EC9D3 Advanced Microeconomics, Part I: Lecture 4

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Intertemporal choice

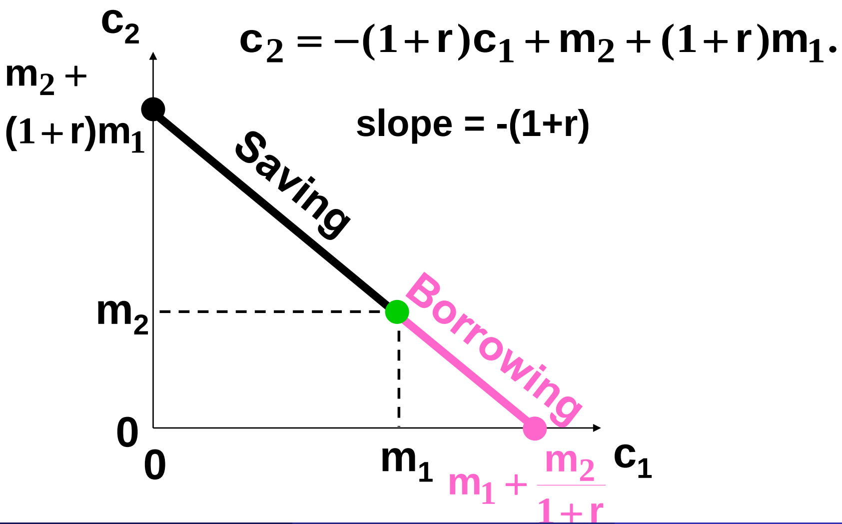
- Intertemporal consumer choice is represented by treating commodities consumed in different periods, $t = 0, 1, ..., T$ as different goods, q_t.

- The intertemporal budget constraint requires that the discounted present value of the spending stream equals the discounted present value of the income stream, y_t, $t = 0, 1, ..., T$;

$$\sum_t^T \frac{\mathrm{p}_t \mathrm{q}_t}{(1+r)^t} \leq \sum_t^T \frac{y_t}{(1+r)^t} = Y$$

where r is the market discount rate.

The Intertemporal Budget Constraint



Intertemporal choice (2)

- We assume separability of preferences across periods.
- Separability restricts the role of memory and anticipation in determining preferences, ruling out habits in consumption.
- Let's assume preferences are strongly intertemporally separable

$$u = \sum_{t}^{T} \phi_t (\mathrm{q}_t)$$

for some concave functions $\phi_t(\cdot)$.

- The MRS between goods consumed in any two periods is independent of the quantities consumed in any third period.

Intertemporal choice (3)

- Let's assume within period preferences are homothetic, and write the problem of allocating spending across periods as:

$$\max_{\mathrm{c}} \sum_{t}^{T} \psi_t\left(c_t\right) \text{ s.t. } \sum_{t}^{T} c_t a_t \frac{\mathrm{p}_t}{(1+r)^t} \leq \sum_{t}^{T} \frac{y_t}{(1+r)^t}$$

  for some concave functions $\psi_t(\cdot)$ and time-specific price indices $a_t(\cdot)$.

- The intertemporal choice problem has the form of a problem of demand with endowments.

- The goods are total consumptions in each period and endowments are incomes in each period.

- Effects of interest rate changes depend on whether one is a net seller or net buyer of these goods, i.e., whether she is a saver or a borrower.

Intertemporal choice (4)

- We write, for some concave function $v(\cdot)$:

$$\psi_t(c_t) = \frac{v(c_t)}{(1+\delta)^t}$$

- $\delta \in (0,1)$ is a subjective discount rate reflecting downweighting of future utility relative to the present and hence capturing impatience.
- Assuming prices $a_t p_t$ are constant, for simplicity, then F.O.C. require

$$\frac{v'(c_t)}{v'(c_s)} = \left(\frac{1+\delta}{1+r}\right)^{t-s}.$$

- If $\delta = r$ then $c_t = c_s$ because of concavity of $v(\cdot)$. Concavity captures the desire to smooth the consumption stream.
- If $r > \delta$, then the chosen consumption follows a rising path.

Producer Theory

- So far, we have considered consumer choice.
- Consider now the other side of the market: *the production side.*
- Individual agent = *the firm.*
- The firm $\Longleftrightarrow$ *production activity*.
- The firm *produces outputs by using inputs* (both measured in terms of flow amounts per unit time).
- Both outputs and inputs are commodities.

Net Outputs and Inputs

In particular let:

- y_j^o = quantity of commodity j produced by the firm as output,
- y_j^i = quantity of commodity j used as input,
- $z_j = y_j^o - y_j^i$ is a net output/input depending on whether the sign of z_j is positive/negative.

Production Plan

Production plan = vector of net outputs and/or inputs of all available commodities

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_L \end{pmatrix}$$

Without loss of generality we assume that:

- the *first h* commodities are *net inputs*
- the *remaining $(L - h)$* commodities are *net outputs.*

Production Plan (2)

- Define:

$$x_1 = -z_1, \ldots, x_h = -z_h, y_1 = z_{h+1}, \ldots, y_{L-h} = z_L$$

- A *production plan* is:

$$z = \begin{pmatrix} -x_1 \\ \vdots \\ -x_h \\ y_1 \\ \vdots \\ y_{L-h} \end{pmatrix}$$

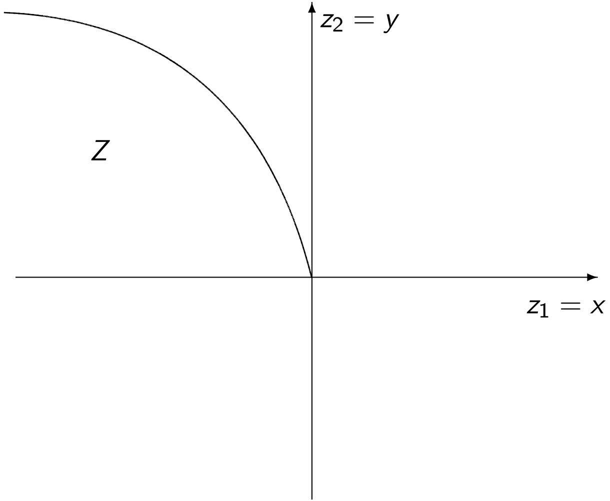
Production Possibility Set

Definition

The *production possibility set $Z \in \mathbb{R}^L$* is the set of all technologically feasible production plans.

- All vectors of inputs and outputs that are technologically feasible.
- Z provides *a complete description of the technology* identified with the firm.

One Input x and One Output y: $z = (-x, y)$



Short Term and Long Term Production Plan

Sometime it is possible to distinguish between:

- *immediately* technologically feasible production plans $Z(\overline{x}_l, \ldots, \overline{x}_h)$;
- *eventually* technologically feasible production plans Z.

If the input x_1 is fixed at the level $\overline{x}_1$ then we can define a *short-run or restricted production possibility set:*

$$Z(\overline{x}_1) = \left\{ z = \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \middle| \; x_1 = \overline{x}_1 \right\}$$

Input Requirement Set and Isoquant

- A special feature of a technology is the *input requirement set:*

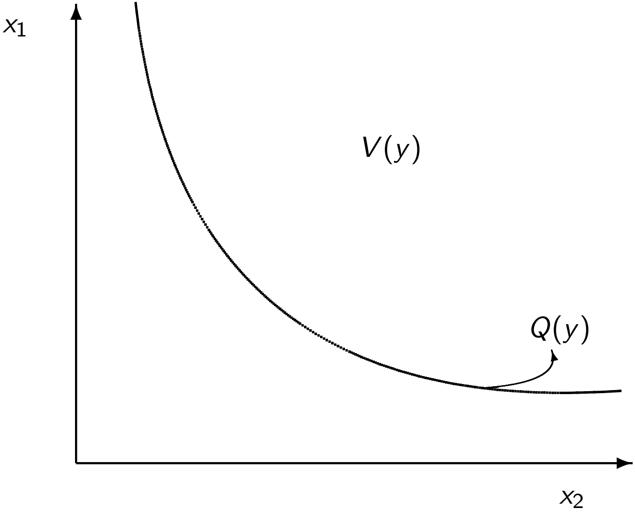
$$V(y) = \left\{ x \in \mathbb{R}^h_+ \middle| \begin{pmatrix} -x \\ y \end{pmatrix} \in Z \right\}$$

- This is the set of all input bundles that produce at least y units of output.

- We define also the *isoquant* to be the set all input bundles that allow the firm to produce exactly y:

$$Q(y) = \left\{ x \in \mathbb{R}^h_+ \middle| x \in V(y) \text{ and } x \notin V(y'),\ \forall y' > y \right\}$$

Input Requirement Set and Isoquant (2)



Production Function

Definition (Production Function)

In the case of only one output, define the production function $f(x)$

$$f(x) = \sup_{y'} \left\{ \begin{pmatrix} -x \\ y' \end{pmatrix} \in Z \right\}$$

as the maximal output associated with the input bundle x.

A production function identifies the maximum output associated with a given level of output.

Technological Efficiency

Definition (Technologically Efficient Production Plan)

The production plan $z = \begin{pmatrix} -x \\ y \end{pmatrix}$ is *technologically efficient*, if and only if there does *not* exist a production plan $z' = \begin{pmatrix} -x' \\ y' \end{pmatrix}$ such that $z' \geq z$ ($z'_i \geq z_i \ \forall i$) and $z' \neq z$.

If z efficient it is not possible to produce more output with a given input or the same output with less input (sign convention).

Examples of Technologies

- *Cobb-Douglas Technology:*

$$f(x_1, x_2) = x_1^{\alpha} \, x_2^{\beta}, \quad \alpha > 0, \ \beta > 0$$

or

$$Z = \left\{ \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \middle| y \le x_1^{\alpha} \, x_2^{\beta} \right\}$$

and

$$Q(y) = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \middle| y = x_1^{\alpha} \, x_2^{\beta} \right\}$$

and

$$V(y) = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \middle| y \le x_1^{\alpha} \, x_2^{\beta} \right\}$$

Examples of Technologies (2)

- *Leontief technology:*

$$f(x_1, x_2) = \min\{ax_1,\ bx_2\}, \quad a > 0,\ b > 0$$

or

$$Z = \left\{ \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \middle| y \leq \min\{ax_1,\ bx_2\} \right\}$$

and

$$Q(y) = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ \middle| y = \min\{ax_1,\ bx_2\} \right\}$$

and

$$V(y) = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ \middle| y \leq \min\{ax_1,\ bx_2\} \right\}$$

Efficiency imposes: $\quad x_1 = y/a, \quad x_2 = y/b.$

Examples of Technologies (3)

- *Perfect Substitutes:*

$$f(x_1, x_2) = ax_1 + bx_2, \quad a > 0,\ b > 0$$

or

$$Z = \left\{ \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \middle| y \leq ax_1 + bx_2 \right\}$$

and

$$Q(y) = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ \middle| y = ax_1 + bx_2 \right\}$$

and

$$V(y) = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ \middle| y \leq ax_1 + bx_2 \right\}$$

Production Possibility Set Assumptions

Conditions we impose on the set PPS Z:

1. *Z is closed* (it contains its boundaries).

   Important property for the definition of production function (sup is a max).

2. $0 \in Z$

   Uncontroversial property in the long run, not necessarily in the short run (inputs used with no outputs).

3. *Free disposal, monotonicity:* if $z \in Z$ and $z' \leq z$ then $z' \in Z$.

   Alternatively: if $x \in V(y)$ and $x' \geq x$ then $x' \in V(y)$.

   Given a production plan if either one increases the quantity of inputs or reduces the quantity of output the new production plan is still feasible.

Production Possibility Set Assumptions (2)

4. *Additivity:* if $z, z' \in Z$ then $z + z' \in Z$.

   For $f(x)$ this property implies $f(x^1 + x^2) \geq f(x^1) + f(x^2)$.

5. *Convexity of $V(y)$:* if $x, x' \in V(y)$ then $tx + (1 - t)x' \in V(y)$ for every $0 \leq t \leq 1$ which means that $V(y)$ is convex set (rescaling of production processes).

   A similar condition may (or may not) be imposed *on the Z:* if $z, z' \in Z$ then $tz + (1 - t)z' \in Z$ for every $0 \leq t \leq 1$, or Z is a convex set.

   Notice: the latter condition is stronger than the former.

Convexity and the Production Possibility Set

Result

The convexity of Z implies the convexity of $V(y)$.

The opposite implication does not hold.

Result

The convexity of $V(y)$ implies that the $f(x)$ is quasi-concave.

Proof: By definition $V(y) = \{x \mid f(x) \geq y\}$ is convex if it is convex for every y, let $y = k$ then $f(x)$ is quasi-concave. □

Convexity and the Production Possibility Set (2)

Result

The convexity of Z implies that $f(x)$ is (weakly) concave.

Proof: Consider

$$z = \begin{pmatrix} -x \\ f(x) \end{pmatrix} \in Z, \qquad z' = \begin{pmatrix} -x' \\ f(x') \end{pmatrix} \in Z$$

Convexity of Z implies that for every $0 \leq t \leq 1$

$$t\,z + (1-t)\,z' = \begin{pmatrix} -(t\,x + (1-t)\,x') \\ tf(x) + (1-t)f(x') \end{pmatrix} \in Z$$

By definition of $f(x)$ this means:

$$tf(x) + (1-t)f(x') \leq f(tx + (1-t)x')$$

for every $0 \leq t \leq 1$, the definition of a concave $f(x)$. □

Returns to Scale

- *Decreasing Returns to Scale:* (DRS) if $z \in Z$ then $t\,z \in Z$ for every $0 \leq t \leq 1$.
- *Increasing Returns to Scale:* (IRS) if $z \in Z$ then $t\,z \in Z$ for every $t \geq 1$.
- *Constant Returns to Scale:* (CRS) if $z \in Z$ then $t\,z \in Z$ for every $t \geq 0$.

Returns to Scale (2)

Result

$0 \in Z$ and Z convex imply DRS.

Proof: follows from the definition of convexity applied at $z' = 0$. $\square$

Result

A technology exhibits CRS if and only if the production function $f(x)$ (if available) is homogeneous of degree 1.

Proof: Assume CRS: this implies that if $z \in Z$ then $t\,z \in Z$, for all $t \geq 0$.

By definition, $z \in Z$ means $y \leq f(x)$ and $t\,z \in Z$ means $t\,y \leq f(t\,x)$.

Returns to Scale (3)

By definition of $f(x)$ choose z, and hence x and y, so that $y = f(x)$.

We can then re-write the condition above as:

$$t\, f(x) \leq f(t\, x)$$

We need to prove that the equality holds.

Suppose it does not. Then there exists y' such that

$$t\, f(x) < y' < f(t\, x)$$

Now $y' < f(t\, x)$ implies by definition of Z that

$$\begin{pmatrix} -t\, x \\ y' \end{pmatrix} \in Z$$

Returns to Scale (4)

and by CRS we get

$$\frac{1}{t}\begin{pmatrix} -t\,x \\ y' \end{pmatrix} \in Z \qquad \text{or} \begin{pmatrix} -x \\ \frac{1}{t}\,y' \end{pmatrix} \in Z$$

which means

$$(1/t)\,y' \leq f(x)$$

or

$$y' \leq t\,f(x)$$

This latter inequality contradicts $t\,f(x) < y'$.

The opposite implication is an immediate consequence of the definition of homogeneity of degree 1. □

Returns to Scale (5)

Weaker conditions apply for DRS and IRS technology.

Result

Consider a technology characterized by a homogenous of degree $\alpha < 1$ ($\alpha > 1$) production function. This technology exhibits DRS (IRS).

The opposite implication does not hold.

Result

Assume that $f(0) = 0$ then we can prove:

- *$f(x)$ concave implies DRS;*
- *$f(x)$ convex implies IRS;*
- *$f(x)$ concave and convex implies CRS.*

Few Definitions

- Marginal Product of input x_i:

$$MP = \frac{\partial f(x)}{\partial x_i}$$

- Average Product of input x_i:

$$AP = \frac{f(x)}{x_i}$$

- Marginal Rate of Technical Substitution between input x_i and x_j:

$$\frac{dx_i}{dx_j} = \frac{\partial f(x)/\partial x_j}{\partial f(x)/\partial x_i}$$

the *slope of the isoquant*.

Few Definitions (2)

- The set of output bundles that are efficient for a given technology is the *Production Possibility Frontier:*

$$PPF(x) = \left\{ y \middle| \nexists z' \in Z \text{ s.t. } z' \geq z = \begin{pmatrix} -x \\ y \end{pmatrix} \right\}$$

- We can then define the Marginal Rate of Transformation between output y_m and y_n:

$$MRT = \frac{dy_m}{dy_n}$$

as the *slope of the PPF*.

The Competitive Firm

Assume that input and output prices are taken parametrically (no influence on such prices).

The basic producer problem is then *profit maximization:*

$$\max_{\{x,y\}} \quad p\,y - \sum_{i=1}^{h} w_i\,x_i$$

$$s.t. \quad \begin{pmatrix} -x \\ y \end{pmatrix} \in Z$$

Profit Maximization

Let

- the h-dimensional vector of input prices be $w = (w_1, \ldots, w_h)$;
- the $(L-h)$-dimensional vector of output prices be $p = (p_1, \ldots, p_{L-h})$.

We can re-write the producer's problem as:

$$\max_{\{z\}} \quad \hat{p}\, z$$
$$s.t. \quad z \in Z$$

where $\hat{p} = (w, p)$ and $z = \begin{pmatrix} -x \\ y \end{pmatrix}$.

Profit Maximization (2)

In the case of a technology that produces only one output the profit maximization problem may be written as:

$$\max_{\{x\}} \quad p\, f(x) - w\, x$$

The necessary first order conditions of this problem are:

$$p\, \nabla f(x^*) \leq w$$

or

$$\frac{\partial f(x^*)}{\partial x_i} \leq \frac{w_i}{p}$$

and

$$\left[\frac{\partial f(x^*)}{\partial x_i} - \frac{w_i}{p}\right] x_i^* = 0.$$

Profit Maximization (3)

In the event that the production possibility set is convex (the production function is concave) the first order conditions are of course both necessary and sufficient.

In other case, the following set of sufficient conditions for a local maximum have to verified:

The Hessian matrix of the production function has to be negative definite at the point x^*.

This condition can be checked by the determinant condition according to which the leading principal minors have to alternate sign starting from the negative one.

Profit Maximization (4)

For the case of two variables the *first order conditions* are for $i = 1, 2$:

$$\frac{\partial f(x^*)}{\partial x_i} \leq \frac{w_i}{p}$$

and

$$\left[\frac{\partial f(x^*)}{\partial x_i} - \frac{w_i}{p}\right] x_i^* = 0$$

while the *second order conditions* are:

$$H = \begin{pmatrix} \frac{\partial^2 f(x^*)}{\partial x_1^2} & \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2} \\ \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2} & \frac{\partial^2 f(x^*)}{\partial x_1^2} \end{pmatrix} \quad \textit{negative definite}$$

Unconditional Factor Demands and Supply Function

Which implies:

$$\frac{\partial^2 f(x^*)}{\partial x_i^2} < 0$$

and

$$\frac{\partial^2 f(x^*)}{\partial x_1^2} \frac{\partial^2 f(x^*)}{\partial x_2^2} - \left(\frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2}\right)^2 > 0$$

The solution to the *profit maximization* problem if it exists provides the *unconditional factor demands:*

$$x^*(p, w)$$

By substitution it is possible to obtain the *supply function* of the producer:

$$y^*(p, w) = f(x^*(p, w)).$$

Properties of Unconditional Factor Demands

1. Non-positive own factor demands price effects (SOC) (generalizes to h inputs):

$$\frac{\partial x_1^*}{\partial w_1} \leq 0 \qquad \frac{\partial x_2^*}{\partial w_2} \leq 0$$

2. Symmetry (generalizes):

$$\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1}$$

3. Complementary inputs (generalizes):

$$\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1} < 0$$

Properties of Unconditional Factor Demands (2)

4. Substitutability of inputs (it does not generalize):

$$\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1} > 0$$

5. Positive output price effects (generalizes):

$$\frac{\partial x_1}{\partial p} > 0 \qquad \frac{\partial x_2}{\partial p} > 0$$

(If x_1 and x_2 are complementary inputs.)

Properties of Supply Function

Obtained differentiating the supply function of the firm:

$$y(p, w) = f(x^*(p, w))$$

6. Own price effect non-negative:

$$\frac{\partial y}{\partial p} \geq 0$$

7. Symmetry:

$$-\frac{\partial x_i}{\partial p} = \frac{\partial y}{\partial w_i}$$

for $i = 1, 2$.

Summary of the Properties

These comparative statics properties of both the unconditional factor demands and the supply function can be summarized as follows:

$$\begin{pmatrix} \dfrac{\partial y}{\partial p} & \dfrac{\partial y}{\partial w_1} & \dfrac{\partial y}{\partial w_2} \\ -\dfrac{\partial x_1}{\partial p} & -\dfrac{\partial x_1}{\partial w_1} & -\dfrac{\partial x_1}{\partial w_2} \\ -\dfrac{\partial x_2}{\partial p} & -\dfrac{\partial x_2}{\partial w_1} & -\dfrac{\partial x_2}{\partial w_2} \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} + & a & b \\ a & + & c \\ b & c & + \end{pmatrix}$$

Other properties

8. Both $x^*(p, w)$ and $y(p, w)$ are homogeneous of degree 0.

Proof: If you increase both input and output prices of a factor $t > 0$ you obtain:

$$\max_x (tp)\, f(x) - (tw)\, x = t\,[p\, f(x) - w\, x]$$

which clearly is solved by the same vector $x^*(p, w)$ that solves:

$$\max_x \; p\, f(x) - w\, x$$

Further, by definition of supply function:

$$y(tp, tw) = f(x^*(tp, tw)) = f(x^*(p, w)) = y(p, w). \qquad \square$$

Profit Function and Its Properties

Definition

Define the profit function as:

$$\pi(p, w) = \max_x \; p\, f(x) - w\, x = p\, f(x^*(p, w)) - w\, x^*(p, w)$$

Properties of the Profit Function:

1. $\dfrac{\partial \pi}{\partial w_i} \leq 0$ and $\dfrac{\partial \pi}{\partial p} \geq 0$.

2. $\pi(p, w)$ homogeneous of degree 1 in (p, w).

   Proof: It follows from the homogeneity of degree 0 of $y(p, w)$ and $x(p, w)$ and the definition of $\pi(p, w)$. □

Hotelling Lemma

3. *Hotelling Lemma* (which proves property 1):

$$\frac{\partial \pi}{\partial p} = y(p, w) \geq 0$$

and

$$\frac{\partial \pi}{\partial w_i} = -x_i(p, w) \leq 0$$

Proof: By Envelope Theorem. □

Convexity of the Profit Function

4. $\pi(p, w)$ convex in (p, w).

Proof: Consider the two price vector (p, w) and (p', w') and for every scalar $\lambda \in (0, 1)$ let $p'' = \lambda\, p + (1 - \lambda)\, p'$ and $w'' = \lambda\, w + (1 - \lambda)\, w'$

Then :

$$\begin{aligned}
\pi\left[p'', w''\right] &= p''\ f(x^*(p'', w'')) - w''\ x^*(p'', w'') \\
&= \lambda\left[p\ f(x^*(p'', w'')) - w\ x^*(p'', w'')\right] \\
&\quad + (1-\lambda)\left[p'\ f(x^*(p'', w'')) - w'\ x^*(p'', w'')\right] \\
&\leq \lambda\, \pi(p, w) + (1-\lambda)\, \pi(p', w')
\end{aligned}$$

which proves convexity of $\pi(p, w)$. □