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EC9D3 Advanced Microeconomics
Additional Questions - Set 4

1. For each of the three properties characterizing majority voting between two alternatives according to the May Theorem (anonimity, neutrality between alternatives, and positive responsiveness) exhibit an example of a social welfare functional $F(R_1, ..., R_I)$ distinct from majority voting and satisfying the other two properties. This shows that none of the three properties is redundant for the result.

2. Aggregate income $\bar{y} > 0$ is to be distributed among a set $\mathcal{I}$ of individuals to maximise the utilitarian social welfare function, $W = \sum_{i \in \mathcal{I}} u_i$. Suppose that $u_i = \alpha_i (y_i)^\beta$, where $\alpha_i > 0$ for all $i \in \mathcal{I}$.

(i) Show that if $0 < \beta < 1$, income must be distributed equally if and only if $\alpha_i = \alpha_j$ for all i and j.

(ii) Now suppose that $\alpha^i \neq \alpha^j$ for all i and j. What happens in the limit as $\beta \to 0$? How about as $\beta \to 1$? Interpret.

3. Atkinson (1970) proposes an index of equality in the distribution of income based on the notion of 'equally distributed equivalent income,' denoted y_e. For any strictly increasing, symmetric, and quasiconcave social welfare function over income vectors, $W(y^1, ..., y^N)$, income y_e is defined as that amount of income which, if distributed to each individual, would produce the same level of social welfare as the given distribution. Thus, letting $\mathbf{e} \equiv (1, ..., 1)$ and $\mathbf{y} = (y_1, ..., y_N)$, we have $W(y_e \mathbf{e}) \equiv W(\mathbf{y})$.

Letting μ be the mean of the income distribution $\mathbf{y}$, an index of equality in the distribution of income then can be defined as follows: $I(\mathbf{y}) \equiv y_e / \mu$

(i) Show that $0 < I(\mathbf{y}) \leq 1$ whenever $y_i > 0$ for all i.

(ii) Show that the index $I(\mathbf{y})$ is always 'normatively significant' in the sense that for any two income distributions, $\mathbf{y}_1, \mathbf{y}_2$ with the same mean, $I(\mathbf{y}_1)$ is greater than, equal to, or less than $I(\mathbf{y}_2)$ if and only if $W(\mathbf{y}_1)$ is greater than, equal to, or less than $W(\mathbf{y}_2)$, respectively.

4. Let x and y be distinct social alternatives. Suppose that the social choice is at least as good as x for individual i whenever x is at least as good as every other social alternative for i. Suppose also that the social choice is at least as good as y for individual j whenever y is at least as good as every other social alternative for j. Prove that $i = j$.

Answers

1. We consider the three pairs of axioms in sequence.

(i) One rule that satisfies anonimity and neutrality, but not positive responsiveness, is "anti-majority." Using the same notation as in the lecture notes, let $n^+(q) = \#\{i : q(i) = 1\}$ and $n^-(q) = \#\{i : q(i) = -1\}$. The anti-majority social welfare rule F is such that: $F(q) = 1$ if and only if $n^+(q) < n^-(q)$, $F(q) = -1$ if and only if $n^+(q) > n^-(q)$, and $F(q) = 0$ if and only if $n^+(q) = n^-(q)$.

(ii) Super-majority quorum rules satisfy anonimity and positive responsiveness, but not neutrality. Letting alternative x denote the status quo, and given any quorum $Q > 1/2$, the Q-quorum social welfare rule F is such that: $F(q) = 1$ if and only if $n^+(q) > Q[n^-(q) + n^+(q)]$, $F(q) = -1$ if and only if $n^+(q) < Q[n^-(q) + n^+(q)]$, and $F(q) = 0$ if and only if $n^+(q) = Q[n^-(q) + n^+(q)]$.

(iii) Weighted majority rules satisfy neutrality and positive responsiveness, but not anonimity. Consider any profile of weights $\mathbf{w} = (w_1, w_2, ..., w_N)$ such that $w_i \geq 0$ for all i and $\sum_i w_i = 1$. Let $n^+(q, \mathbf{w}) = \sum_{i:q(i)=1} w_i$ and $n^-(q, \mathbf{w}) = \sum_{i:q(i)=-1} w_i$. The $\mathbf{w}$-weighted majority social welfare rule F is such that: $F(q, \mathbf{w}) = 1$ if and only if $n^+(q, \mathbf{w}) > n^-(q, \mathbf{w})$, $F(q, \mathbf{w}) = -1$ if and only if $n^+(q, \mathbf{w}) < n^-(q, \mathbf{w})$, and $F(q, \mathbf{w}) = 0$ if and only if $n^+(q, \mathbf{w}) = n^-(q, \mathbf{w})$. The only $\mathbf{w}$-weighted majority social welfare rule F that satisfies anonimity is such that $w_i = 1/N$ for all i.

2. We consider the two parts of the question in sequence.

(i) Let's set up the welfare maximization problem:

$$\max_{(y_1,\ldots,y_I)} \sum_{i\in\mathcal{I}} \alpha_i {y_i}^\beta \text{ s.t. } \sum_{i\in\mathcal{I}} y_i = \bar{y}.$$

Form the Lagrangian:

$$\mathcal{L} = \sum_{i \in \mathcal{I}} \alpha_i {y_i}^{\beta} + \lambda(\sum_{i \in \mathcal{I}} y_i - \bar{y}).$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial y_i} = \beta \alpha_i {y_i}^{\beta - 1} + \lambda = 0,$$

and they lead to the conditions, for every pair of agents i, j:

$$\beta \alpha_i {y_i}^{\beta-1} = \beta \alpha_j {y_j}^{\beta-1}. \tag{1}$$

The equality $y_i = y_j$ is satisfied if and only if $\alpha_i = \alpha_j$.

The second order conditions are satisfied because $0 < \beta < 1$.

(ii) When $\alpha^i \neq \alpha^j$ for all i and j, the condition (1) is satisfied if and only if $y_i/y_j = (\alpha_i/\alpha_j)^{1/(1-\beta)}$. Hence, for $\beta \to 0$, we have $y_i/y_j = \alpha_i/\alpha_j$: the higher weight a_i of individual i relative to the weight a_j of individual j translates proportionally in the ratio of incomes y_i/y_j. Instead, for $\beta \to 1$, we have that $y_i/y_j \to \infty$ if $\alpha_i > \alpha_j$: any higher weight a_i of individual i relative to the weight a_j of individual j lead to an infinitely larger income y_i to i relative to the income y_j to j.

3. We consider the two parts of the question in sequence.

(i) Note that $I(\mathbf{y}) = y_e/\mu = Ny_e/\sum_i y_i$. Because $y_i > 0$ for all i, it must be the case that $y_e > 0$, and hence that $I(\mathbf{y}) > 0$. To see that $I(\mathbf{y}) \leq 1$, we proceed as follows. Let $\Pi(\mathcal{N})$ be the set of all permutations π of the vector of individuals $\mathcal{N} = \{1, 2, ..., N\}$. For any such a permutation π, let $\mathbf{y}'(\pi, \mathbf{y})$ be the vector $\mathbf{y}'$ such that $y'_{\pi(i)} = y'_i$ for all i. Let $\Pi(\mathbf{y})$ be the set of all vectors $\mathbf{y}'(\pi, \mathbf{y})$ for all permutation $\pi \in \Pi(\mathcal{N})$. Because the welfare function W is symmetric, we observe that $W(\mathbf{y}) = W(\mathbf{y}'(\pi, \mathbf{y}))$ for all $\mathbf{y}'(\pi, \mathbf{y}) \in \Pi(\mathbf{y})$. Mixing among the vectors $\mathbf{y}'(\pi, \mathbf{y})$ in $\Pi(\mathbf{y})$ with uniform probability, we obtain the vector $\mu\mathbf{e}$. Hence, the vector $\mu\mathbf{e}$ is a convex combination of the vectors $\mathbf{y}'(\pi, \mathbf{y})$ in $\Pi(\mathbf{y})$. By concavity of W it is then the case that $W(\mu\mathbf{e}) \leq W(\mathbf{y})$. Monotonicity of W and using $W(y_e\mathbf{e}) = W(\mathbf{y})$ imply that $y_e \leq \mu$. Because $I(\mathbf{y}) = y_e/\mu$, we obtain $I(\mathbf{y}) \leq 1$.

(ii) This is an immediate consequence of monotonicity of the welfare function W, and the fact that $\mathbf{y}_1$ and $\mathbf{y}_2$ have the same mean μ. In fact, $I(\mathbf{y}_1) = y_{e1}/\mu \leq (\geq) I(\mathbf{y}_2) = y_{e2}/\mu$ if and only if $y_{e1} \leq (\geq) y_{e2}$ if and only if $W(y_{e1}\mathbf{e}) \leq (\geq) W(y_{e2}\mathbf{e})$.

4. By contradiction, suppose that $j \neq i$. Pick a preference profile R such that the preferences of i and j are strict, x is preferred alternative of i and $y \neq x$ is the preferred alternative of j; i.e., xP_iz for all $z \in X$, and yP_jz for all $z \in X$. Because xR_iz for all $z \in X$, it must be the case that $f(R)R_ix$. Because i's preferences are strict, it must be that $f(R) = x$. Likewise, because yR_jz for all $z \in X$, it must be the case that $f(R)R_jy$, and hence that $f(R) = y$. This contradicts $f(R) = x$ because f is a social choice function.