COMPARATIVE STATICS AND LAWS OF SCARCITY FOR GAMES

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Comparative statics and laws of scarcity for games*

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Summary. A "law of scarcity" is that scarceness is rewarded. We demonstrate laws of scarcity for cores and approximate cores of games. Furthermore, we demonstrate conditions under which all payoffs in the core of any game in a parameterized collection have an equal treatment property and show that equal treatment core payoff vectors satisfy a condition of cyclic monotonicity. Our results are developed for parameterized collections of games and exact bounds on the maximum possible deviation of approximate core payoff vectors from satisfying a law of scarcity are stated in terms of the parameters describing the games. We note that the parameters can, in principle, be estimated. Results are compared to the developments in the literature on matching markets, pregames, and general equilibrium. This paper expands on results published in Kovalenkov and Wooders, *Economic Theory* (to appear).

Keywords and Phrases: *monotonicity, cooperative games, clubs, games with side payments (TU games), cyclic monotonicity, law of scarcity, law of demand, approximate cores, effective small groups, parameterized collections of games.*

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1 Laws of scarcity, parameterized collections of games and equal treatment cores

The importance of the scarcity of a commodity in determining its value in exchange was already recognized by Adam Smith in the paradox that diamonds, although used only for adornment, were expensive, while water, essential to human life, was cheap. This apparent paradox has been much explained in the context of general equilibrium models of economies with private goods. The current paper[1] treats the problem from the perspective of cooperative game theory and demonstrates that if gains to population size are nearly exhausted, then numbers of players who are similar to each other and core payoffs respond in opposite directions. The players could be units of commodities, or people who are endowed with bundles of commodities, or people who just like to get together in groups for the pleasure of each other's company.

More precisely, within the context of parameterized collections of games, we obtain analogues of the celebrated Laws of Demand and of Supply of general equilibrium theory. Roughly, the Law of Demand states that prices and quantities demanded change in the opposite directions while, with inputs signed negatively, the Law of Supply states that quantities demanded as inputs and produced as outputs change in the same direction as price changes.[2] In the framework of a cooperative game, supply and demand are not distinct concepts. Thus, following [36] we refer to our results for games as Laws of Scarcity. If player types are thought of as commodity types while payoffs to players are thought of as prices for commodities, our Laws of Scarcity are closely related to comparative statics results for general equilibrium models with quasi-linear utilities. As we discuss in a section relating our paper to the literature, our results extend the literature in several directions.

Games in a parameterized collection are described by certain parameters: (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness.[3] An equal treatment payoff vector is defined to be a payoff vector that assigns the same payoff to all players of the same approximate type. Our laws of scarcity demonstrate that equal treatment ε-cores satisfy the property that numbers of players who are similar to each other and equal treatment ε-core payoffs respond in nearly opposite directions; specifically, we establish an *exact* upper bound on the extent to which equal treatment ε-core payoffs may respond in the same direction and this bound will, under some conditions, be small. We actually demonstrate a stronger result – equal treatment ε-core vectors and vectors of numbers of players of each

[1] A shorter version of the current paper, with fewer results, is [13].

[2] The Law of Demand therefore rules out "Giffen goods" or treats compensated demands; see Mas-Colell, Whinston and Green [16], Sections 2.F and 4.C. This volume also provides a very clear exposition and further references.

[3] Parameterized collections of games were introduced in [38], [11], [10] and [9].

approximate type satisfy cyclic monotonicity.[4] In addition to cyclic monotonicity, we demonstrate a closely related comparative statics result: When the relative size of a group of players who are all similar to each other increases, then equal treatment ε-core payoffs to members of that group will not significantly increase and may decrease.

The conditions required on a game to obtain our results are that (i) each player has many close substitutes (a thickness condition) and (ii) almost all gains to collective activities can be realized by groups of players bounded in size (a form of small group effectiveness - SGE). The first condition is frequently employed in economic theory. The second condition may appear to be restrictive, but in fact, if there are sufficiently many players of each type, then per capita boundedness (PCB) – finiteness of the supremum of average payoff – and SGE are equivalent.[5] Our results yield explicit bounds, in terms of the parameters describing the games, on the maximal deviation of equal treatment ε-core payoffs from satisfying exact monotonicity. Moreover, our framework allows some latitude in the exact specification of approximate types. These two considerations suggest that in principle our results can be well applied to estimate the effects on equal treatment ε-core payoffs of changes in the composition of the total player set. Note that all the bounds we obtain are exact, and depend on the parameters describing the games and on the ε of the ε-core.

Our results also contribute to a literature relating games, markets and clubs. An advantage of the framework of cooperative games over detailed models of economies is that models of games can accommodate the entire spectrum from games derived from economies with only private goods to games derived from economies with pure public goods. Thus, it is of interest to determine conditions on games ensuring that they are 'market-like' – that they satisfy analogues of well known properties of competitive economies. Important papers in this direction include Shubik [30], which introduced the study of large games as models of large private-goods economies, Shapley and Shubik [29], which demonstrated an equivalence between markets and totally balanced games, and Wooders [36],[37] demonstrating that games with many players are market games. Further motivation for the framework of cooperative games comes from Buchanan [1], who stressed the need for a general theory, including as extreme cases both purely private and purely public goods economies and the need for "a theory of clubs, a theory of cooperative membership."

For our results characterizing ε-cores of games to be interesting, it is important that under some reasonably broad set of conditions, ε-cores of large games are nonempty. Since Shapley and Shubik [28] showing nonemptiness of approximate cores of

[4]Cyclic monotonicity relates to monotonicity in the same way as the Strong Axiom of Revealed Preference relates to the Weak Axiom of Revealed Preference (see, for example, Richter [20], [21]).

[5]This is shown for "pregames" in [37], Theorem 4. Per capita boundedness and small group effectiveness were introduced as conditions limiting returns to coalition size in the study of large games in Wooders [33],[34], [35] where both nonemptiness of approximate cores and the equal treatment property of cores and other properties of large games were investigated.

exchange economies with many players and quasi-linear utilities and Wooders [32], [34], showing nonemptiness of approximate cores of game with many players with and without side payments, there has been a number of further results. For parameterized collections of games, such results are demonstrated in [9], [10], [11] and [38]. Importantly, in [10] equal treatment ε-cores of games with side payments are also shown to be nonempty. The interest of our monotonicity results is further enhanced by results showing that approximate cores have the equal treatment property; in this regard, note that [36] shows that approximate cores of large games treat most similar players nearly equally.[6] In this paper we present an equal treatment result for the "base case" of games with strictly effective small groups. In research in progress, further equal treatment results are demonstrated for parameterized collections of games.

In the next section we define parameterized collections of games. In Section 3, the results are presented. Section 4 consists of an example, applying our results to a matching model with hospitals and interns. Section 5 further relates the current paper to the literature and concludes the paper. In Appendix A we prove that the bounds cannot be tightened. In Appendix B, for the convenience of the reader we describe the pregame framework of the prior literature and make some connections to the framework of parameterized collections of games.

2 Cooperative games

Let (N, v) be a pair consisting of a finite set N, called the *player set*, and a function v, called the *characteristic function*, from subsets of N to the non-negative real numbers with $v(\emptyset) = 0$. The pair (N, v) is a *game* (*with side payments* or a *TU game*). Nonempty subsets of N are called *coalitions* or *groups*. A game (N, v) is *superadditive* if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\{S^k\}$ of S.

In games and economies where the realization of maximum total payoff may require that a group of players sub-divide into smaller coalitions, superadditivity does not necessarily hold. If, however, a game is *essentially superadditive*, that is, a possibility open to a group S is to divide into subgroups and achieve the total payoff realizable by the subgroups, it is natural to apply solution concepts such as the core and approximate cores to the superadditive cover. (See Lemma 0 below.) Thus, we define the *superadditive cover* (N, v^s) of the game (N, v) where:

$$v^s(S) \stackrel{\text{def}}{=} \max \sum_k v(S^k)$$

and the maximum is taken over all partitions $\{S^k\}$ of S. Our results apply to both superadditive games and to superadditive cover games.

[6] These results extend prior results for sequences of games with a fixed distribution of player types in [32], [33], [34] and [31].

Given a nonnegative real number $\delta \geq 0$, two players i and j are *δ-substitutes* if for all groups $S \subset N$ with $i, j \notin S$, it holds that

$$|v(S \cup \{i\}) - v(S \cup \{j\})| \leq \delta.$$

When $\delta = 0$, players i and j are *exact substitutes*.

2.1 Parameterized collections of games

$\delta-$substitute partitions In our approach we approximate games with many players, all of whom may be distinct, by games with player types. This extends the prior model for pregames since the assumption of a compact metric space of player types is not required.

Let (N, v) be a game and let $\delta \geq 0$ be a non-negative real number. Informally, a δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. That is, if all players in a coalition are replaced by δ-substitutes, the payoff to that coalition changes by no more than δ per capita. Formally, a partition $\{N[t]\}$ of N into subsets is a *δ-substitute partition* if all players in each subset are δ-substitutes for each other.[7] The set $N[t]$ is interpreted as an *approximate type*. Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games, say (N, v) and (N, v'), may have the same partitions into δ-substitutes but have no other relationship to each other (in contrast to games derived from a pregame). Examples are provided at the end of this subsection.

(δ,T)- type games. The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game (N, v) is a (δ, T)-*type game* if there exists a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N.

profiles. Profiles of player sets are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let (N, v) be a game and let $\{N[t] : t = 1, .., T\}$ be a partition of N into δ-substitutes. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z^T_+$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z^T_+$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of

[7] The definition of δ-substitutes in our prior papers, including [13], is slightly less restrictive but more complicated.

the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

β-effective B-bounded groups. The following notion formulates the idea of small group effectiveness, SGE, precisely defined in Appendix B, in the context of parameterized collections of games. Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the per capita loss is no more than β.

Let β be a given non-negative real number, and let B be a given integer. A game (N, v) has *β-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $|S^k| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta |S|.$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

parametrized collections of games $\Gamma((\delta, T), (\beta, B))$. Let T and B be positive integers, let δ and β be non-negative real numbers. Define

$$\Gamma((\delta, T), (\beta, B))$$

to be the collection of all (δ, T)-type games that have β-effective B-bounded groups.

Example 1. The following games illustrate the ideas of a δ-substitute partition and β-effective B-bounded groups. Let N be a finite set of players. Suppose that players can be ranked in the $[0, 1]$ interval so that if $i, j \in N$ and $i \geq j$ then i has a higher rank. We consider three different games, all with the same player set and the same ranking.

Let (N, v) be a game where the total payoff to any two players is the sum of their ranks. Suppose also that the payoff $v(S)$ to any other group S is zero. Then for any $\beta \geq 0$ and any $B \geq 2$, the game has β-effective B-bounded groups. Given $\delta \geq 0$, if the distance between the ranks of players i and j is less than δ, then i and j are δ-substitutes, both for the game (N, v) and for the superadditive cover game (N, v^s).

To see that there may be other games with the same partitions of the total player set into δ-substitutes, consider another game (N, v') but where the payoff $v'(\{i\})$ to player i is equal to his rank and the payoff to any other coalition is the given by the superadditive cover of v'. Here for any $\beta \geq 0$ and any $B \geq 1$, B-bounded groups are effective and if the distance between the ranks of i and j less than δ, then i and j are δ-substitutes.

Alternatively, let the payoff to any group consisting of two players be the square of the sum of the ranks of the members of the group (and again take the superadditive cover to create a superadditive game). Then if the distance between the ranks of players i and j is less than δ then i and j are $\delta^2 + 4\delta$ substitutes.

Example 2. This example serves to illustrate how the framework of parameterized collections of games allows new insights that may be hidden within the pregame framework. (Recall that pregames are formally defined in Appendix B.)

Let (N, v) be a game with buyers and sellers, where all sellers sell an identical product, each seller owns one unit of this product and has a reservation price for the unit he owns. Suppose that each buyer only wants to purchase at most one unit of the product and has a reservation price for the unit of product. Suppose that the reservation prices of the buyers are higher than the reservation prices of the sellers, so that always there exist some gains from trade, but the maximal gain from trade is bounded by some constant a. Then for any $\delta > 0$ the game (N, v) belongs to the collection $\Gamma((\delta, T_\delta), (0, 2))$ where T_δ is the smallest integer greater than a/δ.

Now consider instead a production game (N, v') where only two person coalitions are effective and where the worth of any two person coalition is the sum of the fixed productivities assigned to these two players and is less than or equal to a. In spite of the fact that the two games are quite different, the game (N, v') also belongs to the collection $\Gamma((\delta, T_\delta), (0, 2))$.

To put both these sorts of games within one pregame framework would require a topology on the space of player types and would require that the pregame is really just the union of two distinct pregames, one with buyers and sellers and another with production. Also, although it may be intuitive, the pregame framework does not make precise the similarities between the games that drive results, stated in terms of the parameters, applying to both games.

2.2 Equal treatment ε-core

the core and ε-cores. Let (N, v) be a game and let ε be a non-negative real number. A payoff vector x is in the *ε-core* of (N, v) if and only if it is *feasible*, that is, $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon |S|$ for all $S \subset N$. When $\varepsilon = 0$, the ε-core is the *core*.

Lemma 0. Let (N, v) be a not-necessarily superadditive game and let (N, v^s) be its superadditive cover. Let $\varepsilon \geq 0$ be given. Then if x is a payoff vector in the ε-core of (N, v), then x is in the ε-core of (N, v^s).

We leave the easy proof of Lemma 0 to the reader.

the equal treatment ε-core. Given non-negative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N, v) relative to a δ-substitute partition $\{N[t]\}$ of the player set as the set of payoff vectors x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

Our notion of the equal treatment core is motivated by standard economic theory. All units of a commodity may differ; no two workers have exactly the same fingerprints or DNA for example. But yet, nonidentical commodities, if sufficiently similar, are treated as one commodity. The equal-treatment core may be viewed as a stand-in for the competitive equilibrium where similar items are grouped together as the same commodity..

For our comparative statics and monotonicity results, we restrict to payoffs in equal treatment ε-cores. As is well known, even with strictly effective small groups ε-cores do not necessarily treat identical players identically. For example, suppose that (N, v) is an inessential game where $v(S) = |S|$ for all groups $S \subset N$. Then, for any player $i \in N$, the payoff $x \in \mathbb{R}^N$ where $x_i = 1 + \varepsilon(|N| - 1)$ and $x_j = 1 - \varepsilon$ for all $j \neq i$ is in the ε-core.[8] A number of results, however, have shown that under the assumption of per capita boundedness and thickness, bounding the percentages of players of each type strictly away from zero, approximate cores treat most similar players nearly identically.[9] The central result is that with strictly effective groups and sufficiently many players of each type, the core treats identical players identically.[10] We provide a version of this result below for parameterized collections of games with strictly effective small groups.

Proposition 0. Let $(N, v) \in \Gamma((\delta, T), (0, B))$. Let $z \in \mathbb{R}^N_+$ be in the core of (N, v). Suppose that there are more than B δ-substitutes for each player in the game. Then if i,$j \in N$ and i and j are δ-substitutes, it holds that

$$|z_i - z_j| \leq 2\delta.$$

Proof: The proof of this proposition is essentially the same as the proof of Theorem 3 of Wooders (1983), for NTU games. If $\delta = 0$ then, for the special case of TU games, the proof is exactly the same as in the prior paper.

For any $S \subset N$ let $z(S)$ denote $\sum_{a \in S} z_a$. From the assumption that groups bounded in size by B are strictly effective, it holds that for some partition $\{S^k\}$ of N into groups with $\left|S^k\right| \leq B$ for each k,

$$v(N) - \sum v(S^k) = 0.$$

Therefore, since z is in the core,

$$\sum v(S^k) - \sum z(S^k) = 0$$

[8]Such examples go back to earliest versions of [32].

[9]See [32], [31], [36] and [38]. Another related result appears in Kovalenkov and Wooders [9], where conditions are demonstrated under which all payoffs in approximate cores treat similar players equally. These conditions hold for NTU games but not for TU games with unlimited side payments.

[10]Proofs of the more general results in the cited papers follow from "approximating" ε-cores by exact cores of games with admissible sizes of coalitions truncated.

and

$$z(S^k) \geq v(S^k) \text{ for each } k.$$

It follows that

$$z(S^k) = v(S^k) \text{ for each } k.$$

For any $i \in N$ let $S^k(i)$ denote the member of $\{S^k\}$ containing player i. Now suppose, for some players i_0,$j_0 \in N$, that i_0 and j_0 are δ-substitutes and

$$z_{i_0} - z_{j_0} > 2\delta.$$

Let us first show that then there exist two players i_1, $j_1 \in N$, such that i_1 and j_1 are δ-substitutes, $i_1 \notin S^k(j_1)$ and

$$z_{i_1} - z_{j_1} > \delta.$$

If $i_0 \notin S^k(j_0)$ then i_0, j_0 are such two players i_1, j_1. Otherwise, since $\left|S^k(j_0)\right| \leq B$ and since there are more than B δ-substitutes for each player it holds that there is some player l who is a δ-substitute for i_0 and j_0, and $l \notin S^k(j_0)$. Then it follows from the triangle inequality that either $z_{i_0} - z_l > \delta$ or $z_l - z_{j_0} > \delta$. Thus either i_0, l or l, j_0 are such two players i_1, j_1.

Now let us consider $S^* = S^k(i_1) \cup \{j_1\} \backslash \{i_1\}$. Since i_1 and j_1 are δ-substitutes, it holds that

$$v(S^*) \geq v(S^k(i_1)) - \delta.$$

But $z(S^*) < z(S^k(i_1)) - \delta \leq v(S^*)$ and we have a contradiction to the assumption that z is in the core. ■

With the definition of the equal treatment ε-core in hand, we can next address monotonicity properties and comparative statics for this concept. In the following we will simply assume the nonemptiness of equal treatment ε-cores. With SGE along with PCB, for $\varepsilon > 0$ this assumption is satisfied for all sufficiently large games in parameterized collections. Such a result appears in [10], [12].

3 Laws of scarcity

A technical lemma is required. For $x, y \in \mathbf{R}^T$, let $x \cdot y$ denote the scalar product of x and y, i.e. $x \cdot y := \sum_{t=1}^{T} x_t y_t$.

Lemma 1. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and, for $k = 1, 2$, let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\|f^1\right\|.$$

Proof: Since (N, v) has β-effective B-bounded groups, there exists a partition $\{G^{1\ell}\}$ of S^1, such that $|G^{1\ell}| \leq B$ for any ℓ and $\sum_\ell v(G^{1\ell}) \geq v(S^1) - \beta \|f^1\|$. Let us denote the profiles of $G^{1\ell}$ by g^ℓ. Observe that $\sum_\ell g^\ell = f^1$.

Since $f_t^2 \geq B$ for each t, it holds that $g^\ell \leq f^2$ for each ℓ. Therefore for each ℓ there exists a subset $G^{2\ell} \subset S^2$ with profile g^ℓ. Observe that since both $G^{1\ell}$ and $G^{2\ell}$ have profile g^ℓ, it holds that $|v(G^{1\ell}) - v(G^{2\ell})| \leq \delta \|g^\ell\|$. Since x^2 represents a payoff vector in the equal treatment ε-core of (S^2, v) and $G^{2\ell} \subset S^2$ has profile g^ℓ, the total payoff $x^2 \cdot g^\ell$ cannot be improved on by the coalition $G^{2\ell}$ by more than $\varepsilon \|g^\ell\|$. Thus, for each set $G^{2\ell} \subset S^2$ with profile g^ℓ, it holds that $x^2 \cdot g^\ell \geq v(G^{2\ell}) - \varepsilon \|g^\ell\| \geq v(G^{1\ell}) - (\varepsilon + \delta) \|g^\ell\|$. Adding these inequalities we have $x^2 \cdot f^1 \geq \sum_\ell v(G^{1\ell}) - (\varepsilon + \delta) \|f^1\|$. It then follows that $x^2 \cdot f^1 \geq v(S^1) - (\varepsilon + \delta + \beta) \|f^1\|$.

Since x^1 represents a payoff vector in the equal treatment ε-core of (S^1, v), $x^1 \cdot f^1$ is feasible for (S^1, v), that is, $x^1 \cdot f^1 \leq v(S^1)$. Combining these inequalities we have $(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \|f^1\|$. ■

Now we can state and prove our main results.

3.1 Approximate cyclic monotonicity

We derive an exact bound on the amount by which an approximate core payoff vector for a given game can deviate from satisfying exact cyclic monotonicity. The bound depends on:

δ, the extent to which players within each of T types may differ from being exact substitutes for each other;

β, the maximal loss of per capita payoff from restricting effective coalitions to contain no more than B players; and

ε, a measure of the extent to which the ε-core differs from the core.

Our result is stated both for absolute numbers and for proportions of players of each type. If exact cyclic monotonicity were satisfied, then the right hand sides of the equations (1) and (2) below could both be set equal to zero.

Proposition 1. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), .., (S^K, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \|f^1 + f^2 + .. + f^K\| \quad (1)$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq K(\varepsilon + \delta + \beta). \tag{2}$$

That is, the equal treatment ε-core correspondence approximately satisfies cyclic monotonicity both in terms of numbers of players of each type and percentages of players of each type.

Proof: From Lemma 1 we have $(x^k - x^{k+1}) \cdot f^k \leq (\varepsilon + \delta + \beta) \left\|f^k\right\|$ for $k = 1, .., K-1$ and $(x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \left\|f^K\right\|$. Summing these inequalities we get (1).

Alternatively we have $(x^k - x^{k+1}) \cdot \frac{f^k}{\|f^k\|} \leq (\varepsilon + \delta + \beta)$ for $k = 1, .., K - 1$ and $(x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain (2). ■

Remark. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2\right\|.$$

This form of monotonicity is typically called simply *monotonicity* or *weak monotonicity.*

Note that weak monotonicity does not imply cyclic monotonicity.

Corollary. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2\right\| \text{ and } (x^1 - x^2) \cdot (\frac{f^1}{\|f^1\|} - \frac{f^2}{\|f^2\|}) \leq 2(\varepsilon + \delta + \beta).$$

That is, the equal treatment ε-core correspondence is approximately monotonic.

Note that the bound of Proposition 1 and its Corollary holds for any partition of the player set into δ-substitutes.

3.2 Comparative statics

For $j = 1, .., T$ let us define $e^j \in \mathbf{R}^T$ such that $e^j_l = 1$ for $l = j$ and 0 otherwise. Our comparative statics results relate to changes in the abundances of players of a particular type.

Proposition 2. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f^k_t \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then the following holds:

(A) If $f^2 = f^1 + me^j$ for some positive integer m (i.e., the second game has more players of approximate type j but the same numbers of players of other types) then

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta)\frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} = (\varepsilon + \delta + \beta)\frac{2\,\|f^2\| - m}{m}.$$

(B) If $\frac{f^2}{\|f^2\|} = (1-\mu)\frac{f^1}{\|f^1\|} + \mu e^j$ for some $\mu \in (0,1)$ (i.e., the second game has proportionally more players of approximate type j but the same proportions between the numbers of players of other types) then

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta)\frac{2-\mu}{\mu}.$$

That is, approximately the equal treatment ε-core correspondence provides lower payoffs for players of a type that is more abundant.

Proof: (A): Applying Corollary we get $(x^2 - x^1) \cdot me^j \leq (\varepsilon + \delta + \beta)\,\|f^1 + f^2\|$. Since $\|f^2\| = \|f^1\| + m$, this inequality implies our first result.

(B): From Lemma 1 we have $(1-\mu)(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} \leq (1-\mu)(\varepsilon+\delta+\beta)$ and similarly $(x^2 - x^1) \cdot \frac{f^2}{\|f^2\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain $(x^2 - x^1) \cdot (\frac{f^2}{\|f^2\|} - (1-\mu)\frac{f^1}{\|f^1\|}) \leq (2-\mu)(\varepsilon+\delta+\beta)$. Thus we get that $(x^2 - x^1) \cdot \mu e^j \leq (2-\mu)(\varepsilon+\delta+\beta)$. This inequality implies our second result. ■

Obviously, again the bounds provided by Proposition 2 are independent of the specific partition of the player set into δ-substitutes.

3.3 Further Remarks.

1. (B) of Proposition 2 is a strict generalization of (A). ((A) follows from (B) for $\mu = \frac{m}{\|f^2\|}$.) We choose to present (A) in addition to (B) since (A) may be more intuitive. Notice also that although (A) is an immediate consequence of Proposition 1, (B) formally does not follow from Proposition 1.

2. Note that the bounds on the closeness of all our results are computable for a given game and depend only on the parameters describing the game. In the Appendix we demonstrate that all the bounds obtained are *exact*, that is, they cannot be made smaller.

3. For $\varepsilon = \beta = \delta = 0$ the bounds on the closeness of all our approximation results equals zero. Thus for games with finite number of player types and strictly effective small groups (e.g. for matching games with types) we demonstrate that the equal treatment core satisfies cyclic monotonicity and when a type becomes more abundant, players of that type receive (weakly) lower payoffs.

4. The results stated all require that there be at least B players of each type in each game under consideration. With other notions of approximate cores, specifically, the ε-remainder core and the ε_1-remainder ε_2-core, which allow a small percentage of players to be ignored, it may only be required that there are many substitutes for most players in the game; we leave the details to the interested reader. See [11] for definitions and further references.

5. We also leave it to the interested reader to show that results similar to those herein could be obtained for the strong ε-core. This approximate core notion requires that no group of agents can improve on a given payoff by ε in total, that is, given a game (N, v) and $\varepsilon \geq 0$, a payoff vector x is in the *strong ε-core* of (N, v) if and only if $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon$ for all $S \subset N$. For strong ε-cores, the goodness of the approximation improves.

6. In the context of a pregame, as noted earlier, when there are sufficiently many players of each type in the games, then small group effectiveness, SGE, and per capita boundedness, PCB, are equivalent but, in the context of parameterized collections of games, this equivalence no longer holds. SGE, introduced in Wooders ([35], [36], [37]),[11] is a relaxation of "minimum efficient scale," MES (Wooders [34]). MES dictates that *all* gains rather than *almost all* gains to improvement can be realized by groups of players bounded in size.[12] As indicated already by the techniques of Wooders ([33],[34]), when there are sufficiently many players of each type present in the games, sequences of games derived from a pregame satisfying PCB can be approximated by games satisfying MES. (In fact, Shubik and Wooders [31] suggestively call PCB *near minimum efficient scale.*) This is very useful in proving various results since games satisfying MES are especially tractable. It is noteworthy that the results of the current paper do not depend on PCB – a parameterized collection of games does not necessarily have bounded average payoffs. Consider, for example, the collection of games where all players are identical, two-player coalitions are effective, and the per-capita payoff to a two-person coalition in any game in the collection equals the number of players in the game. Clearly, without any loss, coalitions can be restricted to have no more than two players and, even though per capita payoffs

[11] A condition closely related to SGE appears in Wooders and Zame [41]. There, to obtain one of their results, the authors assume that almost all gains to improvement can be realized by groups of players bounded in absolute size. An equivalence between this condition and SGE is demonstrated in [36].

[12] MES has also been called *strict* small group effectiveness. For pregames with side payments it is equivalent to the "exhaustion of gains to scale" in Scotchmer and Wooders [26] and to the "0-exhaustion of blocking opportunities" in Engl and Scotchmer [4],[5].

are unbounded, our results apply to all the games in the collection. Thus, the crucial property is SGE.

7. Cyclic monotonicity has appeared in several papers in economics; see, for example, Kusomoto [14], Epstein [6] and Jorgenson and Lau [7]. All these papers address duality theory in models of private goods economies. Kusomoto's also provides some more general treatment.

4 Matching hospitals and interns; An example

Given the great importance of matching models (see, for example, Roth and Sotomayor [25] for an excellent study and numerous references to related papers), we present an application of our results to a model of matching interns and hospitals. Our example is highly stylized. For a more complete discussion of the matching interns and hospitals problem, we refer the reader to Roth [24].

The problem consists of the assignment of a set of interns $\mathcal{I} = \{1, .., i, .., I\}$ to hospitals. The set of hospitals is $\mathcal{H} = \{1, .., h, .., H\}$. The total player set N is given by $N = \mathcal{I} \bigcup \mathcal{H}$. Each hospital h has a preference ordering over the interns and a maximum number of interns $\overline{I}(h)$ that it wishes to employ. Interns also have preferences over hospitals. We'll assume $\overline{I}(h) \leq 9$ for all $h \in H$. This gives us a bound of 10 on the size of strictly effective groups ($\beta = 0$). For simplicity, we'll assume that both hospitals and interns can be ordered by the real numbers so that players with higher numbers in the ordering are more desirable. The rank held by a player will be referred to as the player's *quality*. More than one player may share the same rank in the ordering. In fact, we assume that the total payoff to a group consisting of a hospital and no more than nine interns is given by the sum of the rankings attached to the hospital and to the interns. Let us also assume that the rank assigned to any intern is between 0 and 1 and the rank assigned to any hospital is between 1 and 2. Thus, if the hospital is ranked 1.3 for example and is assigned 5 interns of quality .2 each, then the total payoff to that group is 2.3.

Since all interns have qualities in the interval $[0, 1)$ and similarly, all hospitals have qualities in the interval $[1, 2]$, given any positive real number $\delta = \frac{1}{n}$ for some positive integer n we can partition the interval $[0, 2]$ into $2n$ intervals, $[0, \frac{1}{n}), .., [\frac{j-1}{n}, \frac{j}{n}), .., [\frac{2n-1}{n}, 2]$, each of measure $\frac{1}{n}$. Assume that if there is a player with rank in the jth interval, then there are at least 10 players with ranks in the same interval.

Given $\varepsilon \geq 0$, let x^1 represent a payoff vector in the ε-core that treats all interns with ranks in the same interval equally and all hospitals with ranks in the same interval equally (that is, x^1 is equal treatment relative to the given partition of the total player set into types). Let us now increase the abundance of some type of intern that appears in N with rank in the jth interval for some j. We could imagine, for example, that some university training medical students increases the number of type j interns by admitting more students from another country. Let x^2 represent an equal treatment payoff vector in the ε-core after the increase in type j interns. It then holds, from result (A) of Proposition 2 that

$$(x_j^2 - x_j^1) \leq (\varepsilon + \frac{1}{n})\frac{\|f^1 + f^2\|}{\|f^2 - f^1\|}.$$

Of course this is not the most general application of our results – we could increase the proportions of players of one type by reducing the numbers of players of other types. Then part (B) of our Proposition could be applied.

It is remarkable that our results apply so easily. For this simple sort of example, it is probably the case that a sharper result can be obtained. This is beyond the scope of our current paper, however. Research in progress considers whether sharper results are obtainable with assortative matching of the kind illustrated by this example – that is, where players can be ordered so that players with higher ranks in the orderings are superior in terms of their marginal contributions to coalitions.

Finally, the parameter values that we have used in this example were chosen for convenience and simplicity. In principle, these could be estimated and various questions addressed. For example, are payoffs to interns approximately competitive? Do non-market characteristics such as ethnic background or gender make significant differences to payoffs?

5 Relationships to the literature and conclusions

5.1 Relationship to the literature on matching markets

Our results may be viewed as a contribution to the literature on comparative statics properties of solutions of games. As noted by Crawford [2], the first suggestion of the sort of results obtained in this paper may be in Shapley [27], who showed that in a linear optimal-assignment problem the marginal product of a player on one side of a market weakly decreases when another player is added to that side of the market and weakly increases when a player is added to the other side of the market. Kelso and Crawford [8], building on the model of Crawford and Knoer [3], show that, for a many-to-one matching market with firms and workers, adding one or more firms to the market makes the firm-optimal stable outcome weakly better for all workers and adding one or more workers makes the firm-optimal stable outcome weakly better for all firms. Crawford [2] extends these results to both sides of the market and to many-to-many matchings.[13] In contrast to this literature, our results are not restricted to matching markets and treat all outcomes in equal treatment ε-cores. Moreover, we demonstrate cyclic monotonicity. Instead of the assumptions of "substitutability" of

[13] And also to pair-wise stable outcomes but this is apparently not so directly related to our paper.

Kelso and Crawford [8], however, we require our thickness condition and SGE. Unlike [8] and [2], our current results are limited to games with side payments – we plan to consider this limitation in future research.

Note that our results imply a certain *continuity* of comparative statics results with respect to changes in the descriptors of the total player set. In particular, the results are independent of the exact partition of players into approximate types. Specifically, given a number T of approximate types and a measure of the required closeness of the approximation, subject to the condition that players of each type are approximate substitutes for each other, our results apply independently of exactly where the boundary lines between types are drawn. Suppose, for example, that we wished to partition candidates for positions as hospital interns into three categories – say "good," "better" and "best." It may be that there is more than one way to partition the set of players into these categories while retaining the property that all players in each member of the partition are approximate substitutes for each other; the exact partition does not affect the results. Relating this feature of our work to general equilibrium theory, a finite set of commodities is typically considered to be an approximation to the real-world situation that all units of each commodity may differ. Descriptions of commodities are incomplete and a "commodity" is a group of objects that satisfy the description. For example, models of labor markets may have two types of workers, "skilled" and "unskilled" but no two workers (or two loaves of bread, or two oranges) may be exactly identical. In the differentiated commodities literature, results addressing this problem show that prices are continuous functions of attributes of commodities (cf., Mas-Colell [15]). Since our framework does not require a topology on the space of player types, continuity takes a different but valid form and is more directly apparent.

5.2 Relationship to the literature on pregames

Besides the matching literature, our results are related to prior results obtained within the context of a pregame, cf. [32], [36]. A pregame specifies a set of compact metric space of player types and a *single* worth function, assigning a worth to each finite list of attributes (repetitions allowed). (Recall that precise definitions appear in Appendix B.) Since there is only one worth function, all games derived from a pregame are related and, given the attributes of the members of a coalition, the payoff to that coalition is independent of the total player set in which the coalition is embedded; widespread externalities are not allowed. In contrast, our results apply to given games and, as in the earlier results for matching models, there is no requisite topological structure on the space of players types. While our results for a given game hold for all games in a collection described by the same parameters, there are no necessary relationships between games. For example, consider the collection of games where two-player coalitions are effective and there are only two types of players. This collection includes two-sided assignment games, such as marriage games and buyer-seller games, and also games where *any* two-player coalition is effective. There appears to be no way in which one pregame can accommodate all the games in the collection. These considerations indicate that the framework of parameterized collections of games is significantly broader than that of a pregame.[14]

In the context of pregames under conditions roughly equivalent to those of Wooders [32] – that *all* gains to coalition formation can be exhausted by coalitions bounded in size – a proof of the comparative statics result and weak monotonicity of core payoffs was provided in Scotchmer and Wooders [26]. Wooders [35],[36] extended the *monotonicity* analysis of Scotchmer and Wooders to hold for arbitrary changes in abundances of players of each type in games satisfying SGE and made the connection to the Law of Demand of economic theory (cf., Hildenbrand 1994). Engl and Scotchmer [4],[5] extended the *comparative statics* analysis of Scotchmer and Wooders to hold for proportions of players of each type and further addressed the relationships between weak monotonicity and the Law of Demand. All of these results, unlike the matching literature, require a fixed set of player types (or a fixed finite set of attributes of players and a single worth function defined over these attributes). The major difference between the results of these papers and those of the current paper are that our assumptions and results (a) treat more general collections of games, (b) apply to individual games, and (c) apply uniformly to all games described by the same parameters.

A major advantage of our approach over the prior approach using pregames is that, except for the special case of pregames satisfying strict small group effectiveness (or, in other words, 'exhaustion of gains to scale by coalitions bounded in size') with a finite number of exact types, *the conditions used in the prior literature cannot be verified* for any finite game.[15] That is, since the conditions are stated on the worth function of the entire pregame, which includes specification of the worths of arbitrarily large groups, or on the closeness of the worth function to the limiting per capita utility function, it is not possible to determine whether the conditions are satisfied. In contrast, given any game, values of parameters describing that game can be computed.[16]

Another major advantage of our approach is that we provide *exact bounds*, in

[14] A short survey discussing parameterized collections of games and their relationships to pregames appears in [39].

[15] *Strict* small group efectiveness dictates that *all* gains to coalition formation can be realized by partitioning the total player set, no matter how large, into coalitions bounded in size. This condition was introduced in Wooders [32] (condition *) and, for NTU games, in Wooders [34], where it was called "minimum efficient scale."

[16] Since there may be many but a *finite* number of coalitions, in fact determining the required sizes of δ and T, β and B may be time-consuming but it is possible. In contrast, to verify that a pregame satisfies SGE or PCB requires consideration of an *infinite* number of payoff sets or, even more demanding, a limiting set of equal treatment payoffs. (In Engl and Scotchmer [4], per capita boundedness is stated in terms of finiteness of limiting equal treatment payoffs.)

terms of the parameters describing a game, on the amounts by which equal treatment ε-core payoff vectors can differ from satisfying cyclic monotonicity. We are unaware of any comparable results in the literature. The prior literature does not indicate the *sensitivity* of the results to specifications of bounds on group sizes and of types of players. Such an analysis is important for empirical testing since, in fact, few commodities are completely standardized. (This may be especially true in estimating hedonic prices as in Rosen 1978,1986 and more recent literature.) Nor does the prior literature provide *empirically testable conclusions* on approximate monotonicity or comparative statics.

5.3 Relationship to the literature on attribute games

One interpretation of a game with side payments, common in the literature, is to regard the players in the game as commodities or inputs. We call this an *attribute game* and the equal treatment core is called the *attribute core.*[17] Our results immediately apply to attribute games.

For a simple example, consider a glove game where each player is a RH glove or a LH glove and the payoff to a coalition consisting of n_1 RH glove players and n_2 LH glove players is $\Psi(n_1, n_2) := \min\{n_1, n_2\}$. Suppose that in total, there are f_1 RH gloves and f_2 left hand gloves. Our laws of scarcity apply equally well to this interpretation of a game. Note that this game is a member of the collection $\Gamma((0,2),(0,2))$.

If ownership of *bundles* of commodities is assigned to individual units (teams or divisions within a firm in the literature on subsidy-free pricing or endowments of individual consumers of commodities in the exchange economy interpretation), then another cooperative game is generated. In this game, essentially some players in the original game are "syndicated," glued together to become one player.

From the data given above, we can construct games where players may be endowed with bundles of gloves. By endowing players in this game with various numbers of RH gloves and LH gloves, we create another game with possibly several types of players. For specificity, suppose:

1. m_1 players of type 1 are endowed with two right hand gloves each;
2. m_2 players of type 2 are endowed with a RH glove and;
3. m_3 players of type 3 are endowed with a LH glove.

For consistency, it must hold that $2m_1 + m_2 = f_1$ and $m_3 = f_2$. (Of course this is only one of many possible games that could be constructed.) Now it is not

[17] Of course this simply gives a name to familiar concepts. The equal-treatment core of a game goes back to some of the first papers introducing the core, cf. Shubik [30].

so immediate that our main results can be applied. However, from the data given, with the three possible endowments of gloves given by 1-3 above, we can determine a number of types T and a bound B so that the game constructed, say (M, w), is a member of the collection $\Gamma((0,T),(0,B))$. It is immediate that $T = 3$. It is fairly obvious and we leave for the reader to verify that $B = 3$ suffices; the largest coalitions that need form in realizing all gains to collective activities consist of one player of type 1 and two players of type 3. Thus, we have that $(M, w) \in \Gamma((0,3),(0,3))$ and our comparative statics and monotonicity results apply to the games in $\Gamma((0,3),(0,3))$.

5.4 Relationship to the literature on general equilibrium

The class of economies treated in the current paper could be considered as a generalization of the standard competitive model by Arrow-Debreu-McKenzie. Moreover we treat the equal treatment ε-core as a "stand-in" for the competitive equilibrium in the general context of the cooperative game theory. Hence, if player types are thought of as commodity types while payoffs to players are thought of as prices for commodities, as in the above subsection, our Laws of Scarcity are closely related to comparative statics results for general equilibrium models.

Indeed, Nachbar [17] has established conditions under which in a general equilibrium model the inner product of endowment changes and normalized competitive equilibrium price changes is negative.[18] The conditions are that (a) the general equilibrium version of Law of Demand holds and (b) goods are normal. The further limitation of Nachbar's result is that the normalization have to be a very specific and unusual. However in case of quasi-linear utility which corresponds to the case of games with side payments treated in the current paper both conditions (a) and (b) are satisfied and normalization became a natural one with a price of numeraire commodity set to one. Thus for quasi-linear utilities Nachbar's result implies a negative monotonicity relation between endowment changes and equilibrium price changes. The results of the current paper show the robustness of this monotonicity conclusion. More precisely our paper considers economies more general than Arrow-Debreu-McKenzie model and identifies conditions that ensure approximate negative monotonicity of payoffs in the equal treatment ε-core with respect to endowment changes.

5.5 An intuition behind the results

Numerous examples of games derived from pregames may lead one to expect our comparative statics result. Consider a glove game, for example where the payoff function can be written as $u(x, y) = \min\{x, y\}$. Suppose initially that the number of RH gloves, say x, is equal to the number of LH gloves, y, and both x and y

[18] This result was generalized independently by Nachbar [18] and Quah [19] to allow discrete changes.

are greater than one. Then the equal-treatment core can be described by the set $\{(p_x, p_y) \in \mathbb{R}^2_+ : p_x + p_y = 1\}$; each RH glove is assigned p_x and each LH glove is assigned p_y and a pair of gloves is assigned 1. Now increase the number of players with RH gloves. The equal treatment core is now described by $\{(0,1)\}$; each RH glove is assigned 0 and each LH glove is assigned 1.

In games with a finite set of player types, defining the core via linear programming also leads to a law of scarcity, quite immediately. Let (N, v) be a game with a finite number T of player types and with m_t players of type t, $t = 1, ..., T$. We take v as a mapping from subprofiles s of m ($s \in \mathbb{Z}^T_+$, $s \leq m$). Then, following Wooders [32], consider the following LP problem[19]:

$$\begin{aligned}&\text{minimize}_{p\geq 0} \;\; p \cdot m \\ &\text{subject to } p \cdot s \geq v(s) \text{ for all } s \leq m\end{aligned}$$

If the game has a nonempty core, then the solution p^* satisfies $v(m) = p^* \cdot m$. Now consider the same problem but with an increased number of players of type $\widehat{t}$ in the objective function for some $\widehat{t} \in \{1, ..., T\}$. Assume that the same inequalities are the only constraints; this imposes a form of strict small group effectiveness on the game – only groups with profiles $s \leq m$ are effective. It is clear that the payoff to players of type $\widehat{t}$ will not increase with the increase in the number of players of that type in the objective function since the constraint set has not changed – the payoff to type $\widehat{t}$ can only decrease. This suggests some of the initial intuition underlying comparative statics results for games.

6 Appendix A: Exact bounds

We construct some sequences of games to demonstrates that all the bounds we obtained in our results are *exact*, that is, the bound cannot be decreased.

I). Let us concentrate first on the central case $\delta = \beta = 0$. Consider a game (N, v) where any player can get only 1 unit or less in any coalition and there are no gains to forming coalitions. This game has strictly effective 1-bounded groups and all agents are identical. Formally, however, we may partition the set of players into many types. Thus $(N, v) \in \Gamma((0, \tau), (0, 1))$ for any integer τ, $1 \leq \tau \leq |N|$. Notice also that for any $\varepsilon \geq 0$ the ε-core of the game is nonempty and very simple: it includes all payoff vectors that are feasible and provide at least $1 - \varepsilon$ for each of the players. All the games that we are going to construct will be subgames of a game (N, v).

a). For the bound in Lemma 1 we can present even a single game with two payoffs vectors that realize this bound. Namely, let $\tau = 1$ (all players are of one type) and

[19] The core has been described as an outcome of a linear programming problem since the seminal works of Gilles and Shapley. Wooders [32] introduces the linear programming formulation with player types.

let us consider any two subgames S^1, S^2 with the same number of players and the equal treatment payoffs $x^1 = 1$ and $x^2 = 1 - \varepsilon$. Then $(x^1 - x^2) \cdot f^1 = \varepsilon \|f^1\|$.

b). For the bound in Proposition 1, for $K \leq |N|$ and some nonnegative integer $l \leq |N| - K$, let us consider $\tau = K$ and the subgroups $S^1, .., S^K$ with the profiles $f^1, .., f^K$ where $f_t^k = l + 1$ for $t = k$ and 1 otherwise. Let also consider payoff vectors x^k where $x_t^k = 1$ for $t = k$ and $1 - \varepsilon$ otherwise. Then $(x^i - x^j) \cdot f^i = \varepsilon l$ for any $i \neq j$. Hence

$$(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K = \varepsilon l K = \varepsilon \left\| f^1 + f^2 + .. + f^K \right\| \frac{l}{l + K}$$

$$\text{and } (x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} = K\varepsilon \frac{l}{l + K}.$$

It is straightforward to verify that for any fixed K both our bounds in Proposition 2 can not be improved for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

c). For the bound in Proposition 2 it is enough to concentrate on (A) since it is a special case of the result (B). For $|N| \geq 2$ let us consider $\tau = 2$ and $l \leq |N| - 2$. Then consider the subgroups S^1, S^2 with the profiles $f^1 = (1, 1)$ and $f^2 = (l + 1, 1)$ and payoff vectors $x^1 = (1 - \varepsilon, 1)$ and $x^2 = (1, 1)$. Then

$$(x_1^2 - x_1^1) = \varepsilon = \varepsilon \frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} \frac{l}{l + 4}.$$

It follows that both our bounds in Proposition 2 can not be improved for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

II). It is easy to modify our example to allow for non-zero δ and β in a such a way that we will have the same profiles as in Part I, but will use the payoffs of $1 + \delta + \beta$ and $1 - \varepsilon$ instead of 1 and $1 - \varepsilon$. This will lead us to the appearance of $\varepsilon + \delta + \beta$ on the places of ε in all bound in Part I. We leave it as a simple exercise for the interested reader.

7 Appendix B: Pregames

In this appendix, for the convenience of the reader in comparing the concepts and in evaluating the contribution of this paper, we review the concept of a pregame.

Let Ω be a compact metric space, interpreted as a set of player "types" or attributes. A *profile* on Ω, interpreted as a description of a group of players in terms of numbers of players of each type in the group, is a function f from Ω to the set Z_+ of nonnegative integers for which the *support* $\sigma(f)$ of f, given by

$$\sigma(f) = \{\omega \in \Omega : f(\omega) \neq 0\},$$

is finite. A profile is simply a function f from Ω to the nonnegative integers with the property that $f(\omega) \neq 0$ for only a finite number of elements ω in Ω. For each $\omega \in \Omega$, we interpret $f(\omega)$ as the number of players of type ω or, in other words, with attributes ω, in the group of players described by f. The set of profiles on Ω is denoted by $P(\Omega)$. We write $f \leq g$ if $f(\omega) \leq g(\omega)$ for each ω in Ω.

By the *norm* of a profile, we mean

$$\|f\| = \sum_{\omega \epsilon \sigma(f)} f(\omega),$$

which is simply the number of players in a group represented by f. This is a finite sum since f has finite support.

A *pregame* is a pair (Ω, Ψ) where Ω is a compact metric space, called the *space of attributes* and $\Psi : P(\Omega) \to R_+$, called the *characteristic function (of the pregame)*, is a function with the following properties:

(a) $\Psi(0) = 0$;

(b) given any $\epsilon > 0$ there is a $\delta > 0$ such that

for each pair of player types ω_1 and ω_2 with $dist(\omega_1, \omega_2) < \delta$

it holds that $|\Psi(f + \omega_1) - \Psi(f + \omega_2)| < \epsilon$ (continuity);

(c) $\Psi(f) + \Psi(g) \leq \Psi(f + g)$ for all profiles f and g, and;

The first condition means that zero players can realize nothing. The second is that players with similar attributes are nearly substitutes. The third expresses the idea that an option open to a group is to split into several smaller groups.

We frequently refer to the elements of Ω as "types". Players of the same type are substitutes.

7.1 Games induced by pregames

To derive a game from a pregame (Ω, Ψ), we specify a finite set N and a function $\alpha : N \to \Omega$, called an *attribute function*. With any subset S of N we can then associate a profile, $prof(\alpha|S)$, given by

$$prof(\alpha|S)(\omega) = |\alpha^{-1}(\omega) \cap S|.$$

The profile $prof(\alpha|S)(\omega)$ simply lists the numbers of players of each type in the subset S. We have now determined a game (N, v_α) where

$$\nu_\alpha(S) = \Psi(prof(\alpha|S))$$

for each $S \subset N$. Let $n = prof(\alpha|N)$. An *equal treatment payoff* is a function $x : \Omega \to R_+$. An equal treatment payoff assigns the same value to all players of the same type. The payoff x is *feasible for* n if

$$\sum_{\omega \in \sigma(n)} n(\omega)\, x(\omega) \leq \Psi(n).$$

For ease in notation, given a profile f and an equal-treatment payoff x define $x(f)$ by

$$x(f) = \sum_{\omega \in \sigma(f)} f(\omega)\, x(\omega).$$

Let f be a profile. When $\sum f^k = f$ for some collection of profiles $f^1, ..., f^K$, not necessary distinct, we say that the collection is *a partition of* f and each member of the collection is called a *subprofile of* f. Obviously, a partition of a profile is related to a partition of a set of players. If (N, v_α) is a game derived from (Ω, Ψ), and $\{S_1, ..., S_K\}$ is a partition of N, then

$$\{f^k : prof(\alpha|S_k) = f^k, k = 1, ..., K\}$$

is a partition of $prof(\alpha|N)$.

7.2 Small Group Effectiveness

A pregame (Ω, Ψ) satisfies *small group effectiveness*, (SGE), if for each positive real number $\beta > 0$ there is an integer $\eta_1(\beta)$ such that for each profile f, for some partition $\{f^k\}$ of f :

(a) $\|f^k\| \leq \eta_1(\beta)$ for each profile f^k in the partition,

(b) $\Psi(f) - \sum_k \Psi(f^k) < \beta\|f\|$.

Small group effectiveness means that given a measure of per capita approximation (a $\beta > 0$) there is an absolute bound on group sizes with the property that almost all gains to collective activities can be realized by groups of players smaller in size than that bound, that is, bounded group sizes nearly exhaust all gains to scale of collective activities.

Let (Ω, Ψ) satisfy small group effectiveness and let β and $\eta_1(\beta)$ satisfy the condition of the definition of SGE. Then it is immediate that any game generated by the pregame has β-effective $\eta_1(\beta)$-bounded groups. Since Ω is a compact metric space it holds that given $\delta > 0$ we can partition Ω into a finite number T of subsets so that all players with attributes in each subset are δ-substitutes. Thus, all games derived from (Ω, Ψ) are in the collection $\Gamma((\delta, T), (\beta, B))$.

When games are required to have many substitutes for each player, small group effectiveness is equivalent to per capita boundedness. A pregame (Ω, Ψ) satisfies *per capita boundedness* if there is a constant A such that

$$\frac{\Psi(f)}{\|f\|} \leq A \text{ for all profiles } f \in P(\Omega).$$

The following result holds more generally but is proven for the case where Ω is a finite set.

Wooders 1994b, *Econometrica,* **Theorem 4.** *With "thickness," SGE=PCB.*

(1) Let (T, Ψ) be a pregame satisfying SGE. Then the pregame satisfies PCB.

(2) Let (T, Ψ) be a pregame satisfying PCB. Then given any positive real number ρ, construct a new pregame (T, Ψ_ρ) where the domain of Ψ_ρ is restricted to profiles f where, for each $t = 1, \cdots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$ (thickness). Then (T, Ψ_ρ) satisfies SGE on its domain.

The equivalence, with thickness, of small group effectiveness with per capita boundedness indicates that SGE is an apparently mild yet powerful condition. But, as we see above, if a pregame satisfies SGE then, given $\beta > 0$, for appropriate choice of δ and T it holds that all games generated by the pregame belong to a parameterized collection of games $\Gamma((\delta, T), (\beta, B))$. Thus, our conditions on parameterized collections of games are less restrictive than those on pregames (as in Wooders and Zame [40], who use a stronger condition than SGE or in Wooders [37],[35]) and, with thickness, less restrictive than the earlier condition of PCB.

The concept of small group effectiveness requires that almost all feasible gains to collective activities can be achieved by groups bounded in absolute size. A related concept requires that almost all improvement be feasible for groups bounded in absolute size. A pregame (Ω, Ψ) satisfies *small group effectiveness for improvement* if for each positive real number $\epsilon > 0$ there is an integer $\eta_2(\epsilon)$ with the following property:

For any profile f and any payoff function $x : \sigma(f) \rightarrow R_+$
if $x(f) + \epsilon \|f\| < \Psi(f)$ then there is a subprofile g of f such that

$$\|g\| \leq \eta_2(\epsilon) \text{ and } x(g) + \tfrac{\epsilon}{2} \|g\| < \Psi(g).$$

The pregame framework may also hide what makes the results work – the facts that there are many close substitutes for most players and that groups bounded is size can nearly exhaust gains to collective activities. In addition, since the pregame framework specifies payoffs for all groups, no matter how large, in general it is difficult, if not impossible to estimate the pregame function Ψ. In contrast, within the framework of parameterized collections, there are only four parameters to be estimated – δ, T, β, and B. The notion of β-effective B-bounded groups makes explicit how close coalitions bounded in size by B are to being able to realize all gains to collective activities for a given game.

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