Supplementary Materials

Proof of Lemma 1. Observe first that

$$p_i(s_k \left| s^{k-1}\right.) = \int_{\Delta^{K-1}} \theta(s_k)\mu_{i,s^{k-1}}(d\theta)$$

for any $1 \leq k \leq t$. Then, we have that

$$\begin{aligned}
\int_{\Delta^{K-1}} P^{\theta}(B)\,\mu_{i,s^t}(d\theta) &= \int_{\Delta^{K-1}} P^{\theta}(B)\frac{\theta(s_t)\mu_{i,s^{t-1}}(d\theta)}{\int_{\Delta^{K-1}}\theta(s_t)\mu_{i,s^{t-1}}(d\theta)} \\
&= \frac{1}{p_i(s_t\,|s^{t-1})}\cdots\frac{1}{p_i(s_1\,|s_0))}\int_{\Delta^{K-1}} P^{\theta}(B)\;\theta(s_t)...\theta(s_1)\,\mu_{i,0}(d\theta) \\
&= \frac{1}{P_i(C(s^t))}\int_{\Delta^{K-1}} P^{\theta}(B_{s^t})\;\mu_{i,0}(d\theta) \\
&= \frac{P_i(B_{s^t})}{P_i(C(s^t))} = P_{i,s^t}(B).
\end{aligned}$$

■

Proof of Lemma 12. Boundedness of $\mathcal{U}$ follows because $Y^{\infty}(\xi)$ is bounded. Convexity follows from the strict concavity of u_i.

To prove that $\mathcal{U}(\xi,\mu)$ is closed, take any sequence $\{w^n\}$ such that $w^n \in \mathcal{U}(\xi,\mu)$ for all n and $w^n \to \overline{w} \in \mathbb{R}^I_+$. Take the corresponding sequence $\{c^n\} \subset Y^{\infty}(\xi)$. Since $Y^{\infty}(\xi)$ is compact under the sup-norm, there exists a convergent subsequence $\{c^{n_k}\}$ such that $c^{n_k} \to \overline{c} \in Y^{\infty}(\xi)$. Thus, it follows by definition that $U_i^{P_i}(c_i^{n_k}) \geq w_i^{n_k}$ for all k and for all i. Since u_i is continuous and $\mathbb{C}(s_0)$ is compact, then $U_i^{P_i}$ is continuous under the sup-norm. Thus, it follows that $U_i^{P_i}(\overline{c}_i) \geq \overline{w}_i$, for all i. Consequently, $\overline{w} \in \mathcal{U}(\xi,\mu)$ by definition and $\mathcal{U}(\xi,\mu)$ is closed. ■

Proof of Lemma 13. That v^* is increasing in α and homogenous of degree one is straightforward. $v^*(\xi,\alpha,\mu)$ is bounded because the constraint set, $Y^{\infty}(\xi)$, is uniformly bounded and $\beta \in (0,1)$. Let $Y^k(\xi) \equiv \{c \in Y^{\infty}(\xi) : c_i(s^t) \equiv c_{i,t}(s) = 0$ for all $t \geq k\}$ be the $k-$truncated set of feasible allocations. Note that $Y^k(\xi) \subset Y^{k+1}(\xi) \subset Y^{\infty}(\xi)$ and define

$$v_k^*(\xi,\alpha,\mu) \equiv \max_{c\,\in\,Y^k(\xi)} \sum_{i\in\mathcal{I}} \alpha_i\; U_i^{P_i}(c_i)$$

Suppose that $\left\{(\mu_i^n)_{i=1}^I\right\}$ is a sequence of probability measures such that μ_i^n converges weakly to $\overline{\mu}_i \in \mathcal{P}(\Delta^{K-1})$ for all i. Given k, note that

$$\sum_{t=0}^{k} \beta^t \int_{\Delta^{K-1}} \left(\sum_{s^t} P^\theta \left(C(s^t) \right) u_i(c_i(s^t)) \right) \mu_i^n \, (d\theta) \to \sum_{t=0}^{k} \beta^t \int_{\Delta^{K-1}} \left(\sum_{s^t} P^\theta \left(C(s^t) \right) u_i(c_i(s^t)) \right) \overline{\mu}_i \, (d\theta)$$

since $P^\theta \left(C(s^t) \right)$ is continuous and bounded for all t and s^t. Thus, it follows from the Maximum Theorem that $v_k^*(\xi, \alpha, \mu)$ is continuous in (μ, α) for all ξ.

Note that $v_k^*(\xi, \alpha, \mu) \leq v_{k+1}^*(\xi, \alpha, \mu) \leq v^*(\xi, \alpha, \mu)$ for all (ξ, α, μ). Hence, $v_k^*(\xi, \alpha, \mu) \to v^*(\xi, \alpha, \mu)$ for each (ξ, α, μ) since there exists some $c^* \in Y^\infty(\xi)$ attaining $v^*(\xi, \alpha, \mu)$. Now we show that this convergence is uniform.

Given any (ξ, α, μ), let $c^* \in Y^\infty(\xi)$ attain $v^*(\xi, \alpha, \mu)$ and define c^{*^k} as its $k-$truncated version. Then,

$$0 \leq v^*(\xi, \alpha, \mu) - v_k^*(\xi, \alpha, \mu) \leq \sum_{i=1}^{I} \alpha_i (U_i^{P_i}(c_i^*) - U_i^{P_i}(c_i^{*^k})) \leq \frac{\beta^k}{1-\beta} \max_i u_i(\overline{y}).$$

Since $\beta \in (0,1)$, this convergence is uniform (i.e., the RHS is independent of (ξ, α, μ)) and thus $v^*(\xi, \alpha, \mu)$ is continuous. ■

Proof of Lemma 14. Observe that $v^*(\xi, \alpha, \mu) \geq \alpha \, u$ for all $\alpha \in \Delta^{I-1}$ holds if and only if

$$\min_{\widetilde{\alpha} \in \Delta^{I-1}} \left[v^*(\xi, \widetilde{\alpha}, \mu) - \sum_{i=1}^{I} \widetilde{\alpha}_i \; w_i \right] \geq 0.$$

Therefore, it suffices to show that $w \in \mathcal{U}(\xi, \mu)$ if and only if $w \geq 0$ and $v^*(\xi, \alpha, \mu) \geq \alpha \, w$ for all $\alpha \in \Delta^{I-1}$.

For any $w \in \mathcal{U}(\xi, \mu)$, (28) implies that $v^*(\xi, \alpha, \mu) \geq \alpha \, w$ for all $\alpha \in \Delta^{I-1}$.

To show the converse, suppose that $w \geq 0$ and $v^*(\xi, \alpha, \mu) \geq \alpha \, w$ for all $\alpha \in \Delta^{I-1}$ but $w \notin \mathcal{U}(\xi, \mu)$. This implies that $\nexists \; \widetilde{w} \in \mathcal{U}(\xi, \mu)$ such that $\widetilde{w} \geq w$. Since $\mathcal{U}(\xi, \mu)$ is convex, it follows by the separating hyperplane theorem that $\exists \; \eta \in \mathbb{R}_+^I / \{0\}$ such that $\eta w \geq \eta \widetilde{w}$ for all $\widetilde{w} \in \mathcal{U}(\xi, \mu)$. Since $\mathcal{U}(\xi, \mu)$ is closed, $\eta w > \eta \widetilde{w}$ for all $\widetilde{w} \in \mathcal{U}(\xi, \mu)$, where η can be normalized such that $\eta \in \Delta^{I-1}$. But then $v^*(\xi, \eta, \mu) \geq \eta \, w > \eta \; \widetilde{w}$ for all $\widetilde{w} \in \mathcal{U}(\xi, \mu)$. This contradicts (28). ■

Proof of Step 3, Theorem 3. We show that there exists some $\alpha_0 = \alpha(s_0, \mu_0)$ such that $A_i(s_0, \alpha_0, \mu_0) = 0$ for all i, given (s_0, μ_0).

Note first that if $\alpha_i = 0$, then $c_i(\xi, \alpha) = 0$ and consequently $A_i(\xi, \alpha, \mu) < 0$ for all (ξ, μ). Define the vector-valued function g on Δ^{I-1} as follows:

$$g_i(\alpha) = \frac{\max[\alpha_i - A_i(s_0, \alpha, \mu_0), 0]}{\sum_{i=1}^{I} \max[\alpha_i - A_i(s_0, \alpha, \mu_0), 0]}, \tag{1}$$

for each i. Note that $H(\alpha) = \sum_{i=1}^{I} \max[\alpha_i - A_i(s_0, \alpha, \mu_0), 0]$ is positive for all $\alpha \in \Delta^{I-1}$. Also, $g_i(\alpha) \in [0, 1]$ and $\sum_{i=1}^{I} g_i(\alpha) = 1$ for all α. Thus, g is a continuous function mapping Δ^{I-1} into itself. The Brower's fixed point theorem implies that there exists some $\alpha_0 = \alpha(s_0, \mu_0)$ such that $\alpha_0 = g(\alpha_0)$.

Suppose now that $\alpha_{i,0} = 0$ for some i. For such α_0, (39) implies that $-A_i(s_0, \alpha_0, \mu_0) \leq 0$. But we have already argued that $-A_i(s_0, \alpha_0, \mu_0) > 0$ if $\alpha_{i,0} = g_i(\alpha_0) = 0$. This would lead to a contradiction and, hence, $\alpha_{i,0} > 0$ for all i. This implies that $\alpha_{i,0} - A_i(s_0, \alpha_0, \mu_0) > 0$ for all i. Therefore,

$$H(\alpha_0)\alpha_{i,0} = H(\alpha_0)g_i(\alpha_0) = \max[\alpha_{i,0} - A_i(s_0, \alpha_0, \mu_0), 0] = \alpha_{i,0} - A_i(s_0, \alpha_0, \mu_0).$$

This implies that $H(\alpha_0) = H(\alpha_0) \sum_{i=1}^{I} \alpha_{i,0} = \sum_{i=1}^{I} \alpha_{i,0} - \sum_{i=1}^{I} A_i(s_0, \alpha_0, \mu_0) = 1$. Therefore, $\alpha_{i,0} = \alpha_{i,0} - A^i(s_0, \alpha_0, \mu_0)$ for all i and thus $A^i(s_0, \alpha_0, \mu_0) = 0$ for all i. ■

Proof of Lemma 20. Notice that for $t > N$,

$$s \in \Omega_{t-N} \cap \Omega_{1,t-1}^{N-1} \quad \Rightarrow \quad E^{P^{\theta^*}}\left[1_{\Omega_{t-N} \cap \Omega_{1,t}^{N}} \middle| \mathcal{F}_{t-1}\right](s) = P^{\theta^*}\left[s_t = 1 \middle| \mathcal{F}_{t-1}\right](s) = \theta^*(1) > 0, \tag{2}$$

where we use the convention that $\Omega_{1,t}^{0} = \Omega$ to handle the case where $N = 1$.

For $s \in \{\Omega_t \text{ i.o.}\}$ arbitrarily chosen, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $s \in \Omega_{t_k}$ for every $k = 1, 2, \cdots$. Since $\Omega_{1,t}^{0} = \Omega$, $s \in \Omega_{(t_k+1)-1} \cap \Omega_{1,(t_k+1)-1}^{1-1}$. Therefore, (40) implies that

$$\begin{aligned} \sum_{t=1}^{\infty} E^{P^{\theta^*}}\left[1_{\Omega_{t-1} \cap \Omega_{1,t}^{1}} \middle| \mathcal{F}_{t-1}\right](s) &\geq \sum_{k=1}^{\infty} E^{P^{\theta^*}}\left[1_{\Omega_{(t_k+1)-1} \cap \Omega_{1,t_k+1}^{1}} \middle| \mathcal{F}_{t_k}\right](s) \\ &\geq \sum_{k=1}^{\infty} P^{\theta^*}\left[s_{t_{k+1}} = 1 \middle| \mathcal{F}_{t_k}\right](s) = +\infty, \end{aligned}$$

and it follows by Lemma 20 that $\sum_{t=1}^{\infty} 1_{\Omega_{t-1} \cap \Omega_{1,t}^{1}}(s) = +\infty \quad P^{\theta^*} - \text{a.s. on } \{\Omega_t \text{ i.o.}\}$.

Suppose that the result holds for $N-1$. So, P^{θ^*}-a.s. on $\{\Omega_t \text{ i.o.}\}$, there exists $\{t_k\}_{k=1}^{\infty}$ such that $s \in \Omega_{t_k-(N-1)} \cap \Omega_{1,t_k}^{N-1} = \Omega_{(t_k+1)-N} \cap \Omega_{1,(t_k+1)-1}^{N-1}$ so that

$$\begin{aligned} \sum_{t=N}^{\infty} E^{P^{\theta^*}}\left[1_{\Omega_{t-N} \cap \Omega_{1,t}^{N}} \middle| \mathcal{F}_{t-1}\right](s) &\geq \sum_{k=1}^{\infty} E^{P^{\theta^*}}\left[1_{\Omega_{(t_k+1)-N} \cap \Omega_{1,t_k+1}^{N}} \middle| \mathcal{F}_{t_k}\right](s) \\ &\geq \sum_{k=1}^{\infty} P^{\theta^*}\left[s_{t_{k+1}} = 1 \middle| \mathcal{F}_{t_k}\right](s) = +\infty, \end{aligned}$$

and it follows by Lemma 20 that $\sum_{t=N}^{\infty} 1_{\Omega_{t-N} \cap \Omega_{1,t}^{N}}(s) = +\infty \quad P^{\theta^*} - \text{a.s. on} \{\Omega_t \text{ i.o.}\}$. That completes the induction argument and the proof. ■