between 12 and 56 finite players, with a median of 27, due to data limitations. In all but a very few of these cases it would have been impossible to choose M equal to the set of finite players. Therefore I set M={1, 2, 3, 4, 5}, m = 5, with the remaining finite players modelled by the probabilistic voting assumptions.

Banzhaf indices for oceanic games were studied by Dubey and Shapley (1979) who showed that under suitable conditions they can be obtained as the Banzhaf indices for the modified, finite game consisting only of the major players M with weights $w_1, w_2, \ldots, w_m$ and a modified quota q - (1 - w(M))/2. This result depends on the quota q (that is, in the original game). For certain values of q the power indices are zero in the limit (the so-called "pitfall" points where the number of minor-player swings become so numerous that the Banzhaf indices for major players go to zero). However, where there is a simple majority rule, with q always equal to w(N)/2, this problem does not arise. Any algorithm for Banzhaf indices of finite games can be used. In the corporate control examples I used M equal to all the finite players and the modified MLE algorithm.

4 Determining Weights: the "Inverse Problem"

In designing a system of weighted voting, weights ought to be allocated to members in such a way as to bring about the desired distribution of voting power. This issue is discussed in Nurmi (1981). Power indices enable this to be done numerically by means of an iterative process by which the weights are successively updated. Starting with an initial guess, the power indices are recalculated each time the weights are modified until they achieve preassigned values. In this section I describe the approach. The notation assumes that the power indices used are the normalised Banzhaf indices but the method can be applied to others if desired. It assumes the power indices are normalised but it would be desirable to be able to relax this restriction. An iterative procedure similar to the one described here was proposed in Laruelle and Widgren (1998).

The values required for the power indices are fixed as a design property of the voting system. For example one criterion that has been suggested should be used for international organisations is the equalisation of voting power among citizens of different countries. This has been suggested as a basic principle for reweighting the votes in the EU Council of Ministers by Felsenthal and Machover (2000, 2001), Laruelle and Widgren (1998) and Leech (forthcoming). In Leech (2002c) I have used the criterion of equalising power in the governing body of the IMF to members' IMF quotas (mainly based on financial contributions to the organisation).

Let it be required that member i should possess a voting power of t_i, where $\sum t_i = 1$. The problem is to find weights that have associated power indices, β_i, such that $\beta_i = t_i$, for all i.

Denote the required power, the weights and corresponding power indices, as functions of the weights, by the n-vectors t, w and β(w).

Now we can compute the power indices for given weights and compare them with their desired values. Any suitable power indices algorithm can be used for this: I have used the modified MLE approximation method with the IMF governors and the direct enumeration method with the EU Council. Using a suitable updating rule to change the weights provides an iterative algorithm which should converge to the desired power distribution to an accuracy defined by a suitable stopping rule.

Let the weights after p iterations be denoted by the vector $w^{(p)}$; the initial guess is $w^{(0)}$. The corresponding power indices are the vectors of functions $\beta(w^{(p)})$. The iterative procedure can then be written in terms of an updating rule:

$$w^{(p+1)} = w^{(p)} + \lambda(t - \beta(w^{(p)})) \qquad (29.)$$

for some appropriate scalar $\lambda > 0$. More generally we might replace the scalar λ with a matrix , but in my work I have only used the simple formulation described here and found it adequate.

If the iterative procedure (29) converges to a vector, w*, then that is taken to be the desired weight vector, since then: w*=w* + λ(t - β(w*)) and t = β(w*). Convergence can be defined in terms of a measure of the distance between $\beta(w^{(p)})$ and t and a stopping criterion. The simple sum of squares measure $\sum (\beta_i^{(p)} - t_i)^2$ with a suitable stopping criterion has been found to

work well. My experience is that accuracy of convergence varies according to the game. The algorithm is illustrated in Figure 1.
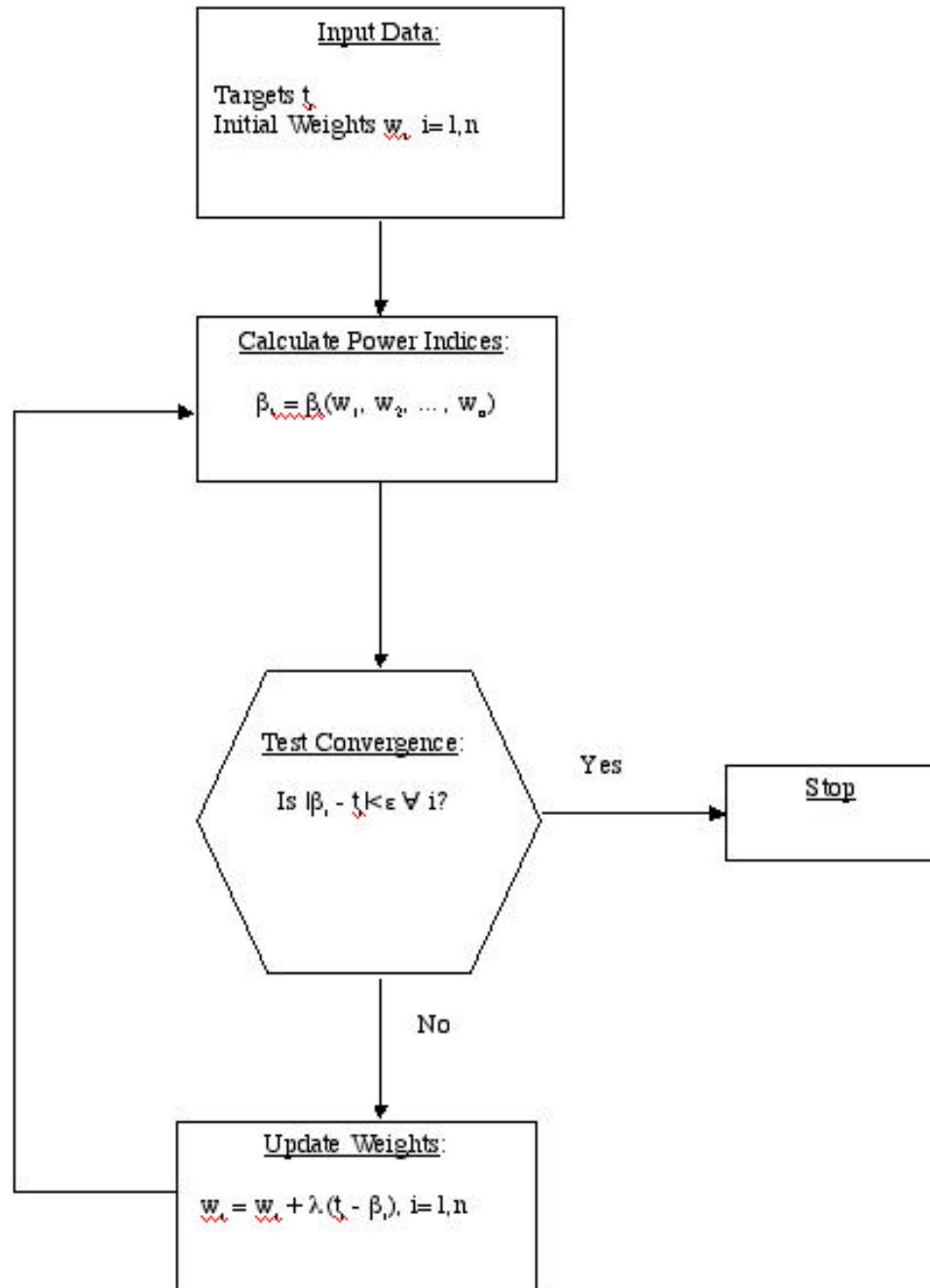
This algorithm is not unproblematical since the indices are not real numbers, but rational numbers, and in general therefore the distance between $\beta(w^{(p)})$ and t has a lower bound determined theoretically as a property of the game. More importantly, the existence of voting paradoxes in the relationships between weights and indices (described by Felsenthal and Machover (1998)), points to the relationship between w and β being possibly not continuous. This suggests that the weights w* computed may not be unique.

For the 15-member EU with the Nice Treaty triple-majority decision rule (the game labelled N15 in Felsenthal and Machover (2001)), the algorithm was found to converge to an accuracy, in terms of this stopping criterion, of the order of 10^{-8}, but it was not possible to get full convergence using a smaller value. For N27 it easily converged with respect to a stopping rule of the order of 10^{-10} but no smaller. The power indices were computed exactly using the direct enumeration program ipnice (used by Felsenthal and Machover (2001) and Leech (forthcoming)). In my work on the IMF Board of Governors, reported in Leech (2002c), the same iterative algorithm was used to compute fair weights for the International Monetary Fund Board of Governors with n=178. But in this case the power indices were calculated using a different program suitable for large n (which implements the modified MLE approximation algorithm); the accuracy achieved in terms of the sum of squares stopping rule was at least 10^{-15}. There are indications therefore that the iterative algorithm for solving the "inverse problem" works better for

larger n. This might be due to the fact that the mapping from β (w) is closer to being a continuous one in this case.

Figure 1:Flowchart of an Iterative Algorithm to Determine Weights

Table 5 shows the results of applying the iterative procedure to the choice of weights in the IMF Governors. As is to be expected, the resulting weights are very different for the two majority requirements q. For ordinary decisions, the voting weight of the United States should be reduced to under15 percent, and the voting weight of the other member countries increased slightly .in order to achieve the levels of voting power given in the appendix to the IMF Annual Report for 1999: United States 17.55, Japan 6.3, Germany 6.15, etc. However for 85% special majority decisions, in order to achieve these values for the power index, the weight of the United States would have to be increased to almost 70 percent and those of all other countries reduced substantially.

Table 5 Fair Weights in the IMF Board of Governors

	Banzhaf Power β_i	Weight w_i* q=50%	Weight w_i* q=85%
USA	17.55	14.06	69.78
Japan	6.30	6.53	2.20
Germany	6.15	6.38	2.16
France	5.08	5.27	1.82
UK	5.08	5.27	1.82
Italy	3.34	3.48	1.23
Saudi	3.31	3.45	1.21
Canada	3.02	3.15	1.11
Russia	2.82	2.94	1.04
Netherlands	2.45	2.56	0.91
China	2.22	2.32	0.82
India	1.97	2.06	0.73
Switzerland	1.64	1.72	0.61
Australia	1.54	1.61	0.57
Belgium	1.48	1.54	0.55
Spain	1.45	1.52	0.54
Brazil	1.45	1.51	0.54
Venezuela	1.27	1.32	0.47
Mexico	1.23	1.29	0.46
Sweden	1.14	1.19	0.43
Argentina	1.01	1.06	0.38
Indonesia	0.99	1.04	0.37
Austria	0.90	0.94	0.33
...	...	...	...
…	…	…	…
…	…	…	…

Table 6 shows the results of applying the iterative algorithm to compute the fair weights for QMV in the European Council under the Nice rules when n=15. Column (4) shows the power indices for the Nice weights in columns (1) and (3). The targets are in column (5) and the resulting weights in column (6). The only member countries whose weights change substantially

are Germany and Spain: Germany's weight has now increased to 15.12 and Spain's reduced to

9.34 percent of the votes. The conclusion is that the Nice weights are surprisingly close to being

fair.

Table 6 Fair Voting Weights under the Nice Treaty with n=15

N15	q_1=169	q_2=62%				
(1)	(2)	(3) $w^{(0)}$	(4) $\beta(w^{(0)})$	(5) t	(6) w*	(7)
Weight	Country	Weight%	Bz Index %	vPop%	Fair Weight %	Pop%
29	Germany	12.24	12.11	13.97	15.12	21.858
29	UK	12.24	11.99	11.87	12.06	15.786
29	France	12.24	11.99	11.84	12.05	15.711
29	Italy	12.24	11.99	11.70	11.99	15.350
27	Spain	11.39	11.11	9.68	9.34	10.496
13	Netherlands	5.49	5.50	6.12	5.98	4.199
12	Greece	5.06	5.16	5.00	4.64	2.806
12	Belgium	5.06	5.16	4.93	4.61	2.721
12	Portugal	5.06	5.16	4.87	4.58	2.659
10	Sweden	4.22	4.30	4.59	4.47	2.359
10	Austria	4.22	4.30	4.38	4.41	2.153
7	Denmark	2.95	3.09	3.55	3.22	1.416
7	Finland	2.95	3.09	3.50	3.20	1.375
7	Ireland	2.95	3.09	2.98	3.03	0.998
4	Luxembourg	1.69	1.96	1.01	1.29	0.114
237		100.00	100.00	100.00	100.00	100.00

Bz: Banzhaf; q_1= the threshold in terms of weighted votes, q_2 = the population condition.

5 Conclusion

This paper has described five algorithms for computing the Banzhaf and Shapley-Shubik power indices for finite voting games and compared their advantages and disadvantages for the analysis of different voting situations. It is hoped that the better availability of computer algorithms will stimulate more empirical research aimed at giving a better understanding of the analysis of voting power.

It has also discussed the somewhat harder problem of computing the weights with the property that the power indices are equal to certain given predetermined values. This problem is intimately related to the use of power indices as an aid to the design of voting systems.

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Some Web Links

There are some good websites about power indices. These are a few interesting and useful sites.

An interesting on-line power indices calculator by Kazuo Morota and Yashuaki Oisho, Department of Mathematical Engineering and Information Physics, The University of Tokyo. http://www.misojiro.t.u-tokyo.ac.jp/~tomomi/cgi-bin/vpower/index-e.cgi (generating functions).

The Voting Power and Power Index Website, Antti Pajala, University of Turku, http://powerslave.val.utu.fi/. Lots of information about power indices including on-line computation (for small games, n=20, direct enumeration method).

Thomas Brauninger and Koenig's computer programme IOP http://www.uni-konstanz.de/FuF/Verwiss/koenig/IOP.html (direct enumeration).

European Voting Games website by Jesus Maria Bilbao and Carmen Herrero which contains much information about the EU, voting power analysis, papers, computing and many links. http://www.esi2.us.es/~mbilbao/eugames.htm

The Banzhaf Power Index Calculator, http://www.math.temple.edu/~cow/bpi.html An on-line calculator from Temple University. (generating functions)

The mathematics of voting power: an introduction to Banzhaf power indices, by Bjørn K. Alsberg. http://pcf1.chembio.ntnu.no/~bka/div/MAKT/Powerweb.htm

Banzhaf Power Index by Mark Livingston, http://www.cs.unc.edu/~livingst/Banzhaf/ Much information about the Banzhaf index and links (uses the Monte Carlo method).

Voting Power Project at LSE http://www.lse.ac.uk/Depts/cpnss/projects/vp.html